# Supersymmetric quantum mechanics, phase equivalence, and low energy scattering anomalies

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Supersymmetric quantum mechanics links two Hamiltonians with the same scattering (phase equivalence) but different number of bound states. We examine the Green's functions for these Hamiltonians as a prelude to embedding the two-body dynamics in a many-body system. We study the effect of the elimination of a two-body bound state near zero energy for the Efimov effect and Beg's theorem.

There are many applications where it is convenient to represent scattering off a composite object by a local potential; sometimes the projectile itself is also composite. The phenomenological local potential one naturally constructs may permit bound states of the projectile-target system that are in fact forbidden by the Pauli principle. Supersymmetric quantum mechanics[1] (SSQM) can be used to construct the unique local potential that has exactly the same scattering amplitude as the first potential, but does not support the forbidden bound state. This was first discussed by Baye[2] in the problem of alpha-alpha scattering and has subsequently been studied in detail in a number of formal[3] and numerical[4] examples.

Suppose one wants to employ this phase-equivalent potential in a few- or many-body system. For example, we have recently discussed[3] how SSQM can be used to construct a nucleon-alpha local potential without the bound state as a prelude to application to deuteron-alpha scattering. There are results in few- or many-body scattering, for example, the Efimov effect,[5,6] or Beg's theorem,[7] that are supposed to depend only on features of the physical scattering and be independent of the potential. Are they in the case of phase equivalent potentials? The purpose of this note is to explain how they are *not*, particularly in the case of scattering with a low energy anomaly, that is with a nearby zero energy bound state.

It is generally believed that in potential scattering a very large positive scattering length is the sign of a virtual (near bound) state while a very large negative scattering length is the sign of a bound state near threshold. For a very small binding energy $B$ ($B > 0$), the scattering length is given by $a = -1/\sqrt{B}$ (we take $\hbar = 2m = 1$). Efimov's effect,[5] the existence of a logarithmically divergent number of three-body bound states in a system of three identical bosons interacting by pairwise potentials in the limit that $B \to 0$, is said to depend only on the divergence of the scattering length and not on the details of the potential. Hence one might expect that the phase-equivalent potential shows the effect as well, since the scattering length is the same (by the definition of phase-equivalence). But the phase-equivalent potential has no two-body bound state. It is in fact mostly repulsive, in particular at very small and large $r$. How then can it have any three-body bound state, let alone an infinite number? We will show that it does not. In essence the phase-equivalent potential is pathological. It can have a very large scattering length and no bound state. It avoids Levinson's theorem[8] by having a $6/r^2$ repulsive behavior at the origin for an $s$-wave potential. As we shall see it also becomes long range as $B \to 0$.

Beg's theorem[7] also raises questions for phase equivalence. The theorem states that scattering of a projectile from a set of static objects depends only on the on-shell two body scattering input if the range of the force is finite and the objects are separated by more than that range. We shall see what goes wrong with this argument for phase equivalence and in particular when the original potential has a bound state near zero energy.

Since our argument is framed in the language of SSQM,[1,9] we begin by reviewing it briefly. Consider the $s$-wave radial Schrödinger Hamiltonian for a local potential $V_1(r)$; with $\hbar = 2m = 1$ and $r$ running from 0 to $\infty$, we have

$$H_1 = -\frac{d^2}{dr^2} + V_1(r) \tag{1}$$

with eigenstates

$$H_1 \Psi_n^{(1)} = E_n \Psi_n^{(1)} , \tag{2}$$

where $n = 0, 1, \ldots$, includes the continuum. We assume that there is a lowest bound state $\Psi_0^{(1)}$ at energy $E_0 = -B = -K^2$ ($B > 0$). Then we can write Eq. (1) in factored form[9]

$$H_1 = A_1^+ A_1^- - B , \tag{3}$$

where

 

$$A_1^{\pm} = \pm\frac{d}{dr} + W_1 \ . \tag{4}$$

The superpotential $W_1$ is given in terms of the bound-state wave function by

$$W_1 = \frac{d\Psi_0^{(1)}/dr}{\Psi_0^{(1)}} \ . \tag{5}$$

The supersymmetric partner Hamiltonian to $H_1$, $H_2$ is defined by

$$H_2 = A_1^- A_1^+ - B = H_1 - 2\frac{dW_1}{dr} \ , \tag{6}$$

and it is easy to show that $H_2$ is isospectral (same bound states and continuum) to $H_1$

$$H_2\Psi_n^{(2)} = E_n\Psi_n^{(2)} \tag{7}$$

for $n = 1, 2, \ldots$, i.e., except for the state at $-B$ that is present in $H_1$ but absent in $H_2$ since

$$A_1^-\Psi_0^{(1)} = 0. \tag{8}$$

The eigenfuntions of $H_2$ for energies $E_n, n = 1, 2, \ldots$, are given in terms of those of $H_1$ at the corresponding energy, up to a normalization factor, by

$$\Psi_n^{(2)} = A_1^-\Psi_n^{(1)} \ . \tag{9}$$

In general, $2dW_1/dr$ is nontrivial and $H_2$ is a different Hamiltonian from $H_1$.

By considering asymptotic states, the $S$ matrix $S_1$ of $H_1$ and that $S_2$ of $H_2$ at energy $E$ ($E = k^2$) are related by[3,9]

$$S_2 = \frac{k - iK}{-k - iK}S_1 \ . \tag{10}$$

This relation shows that the bound state pole of $S_1$ at $k = iK$ is removed in $S_2$. These facts follow from the asymptotic value of the superpotential (5), $W_1(\infty) = -K$.

We can rewrite Eq. (6) in the factorized form

$$H_2 = A_2^+ A_2^- - B \tag{11}$$

with $A_2$ given by

$$A_2^{\pm} = \pm\frac{d}{dr} + W_2 \tag{12}$$

and the superpotential $W_2$ given by

$$W_2 = \frac{d\Psi_0^{(2)}/dr}{\Psi_0^{(2)}} \tag{13}$$

with, now,

$$H_2\Psi_0^{(2)} = -B\Psi_0^{(2)} \ . \tag{14}$$

However, there is no true bound-state eigenfunction of $H_2$ at the energy $E = -B$. Hence $\Psi_0^{(2)}$, while regular at the origin $r = 0$, is non-normalizable and grows like $e^{Kr}$ at large $r$. Nevertheless, the above construction is legitimate and $W_2$ is bounded. The important feature is not the exponential growth of $\Psi_0^{(2)}$ at large $r$, which cancels in $W_2$, but rather that $\Psi_0^{(2)}$ has no nodes because $-B$ is below the bound-state spectrum of $H_2$. We can now construct the supersymmetric partner $H_3$ of $H_2$:

$$H_3 = A_2^- A_2^+ - B = H_2 - 2\frac{dW_2}{dr} = H_1 - 2\frac{dW_1}{dr} - 2\frac{dW_2}{dr} \tag{15}$$

with

$$H_3\Psi_n^{(3)} = E_n\Psi_n^{(3)} \tag{16}$$

for $n = 1, 2, \ldots$, and (up to a normalization factor)

$$\Psi_n^{(3)} = A_2^-\Psi_n^{(2)} = A_2^- A_1^-\Psi_n^{(1)}. \tag{17}$$

Clearly, $H_3$ has the same spectrum as $H_2$ with no missing states, but also with no extra states; in particular, it has no bound state at $-B$. Since the asymptotic value of the superpotential in Eq. (13) is $W_2(\infty) = +K$, we obtain for the $S$ matrix $S_3$, using Eq. (10)

$$S_3 = \frac{k + iK}{-k + iK}S_2 = S_1 \ . \tag{18}$$

Hence $H_3$ and $H_1$ are phase-equivalent Hamiltonians; i.e., they have the same $S$ matrix even though they correspond to different potentials. In particular, $H_1$ has a bound state at $E = -B$ while $H_3$ does not.

To better understand what is happening and to apply this understanding to the Efimov effect or to Beg's theorem, let us examine the full interacting Green's functions of each of the Hamiltonians in Eqs. (3), (6), and (15) and relate them. In terms of the properly normalized states of $H_2$, the corresponding partial-wave Green's function $G_2^+(p)$ at energy $E = p^2$ can be written as

$$G_2^+(p) = \sum_n \frac{\Psi_n^{(2)}(\Psi_n^{(2)})^\dagger}{p^2 - E_n} \ . \tag{19}$$

If for simplicity we take $H_2$ to have only a continuum spectrum (we have removed the bound state at $E = -B = -K^2$) with $E_n = k^2$, we have

$$\Psi_n^{(2)} = \frac{A_1^-\Psi_n^{(1)}}{ik - K} \ , \tag{20}$$

where the factor $ik - K$ comes from the action of $A_1^-$ on the incoming wave at very large $r$. It is such that $\Psi_n^{(2)}$ has a properly normalized incoming wave if $\Psi_n^{(1)}$ does. We then get

$$G_2^+(p) = \sum_n \frac{A_1^-\Psi_n^{(1)}(\Psi_n^{(1)})^\dagger A_1^+}{(p^2 - E_n)(E_n + B)} \ . \tag{21}$$

Writing

$$\frac{1}{E_n + B}\frac{1}{p^2 - E_n} = \frac{1}{p^2 + B}\left(\frac{1}{p^2 - E_n} + \frac{1}{E_n + B}\right) \ , \tag{22}$$

we obtain finally

$$G_2^+(p) = A_1^- \left( \frac{G_1^+(p) - G_1^+(iK)}{p^2 + B} \right) A_1^+ , \tag{23}$$

where we have been a bit cavalier about interchanging the order of the summation and operation with $A_1^\pm$. This is particularly clear from the appearance of $G_1^+(iK)$, which is infinite since $G_1^+$ has a pole at $p = iK$, but since the operator $A_1^-$ annihilates the residue at that pole (the bound-state wave function) there is no singularity. In fact, it is easy to show that $A_1^- G_1^+(iK) A_1^+$ is equal to the unit operator.

An exactly equivalent analysis gives

$$G_3^+(p) = A_2^- \left( \frac{G_2^+(p) - G_2^+(iK)}{p^2 + B} \right) A_2^+ , \tag{24}$$

where now the $G_2^+(iK)$ term is neither singular nor trivial.

To understand what happens to these Green's functions, in particular, when there is a bound state near zero energy in $H_1$, consider the Green's functions in coordinate space. Suppose the potential $V_1(r)$ has a finite and short range $\mu$. For $r, r' > \mu$ the outgoing-wave Green's function $G_1^+(p)$ is

$$G_1^+(r, r'; p) = \frac{-i}{2p} [e^{ip|r-r'|} - S_1(p) e^{ip(r+r')}] . \tag{25}$$

For $r > \mu$, the lowest-energy wave function is $\Psi_0^{(1)}(r) \propto e^{-Kr}$ so that in this region Eq. (5) reduces to $W_1(r) = -K$. It is then easy to show using Eq. (21) or (23) that

$$G_2^+(r, r'; p) = \frac{-i}{2p} [e^{ip|r-r'|} - S_2(p) e^{ip(r+r')}] \tag{26}$$

for $r, r' > \mu$, with $S_2$ is given by Eq.(10). The solution $\Psi_0^{(2)}(r)$ of Eq. (14) reads for $r > \mu$

$$\Psi_0^{(2)}(r) = \alpha e^{Kr} + \beta e^{-Kr} . \tag{27}$$

Hence, the superpotential $W_2(r)$ of Eq. (13) for $r > \mu$ is

$$\begin{aligned} W_2(r) &= K \frac{\alpha e^{Kr} - \beta e^{-Kr}}{\alpha e^{Kr} + \beta e^{-Kr}} \\ &= K \left( 1 - 2 \frac{\beta e^{-Kr}}{\alpha e^{Kr} + \beta e^{-Kr}} \right) . \end{aligned} \tag{28}$$

That is, $W_2(r)$ goes to $+K$ for large $r$ with corrections of the order of $2K\beta e^{-2Kr}/\alpha$. We then find that $G_3^+(r, r'; p)$ given by Eq. (24) is for $r, r' > \mu$ ($r_>$ and $r_<$ denote, when required, the larger and the smaller of $r$ and $r'$),

$$\begin{aligned} G_3^+(r, r'; p) = -\frac{i}{2p} \Bigg\{ & e^{ip(r_> - r_<)} \frac{[p - iW_2(r_<)][p + iW_2(r_>)]}{p^2 + K^2} + \frac{p}{iK} e^{-K(r_> - r_<)} \frac{[K - W_2(r_<)][K + W_2(r_>)]}{p^2 + K^2} \\ & + \Gamma_1(K) \frac{2p}{p^2 + K^2} e^{-K(r+r')} \frac{[K + W_2(r)][K + W_2(r')]}{(2K)^2} \\ & - S_1(p) e^{ip(r+r')} \frac{[p + iW_2(r)][p + iW_2(r')]}{(p + iK)^2} \Bigg\} , \end{aligned} \tag{29}$$

where $\Gamma_1(K)$ is the residue of $S_1(p)$ at the pole at $k = iK$. This term comes from $S_2(p)$. Because $W_2(r) \to K$ as $r \to \infty$ we see that for fixed $K$ $G_3^+(r, r'; p) \to G_1^+(r, r'; p)$ as $r, r' \to \infty$. This is phase equivalence at the Green's-function level. However, for fixed $r$ and $r'$ it is easy to verify that $G_3^+(r, r'; p)$ has no pole at $p = iK$, that is, that $G_3^+(r, r'; iK)$ is perfectly finite, while of course $G_1^+(r, r'; iK)$ is not. Hence the limits $r, r' \to \infty$, $p \to iK$ do not commute (we have seen a similar feature of SSQM before[9]). We have found explicit, analytic calculation of these Green's functions using a delta shell potential helpful.

From Eq. (29) we can begin to understand what happens in composite systems. While, for fixed $K$, $G_3 = G_1$ asymptotically, one is never really asymptotic in the composite system. In particular, the range of the differences between $G_3$ and $G_1$ is set by $1/K$ and not by the range of the potential $V_1$. Hence in finite systems $H_3$ is not equivalent to $H_1$. What goes wrong in Beg's theorem is particularly easy to see from Eqs. (29) and (25). If $V_1$ has range $\mu$ and the scatterers are static and separated by distances greater than $2\mu$, all that ever enters in the propagation of the scattered wave from scatterer to scatterer is $G_1$ of (25), and it depends only on on-shell scattering [the actual case is more complex because (25) is only for one partial wave, but adding the partial waves up will not change the substance of the argument]. But $G_3$ is not asymptotic for $r, r' > \mu$. If the distance between the scatterers is not large compared with $1/K$, propagation from scatterer to scatterer is substantially different in $G_3$ and $G_1$ and Beg's theorem is different for $H_3$ from $H_1$. In particular, as $K \to 0$ the interscatterer distance must go to infinity for the scattering to depend only on the on-shell two-body data.

For the Efimov effect,[5] the situation is somewhat more complex. Amado and Noble[6] show that if the homogeneous (bound-state) three-body equation is written

$$\phi = \mathcal{K}\phi , \tag{30}$$

where $\phi$ is a three-body wave function and $\mathcal{K}$ is an inte-

gral operator that depends parametrically on the three-body center-of-mass energy $E$, the Efimov effect arises because an infinite number of the eigenvalues of $\mathcal{K}$, $\lambda_n$, ($\mathcal{K}\chi_n = \lambda_n\chi_n$) exceed 1 as $B \to 0$ and $E \to 0$. This fact is manifested in the divergence of the trace of $\mathcal{K}$ in this limit since $\mathrm{tr}\mathcal{K} = \sum_n \lambda_n$. They write Eq. (30) explicitly as ($\hbar = 2m = 1$)

$$\phi(\mathbf{p},\mathbf{k}) = \int d^3p' \frac{2\langle \mathbf{k} + \mathbf{p}/2|t(E - 3/2p^2)|\mathbf{p}' + \mathbf{p}/2\rangle}{E - 2p^2 - 2k^2 - 2\mathbf{p}\cdot\mathbf{k}}\phi(\mathbf{p}',\mathbf{p}) \tag{31}$$

and argue that since the divergence of the trace is a low energy or small momentum phenomenon, dominated by the two-body virtual or bound-state pole of the two-body $t$-matrix near zero energy, one can study the effect by using a separable potential with that pole. The vertex functions of that separable potential contribute nothing to the low energy singular behavior of the integral. They do ensure convergence at high momenta. The singular (as $E \to 0, B \to 0$) contribution of the separable $t$ matrix to the trace can be expressed in terms of the full two-body Green's function at fixed $r$ and $r'$, the $r$ dependence being taken up in the irrelevant vertex functions. The divergent part of the trace of the kernel of Eq. (31) comes then from the lower limit of the following integral

$$\mathrm{tr}\mathcal{K}(E) = \int d^3k\, \mathcal{K}(\mathbf{k},\mathbf{k}) \sim \int \frac{k^2 dk}{E - 6k^2} G\sqrt{E - \tfrac{3}{2}k^2}. \tag{32}$$

If we use $G_1$ of (25) with $B = 0$ in (32) we note that since $S = -1$ for $K = 0$ and $p$ small, $G_1(p) \sim -i/p$ near $p = 0$, and the trace diverges logarithmically as $E \to 0$. This is the Efimov effect. For $G_3(p)$ at $K = 0$ and $r, r' > \mu$ we find for small $p$,

$$G_3(r,r';p) \simeq \frac{-i}{p}[1 - rW_2(r)][1 - r'W_2(r')]. \tag{33}$$

The $i/p$ term appears to again trigger the Efimov divergence, but in fact the $1 - rW_2$ factors are zero at $K = 0$ and there is no divergence. In deriving (33) we have assumed that vertex factors will keep the domain of $r, r'$ finite so that the exponentials can be expanded. As we have stressed above it is precisely this finite $r, r'$ dependence that makes $G_3 \neq G_1$. To study the vanishing of (33) we need to know $W_2(r)$ for $K = 0$. Equation (28) does not give a clear limit. At $B = 0$ the solution of (14) that is regular at the origin becomes the zero energy scattering solution. Hence $\Psi_0^{(2)}$ with $B = 0$ goes over to that solution. Outside the range of force ($r > \mu$), the most general form of $\Psi_0^{(2)}$ at $B = 0$ is $r + a_2$, where $a_2$ is the scattering length of $H_2$. But at $K = 0$, $S_1 = -1$ at low energy, with corrections of order $k^2$. Hence from (10), $S_2 = 1$ up to order $k^2$, which means that $a_2 = 0$. If $\Psi_0^{(2)} = r$ for $r > \mu$ then $W_2(r) = 1/r$. Thus the coefficient of $i/p$ in $G_3$ vanishes at $K = 0$ and $\mathcal{K}$ has finite trace.

There is no Efimov effect for $H_3$. This is reassuring since for most simple models $V_3$ turns out to be completely repulsive, if $V_1$ has only one bound state. Even one three-body bound state, let alone an infinite number, would be surprising for a repulsive two-body potential. Furthermore, for $H_1$, the many bound states become few with increasing coupling strength because for $B > 0$, there is a scattering threshold at $E = -B$ that covers the Efimov poles. Amado and Noble have stressed that the Efimov effect arises from the singular collision of the two-body threshold with the three-body threshold at $E = 0$. For $H_3$ there is no such threshold, and hence if we had an infinite number of bound states, we would not be able to get rid of them. Thus the phase-equivalent Hamiltonian does not lead to a large number of three-body bound states, even though it has a large negative two-body scattering length. Nevertheless, the point $E = 0, B = 0$ is a singular point of the integral equation and care must be exercised in numerical solutions in the vicinity of that point. This is clear from the fact that at $B = 0$, $W_2 = 1/r$ so that at large $r$, $V_3$ has a $2/r^2$ long-range repulsion.

In conclusion, we have seen that, using SSQM, it is possible to relate two local partial-wave Hamiltonians that yield the same on-shell scattering ($S$ matrix) while one has a bound state and the other does not. This can be done even if that bound state is very near threshold so that the scattering length is very large. We have examined situations in which these two dynamical schemes are the input to few- or many-body scattering to see whether features such as the Efimov effect or Beg's theorem, which are purported to depend only on the on-shell scattering data, are identical for the two Hamiltonians. We find that they are not. For example, there is no Efimov effect for the two-body dynamics that does not have the bound state, even though the scattering length is very large. We have done our analysis by using the interacting Green's functions of the various Hamiltonians, exploiting the formal algebraic relations among the sytems.

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