Nuclear medium modification of nucleons via a Bethe ansatz model

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The modification of the internal structure of nucleons inside the nucleus is investigated within the context of a constituent quark model, in which quarks interact forming bound clusters that act as nucleons. The elastic form factor, the momentum distribution, the correlation function, and the Coulomb sum rule are calculated. The contribution to these structure functions arising from nucleon degrees of freedom is identified by means of an impulse approximation, in which the quark structure of nucleons is assumed to be unaffected by other nucleons, and the dynamics of nucleons is given by an effective theory, extracted from our quark model. The effect of the modification of the nucleon is found to be noticeable on the elastic form factor at large momentum transfer, and on the Coulomb sum rule. A cluster approximation is also used to study the extent to which these effects are due to exchange of quarks.

I. INTRODUCTION

There is considerable interest in finding evidence for the influence of the nuclear environment on the internal structure of nucleons. One of the most promising tools is the scattering of weak probes, e.g., electrons by nuclear targets. There are already two unexpected experimental results that may well be the outcome of modification of the nucleon's structure in the nuclear medium: the *EMC effect* (for a review, see Ref. 1) and the depletion of the longitudinal response function of nuclear targets.[2]

The structure functions for elastic and inelastic scattering contain information on both quark and nuclear degrees of freedom. (For this work, the substructure of the nucleus is assumed to be given in terms of quark degrees of freedom.) The problem is how to separate both contributions to the structure function. What is usually done is to compare the experimental data to the results from the *impulse approximation* with free-nucleon structure; the differences are assigned to the modification of nucleon structure in the nucleus. One is left with the problem of interpreting any effect in terms of the dynamics of the system.

In this work we use a dynamical model of a nuclear system to test the sensitivity of scattering structure functions to nuclear effects on the internal structure of nucleons. Having a dynamical model, one can investigate the causes of any effects obtained. However, our model does not attempt to incorporate all the features of the theory of strong interactions.

The model nucleus studied here is a system of quarks in which bound clusters of quarks act as nucleons. The spectrum of the system includes nuclear excitations, excitations of the internal structure of clusters, and even cluster breakup. We report on the results for a deuteronlike system, that is, a system with just enough quarks to form two nucleons.

The quark system has been chosen to be of the type that can be solved exactly using the *Bethe ansatz* technique (Sec. III): a one-dimensional, nonrelativistic gas of fermions with a two-body, zero-range interaction. In a previous work[3] this same kind of system was employed to study the ground state of nuclear matter in terms of clusters of quarks. The results of that work provided some knowledge of the spectrum of the nuclear system under consideration.

In order to carry out the study of scattering from the nuclear system, we need also a more detailed knowledge of the states of the system in the finite case (finite in both number of quarks and length). These states can be obtained using the Bethe ansatz method, which in this case is well known.[4,5] However, this form (in configuration space) of the states makes the calculation of structure and correlation functions very difficult. We will adopt another approach proposed by Sasaki and Kebukawa[6] which consists in the use of momentum representation and second quantization. By reducing the results of Ref. 7 to a simpler form, we are able to calculate correlation and structure functions in an analytical way (details are given in Refs. 8 and 9).

For the purpose of our study, the results obtained from the quark model are regarded as "experimental" data. The part that is due to nuclear degrees of freedom will be "predicted" by introducing a nucleon impulse approximation based on the assumption that the structure of nucleons is not affected by the nucleus. The nuclear states are given by an effective Hamiltonian which reproduces the energy spectrum of the quark model, corresponding to states of nucleons without internal excitation. The differences between the exact results of the model and the "theoretical" predictions from the impulse approximation are the signature of modification of nucleon structure by the nucleus.

The impulse approximation just mentioned ignores the

exchange of quarks between nucleons due to the antisymmetry of the nuclear state in quark variables. This (Pauli) exchange can be included by introducing a *cluster approximation,* based on the impulse approximation for the nuclear wave functions. Now, however, the state is completely antisymmetrized under exchange of quarks among the nucleons. This approach has been used by several authors[10] to investigate the effects of quark exchange, but with nucleon wave functions unconnected to quark dynamics (e.g., harmonic oscillators). However, our method has the advantage that the effective theory that we use is derived from our model quark Hamiltonian, so that we can study other quark effects in addition to Pauli exchange.

The method of study that we describe here follows the same lines of another model which has been recently developed and published in a series of papers.[11] The details of that model differ considerably from ours in the quark interaction, mechanism for confinement, and definition of an effective theory. The methods of solution also differ, in that we use analytic methods, in contrast to the Monte Carlo techniques of Ref. 11. We have taken a similar approach to theirs in comparing quark results to a nucleon impulse approximation.

We will start by introducing the model in Sec. II. Section III contains a brief summary of the Bethe ansatz technique as found in the literature. In Sec. IV we study the systems of one and two clusters, which constitute our model nucleon and deuteron, and we derive the complex solutions to the Bethe equations of four fermions in the finite case. The momentum representation of the states of the model deuteron is introduced in Sec. V. These states are then used in Sec. VI to calculate the experimental value of the structure functions. The effective theory (with nucleon degrees of freedom) is extracted in Sec. VII, and used in Secs. VIII and IX to calculate the theoretical predictions from the impulse and cluster approximation. Conclusions are given in Sec. X.

II. QUARK MODEL OF THE NUCLEUS

We consider a model system of N fermion quarks moving in one spatial dimension, interacting through an attractive zero-range potential. The nonrelativistic Hamiltonian is (here we use natural units with $c=\hbar=1$)

$$\hat{H}=\sum_{i=1}^{N}\frac{\hat{p}_i^{\,2}}{2m}+g\sum_{i<j}^{N}\delta(x_i-x_j)\ , \tag{1}$$

in the coordinates x_i and momenta p_i of the quarks, with mass m; the strength of the potential is given by g, which is negative. We include only one internal degree of freedom carried by the quarks, which we call *color.* The solutions of the Schrödinger equation for such a system are obtainable from the Bethe ansatz, as given in the following section. We first describe a number of properties of the states of this system, which characterize our model.

If there are n_c different colors, then bound states of up to n_c quarks exist; the bound state of $n=n_c$ quarks is color neutral [a singlet with respect to the SU(n_c) color symmetry]. These color-neutral bound states or *clusters* are our model nucleons. A nuclear system consists of A such nucleons, or $N=n_c A$ quarks, and is also color neutral. However, the effective interaction of these nucleons is repulsive, and so the nuclear system has no bound states.

We therefore confine the N quarks within a length L by imposing *periodic* boundary conditions on the quark variables. This has the following properties: First, the solutions of the Schrödinger equation are still of the Bethe ansatz form, with discrete quantum numbers (Bethe *momenta*). Second, the system is translationally invariant, which is appropriate for the applications of scattering theory, used later in this work. Third, since the quarks are confined within a length L, the color is also confined to the nuclear size, even though individual nucleons may dissociate *within* a nucleus.

We shall take the number of colors $n_c=2$. It would be natural to take $n_c=3$, in correspondence with QCD, for which the nucleons have fermionic behavior. This has been done previously[3] for a model of the ground state of nuclear matter. However, the system with $n_c=2$ actually has very similar dynamics to that with $n_c=3$, even though the $n_c=2$ clusters are quasibosons, in the sense of having even properties under exchange of coordinates. (This can be seen by comparing Ref. 3 with the results for $n_c=2$ in the attractive case given by Gaudin.[4]) The reason for the similarity is the repulsion among clusters, which makes the spectrum fermionic, independent of statistics. Since the $n_c=2$ case is considerably more tractable for the calculation of structure functions, we shall use $n_c=2$ in this work.

We take the $A=2$ system, with four quarks, as our model nuclear target; in the ground state, it resembles a deuteron. The excitation properties of this system are sufficiently complex to allow for investigation of both quark and nuclear degrees of freedom. This will become clear by describing the full excitation spectrum.

First, one nucleon cluster (without boundary conditions) has one bound state, color singlet, as mentioned earlier. The binding energy of our model nucleon is $E_0=-B/2$ [see (17) below], with the unit of energy B defined as in (8). There is a continuum of unbound two-quark states, $E>0$, color neutral (but singlet or triplet), as well as a continuum of cluster states with momentum Q and energy $E_Q=-B/2+Q^2/4m$. With the periodic boundary conditions on length L, the spectrum becomes discrete, with total momentum Q quantized. Again, there is one bound state, $E_0<0$, but with $E_0\neq -B/2$, because of effects of the boundary condition [see Fig. 1 and, after (17)]. There are excited states ($Q\neq 0$) of one cluster, as well as two quark excited states, whose energies depend on L.

Next, for the $N=4$ system, with no boundary condition, the spectrum includes states of two moving clusters, one cluster and two unbound quarks, or four unbound quarks, all states in the continuum, with total energy

$$E=-\frac{\nu}{2}B+\text{kinetic energy}\ , \tag{2}$$

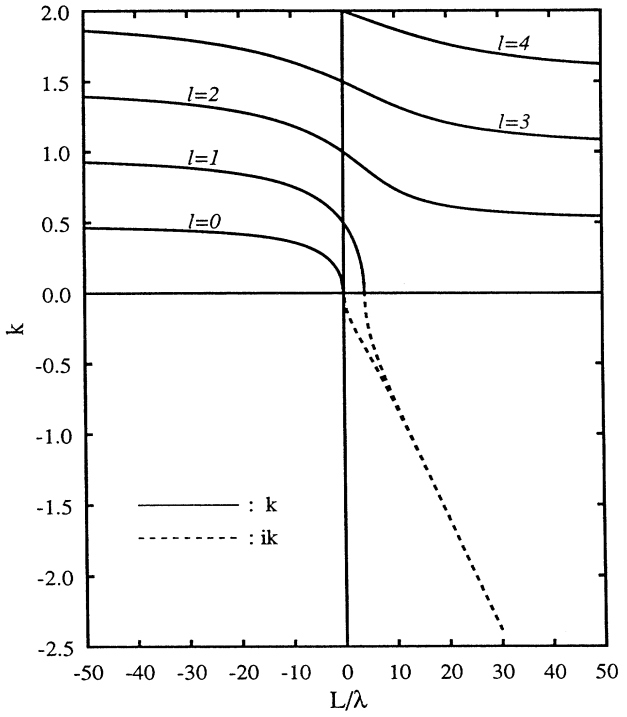

FIG. 1. Bethe momentum k of the two-boson system or the two-fermion color singlet, in units of $2\pi/L$. The dashed lines correspond to purely imaginary k, in which case ik has been plotted. Positive values of the dimensionless parameter L/λ correspond to attractive interaction, while a repulsive interaction leads to L/λ negative.

where ν is the number of clusters (2,1,0). With the imposition of the periodic boundary condition, the spectrum becomes discrete, the energies depending on L. One can classify the spectrum by the number of clusters ν; the energies will approach (2) as $L\rightarrow\infty$. The properties just discussed for the $N=2$ and 4 systems are shown explicitly in Sec. IV.

III. BETHE ANSATZ

A. Infinite length

The Hamiltonian (1) represents one of the few examples found in the literature of a system of particles interacting via a two-body potential, which can be solved exactly.[12] Gaudin[4] and Yang[5] have shown how to construct the eigenstates of this Hamiltonian in the case of fermions with one internal degree of freedom (they considered fermions with spin $\frac{1}{2}$ which is equivalent to our model in the two-color case).

The spatial wave function may be expressed as a *finite* superposition of plane waves in the form

$$\psi(x_1,\ldots,x_N)=\Psi(X)\sum_{\mu,\nu\in S_N}a_\mu^\nu\theta_\nu(x_1,\ldots,x_N)\times\exp\left[i\sum_{j=1}^{N}k_{\mu_j}x_{\nu_j}\right], \quad (3)$$

where $\mu=(\mu_1,\ldots,\mu_N)$ and $\nu=(\nu_1,\ldots,\nu_N)$ are permutations of order N, and the sums run over the $N!$ permutations of the group S_N. The step function $\theta_\nu(x_1,\ldots,x_N)$ has been defined for each permutation ν of the coordinates as

$$\theta_\nu(x_1,\ldots,x_N)\equiv\begin{cases}1, & \text{if } x_{\nu_1}<x_{\nu_2}<\cdots<x_{\nu_N},\\ 0, & \text{otherwise}.\end{cases} \quad (4)$$

The function $\Psi(X)=\exp(iQX)$ with X, the center-of-mass coordinate, and Q, the total momentum of the system, has been factored for convenience.

The wave function (3) gives the exact form of the eigenstates of the Hamiltonian (1): it is referred to as the *Bethe ansatz* in the literature. The set of numbers $\{k_1,k_2,\ldots,k_N\}$ are called *Bethe momenta.* They completely define the states and have the following properties: All of them must be different, and their sum must be zero, while the sum of their squares is related to the eigenstates of the Hamiltonian by the relation

$$E=\frac{1}{2m}\left[\frac{Q^2}{N}+\sum_{j=1}^{N}k_j^2\right]. \quad (5)$$

The first term is the translational energy of the system as a whole, and the second term gives the energy in the frame moving with the system.

In the repulsive case (g positive), any set of N different real numbers $\{k_1,k_2,\ldots,k_N\}$ leads to an eigenstate; therefore, the energy spectrum is a continuum of positive numbers. On the other hand, in the attractive case, the momenta k can come in complex-conjugate pairs with a discrete imaginary part, leading to a spectrum with a discrete negative part. The $(N!)^2$ coefficients a_μ^ν are completely specified by the condition that ψ, as defined by (3), be an eigenfunction of the Hamiltonian (1) and by imposing the symmetries of an irreducible representation of the symmetric group over the function ψ.[5]

There are several other examples of integrable systems in quantum field theory which have solutions of the Bethe ansatz type. An alternate, more rigorous approach to solving integrable systems is the *inverse scattering method* (for a discussion of that method and a comparison with the Beth ansatz technique, see Ref. 13). However, for the case that interests us—a fermionic field with an internal degree of freedom and inside a finite length—inverse scattering techniques have not been extended yet, and the only approach found in the literature is the Bethe ansatz.

B. Finite length

If the particles are confined to a finite length L with periodic boundary conditions, the Bethe momenta are no longer arbitrary, but are the solutions of a system of coupled equations. The system of equations has been derived and discussed in Refs. 4 and 5; however, we find it con-

venient for our calculations later on this paper to rewrite them in a system of dimensionless units.

We will give the momenta k_j and Q as multiples of $2\pi/L$. Note that the periodic boundary condition then requires Q to be an integer, while the k_j can in principle be any real numbers (or even complex in the attractive case), as long as they are different and their sum vanishes. The energy (5) can then be written as

$$E=B\left[\frac{2\pi\lambda}{L}\right]^2\left[\frac{Q^2}{N}+\sum_{j=1}^{N}k_j^2\right], \tag{6}$$

where the constants λ and B have been defined as

$$\lambda\equiv-\frac{1}{mg}, \tag{7}$$

$$B\equiv\frac{mg^2}{2}. \tag{8}$$

In the case of attractive interaction, g is negative and the constant λ is positive; it is a length associated with the system, and as we will see below, it gives the size of the clusters. The constant B has units of energy, and as we will show shortly, it is twice the binding energy of a cluster. For a given value of the dimensionless parameter L/λ, the constants B and $1/\lambda$ fix the energy and momentum scales, respectively. If the dimensionless momenta are multiplied by $2\pi\lambda/L$, then they are given in units of $1/\lambda$; to find their value in natural units, one must know the value of λ in natural units. The energy can be given in units of B, through (6).

The equations that determine the allowed values of k_j take different forms depending on the symmetry of the wave function (3). In the case of fermions with two colors, the wave function can have the space symmetries of the irreducible representations $[1^N]$ or $[2M,1^{N-2M}]$ (that is, Young *tableaux* with one or two columns). For the symmetry $[1^N]$ (completely antisymmetric wave function), the momenta k_j become

$$k_j=n_j-\epsilon. \tag{9}$$

Here the numbers n_j are all different integers whose average ϵ can take any of the following N values:

$$\epsilon=\frac{1}{N}\sum_{j=1}^{N}n_j=0,\frac{1}{N},\frac{2}{N},\ldots,\frac{N-1}{N}. \tag{10}$$

The total momentum Q must be an integer, but it is not completely arbitrary since it depends on ϵ:

$$Q=N(\epsilon+q)\quad(q\text{ integer}). \tag{11}$$

This solution is the same as for a system of noninteracting fermions (in the zero-momentum frame and dimensionless units).

Next, for the symmetry $[2^M,1^{N-2M}]$, the equations that determine k_j are (Refs. 4 and 5)

$$k_j=n_j-\epsilon+\sum_{\alpha=1}^{M}f(k_j-\Lambda_\alpha),\quad j=1,\ldots,N, \tag{12}$$

$$\sum_{j=1}^{N}f(k_j-\Lambda_\alpha)=\eta_\alpha+\sum_{\beta=1(\neq\alpha)}f\left[\frac{\Lambda_\beta-\Lambda_\alpha}{2}\right],\quad\alpha=1,\ldots,M, \tag{13}$$

where we have defined the function $f(z)$ as

$$f(z)\equiv-\frac{1}{\pi}\cot^{-1}\left[\frac{4\pi\lambda z}{L}\right], \tag{14}$$

with a branch cut along the segment from i to $-i$, and the real part of its image between $-\frac{1}{2}$ and $\frac{1}{2}$. The M numbers Λ are certain auxiliary real momenta, and the numbers η_α are all integers and had to be introduced in order to ensure that the real part of both sides in the second equation are within the same interval $(m-\frac{1}{2},m+\frac{1}{2})$ for integer m.

Gaudin[4] has studied the solution of these equations in the attractive case and for small interaction strength g. He has proven that for a given set of integers $\{n_1,\ldots,n_N\}$ there is a unique solution $\{k_1,\ldots,k_N\}$. The quantum numbers n_j can be equal by pairs, and there can be up to M pairs. Each pair leads to a pair of complex Bethe momenta which are complex conjugate of each other.

IV. QUARK MODEL OF THE DEUTERON

We apply the results of the previous section to the two cases mentioned in Sec. II: $n=2$ and 4, which lead to our model nucleon and deuteron. There are three physical parameters in the model: the nuclear length L, the mass of the quarks m, and the strength of the interaction g. The functional form of the states with a given set of quantum numbers is completely determined by the dimensionless parameter L/λ. As we mentioned above, the parameters λ and B are needed to transform from dimensionless units into natural units. Since the Hamiltonian commutes with color operators, its eigenstates can be classified according to the color-irreducible representation to which they belong. States in a $(2C+1)$-dimensional color multiplet are attained only if there are at least $M=(N/2-C)$ particles of each of the two colors. The configuration space part of the wave function must then have the symmetries of the Young tableau $[2^M,1^{N-2M}]$.

If the number of quarks is $N=2$, then the states can belong either to a color singlet or triplet. The spatial wave function for the triplet must be completely antisymmetric, and then the Bethe momenta are those of the noninteracting system (9). In the color singlet, k_1 and k_2 are given by (12) and (13) with $n=2$ and $M=1$; the result is identical to the case of two bosons.[12] The Bethe momenta are $k_1=-k_2\equiv k$, with k defined by the equation

$$k=\frac{l}{2}+f(k). \tag{15}$$

Figure 1 shows the solution for the lowest values of the

integer quantum number l as a function of the dimensionless parameter L/λ. When the interaction is repulsive (negative L/λ), all solutions are real. As the interaction strength g decreases, k approaches the values of the noninteracting system. If the interaction is attractive (L/λ positive), the momentum k for $l=0$ becomes purely imaginary (dashed line in Fig. 1). That solution constitutes a cluster of the two fermions, which is the ground state of our model nucleon. Note that if the interaction is attractive enough such that L/λ is greater than 4, the state $l=1$ also becomes a bound system with complex k. This second bound state is not just the same cluster ground state with a momentum boost; the states with $l=0$ and 1 cannot take the same values of total momentum, since $l=Q$ (mod2). The existence of a second bound state is a peculiarity introduced by the periodic boundary conditions.[12]

The free nucleon is obtained if the limit $L\rightarrow\infty$ is taken. In that limit, $\alpha\equiv -ik$ becomes $L/4\pi\lambda$ (or $1/2\lambda$ in natural units), and the density of the free nucleon, derived from the Bethe ansatz, is

$$\rho(r)=\frac{2}{\lambda}e^{-2|r|/\lambda}\ ; \tag{16}$$

therefore, the parameter λ gives a good estimate of the size of the isolated nucleon, and L/λ is the ratio of the nuclear size to the size of the free nucleon. From (6), the binding energy of the free nucleon is

$$E_0=-2B\left[\frac{2\pi\lambda}{L}\right]^2\alpha^2=-\frac{B}{2}\ . \tag{17}$$

However, for finite L/λ, $E_0\neq -B/2$ (see Fig. 1).

The system of $N=4$ quarks can form a color singlet, triplet, or quintuplet; the corresponding spatial wave functions must have symmetry [2,2], [2,1,1], and [1,1,1,1]. If the four quantum numbers n_j are all different, the solutions to (12) and (13) correspond to "free" quarks: no clusters. However, as Gaudin has shown[4] for the limit $g\rightarrow 0^-$ (weak attraction), the equations also admit solutions in which the n_j can be equal by pairs: Each pair represents a cluster.

In general, for any value of $g<0$, the equations for the momenta of the two-cluster system, which constitutes our model deuteron, are derived from (12) and (13), when the quantum numbers are equal by pairs. Since $n_1=n_2$ and $n_3=n_4$, (10) requires that $\epsilon=0$ or $\frac{1}{2}$. The coupled equations become

$$K=\frac{m}{2}+f(K+\Lambda)+f(K-\Lambda)\ , \tag{18}$$

$$f(\Lambda)=2\,\mathrm{Re}[f(K+\Lambda)-f(K-\Lambda)]\ , \tag{19}$$

$$k_1=k_2^*=-k_3^*=-k_4\equiv K\ . \tag{20}$$

There are two pairs of complex-conjugate momenta. The imaginary parts of each complex-conjugate pair are the same; consequently, the internal structure of the two clusters is identical. There is only one internal quantum number m which is a positive integer such that $m(\mathrm{mod}2)=2\epsilon$. The states are labeled by the numbers m and Q. According to (11), the total momentum of the

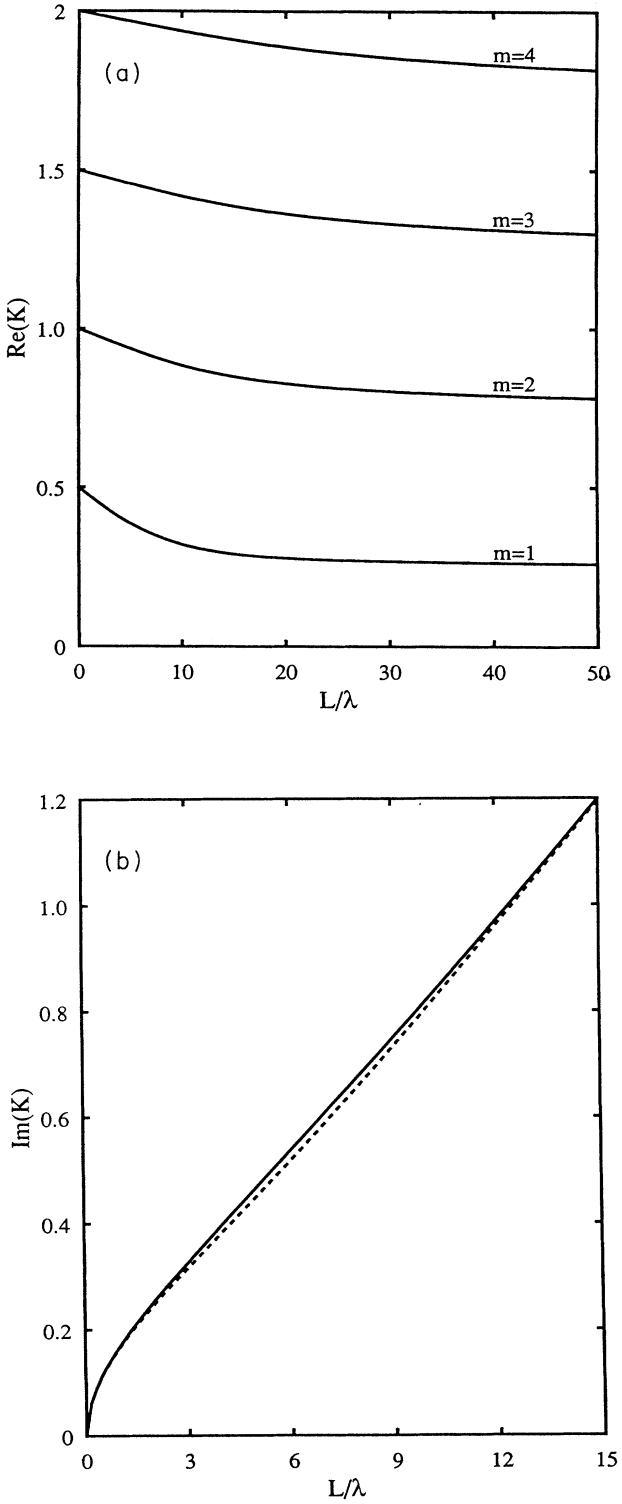

FIG. 2. Bethe momentum of the deuteron system (units of $2\pi/L$). (a) The real part is shown for the first four states. (b) The imaginary part is shown for the ground state (solid line); the dashed line is the Bethe momentum α of one cluster. The imaginary part of K in the excited states lies between the solid and dashed lines, and approaches the dashed line as the quantum number m increases.

two-cluster system can take the values

$$Q=4q+2m(\text{mod}2) \quad (q \text{ integer}) ; \tag{21}$$

therefore, there are two classes of solutions corresponding to even and odd quantum numbers m. Our model deuteron ground state is the lowest-energy two-cluster state in the c.m. frame ($Q=2$, $m=1$); this is the lowest-energy $N=4$ state in that frame.

We have solved Eqs. (18) and (19) using the Newton-Raphson algorithm. The results for some values of the parameter L/λ and the quantum number m are shown in Fig. 2. Note that the total momenta of the clusters are $\pm 2\,\mathrm{Re}(K)$. Comparing Fig. 2(a) with Fig. 1, it can be seen that the curves for $2\,\mathrm{Re}(K)$ are very similar to the subset $l=2,4,6,\ldots$ of the curves of k for the system of two bosons in the latter figure. The similarity between the spectra of the two clusters and the system of two bosons with zero-range interaction will be exploited in Sec. VII to construct an effective theory.

The imaginary part of K for the ground state goes asymptotically to $L/4\pi\lambda$ (or $1/2\lambda$ in natural units). The dashed line in Fig. 2(b) corresponds to the Bethe momentum α of the free nucleon. The clusters (nucleons) are then more tightly bound in the deuteron than when they are isolated. As $L\rightarrow\infty$, the nucleons look more like the free nucleon. For the excited states ($m>1$), the value of $\mathrm{Im}(K)$ lies in between the dashed and solid lines in Fig. 2(b). It becomes closer to the solution for the free nucleon as m increases. All excited states then have clusters with a different internal structure, which is less distorted from the structure of the free nucleon, the higher the quantum number m.

The energy in the c.m. frame is obtained from (6), which as a function of K becomes

$$E_m=B\left[\frac{4\pi\lambda}{L}\right]^2[(\mathrm{Re}K)^2-(\mathrm{Im}K)^2] . \tag{22}$$

In Fig. 3 we have plotted the energy in the first four levels for some range of the parameter L/λ. For a fixed value of the nucleon's size, λ, as L goes to infinity, the energy goes asymptotically to $-B$, which in our model corresponds to the internal binding energy of the two nucleons. In a real deuteron the nuclear energy is small compared to the internal binding energy of the nucleons; this is true in our model only for $L>5\lambda$, where the ground-state energy approaches $-B$. Therefore, on the basis of deuteron energetics, we shall fix the value of the parameter $L/\lambda=5$ for the ground state.

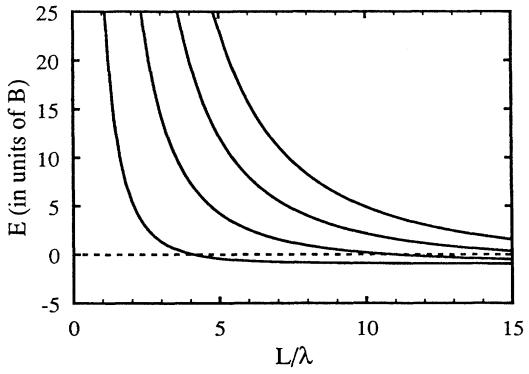

FIG. 3. Energy spectrum of the deuteron system as a function of the dimensionless parameter L/λ. The first four states are shown ($m=1,2,3,4$ from the bottom) and the energy is given in units of B (≈ 42 MeV).

If the size of the nucleon, λ, is taken to be of the order of 1 F, we may estimate the value of B. First, note that the mass of the quark is $m=1/(2B\lambda^2)$, while the binding energy of the quarks in the nucleon is $-B/2$. With a nucleon mass of 940 MeV, B must be of the order of 42 MeV.

V. MOMENTUM REPRESENTATION

For the calculation of structure functions, it is useful to work directly in a momentum representation. Indeed, the wave function (3) in position space is cumbersome for this purpose, because of the changes of form at each of the boundaries $x_i=x_j$. Fortunately, the momentum representation of the Bethe ansatz wave functions has been studied by Sasaki and Kebukawa, in a series of papers.[6,7] By starting with the case of only two particles, which can be solved easily with the techniques of one-particle quantum mechanics, they then inferred what the general solution for any N should be. They also used second quantization, which is a more convenient approach to systems of identical particles, and which we adopt here.

For the case of fermions with two colors, the Hamiltonian (1) becomes

$$\hat{H}=B\sum_p\left[\frac{2\pi\lambda p}{L}\right]^2(a_p^\dagger a_p+b_p^\dagger b_p) - \frac{2B\lambda}{L}\sum_{p,q,r}a_{p+q}^\dagger b_{r-q}^\dagger b_r a_p , \tag{23}$$

where the operators $a_p^\dagger$ and $b_p^\dagger$ create quarks of colors a and b, respectively, and with dimensionless integer momentum p. Their Hermitian conjugates annihilate the same quarks.

The states of two fermions coupled into a color singlet have the general form

$$|\phi\rangle=\sum_{p_1,p_2}\delta(p_1+p_2-Q)\phi\left[\frac{p_1-p_2}{2}\right]a_{p_1}^\dagger b_{p_2}^\dagger|0\rangle , \tag{24}$$

with an even wave function $\phi(p)$. The wave function that makes these states eigenstates of the Hamiltonian is

$$\phi(p)=\frac{\beta}{p^2-k^2} , \tag{25}$$

where k is the Bethe momentum defined by (15), and β is a normalization constant.

The state of one cluster of fermions is obtained from (24) and (25), when the quantum number l of (15) is made equal to zero, and the momentum k then becomes purely imaginary ($k\equiv i\alpha$). Thus the wave function for a cluster

of quarks is

$$\omega(p)=\frac{\beta}{p^2+\alpha^2} . \tag{26}$$

The state (24) with the wave function (26) gives the state of the nucleon. The value of α is shown in Fig. 2(b) as the dashed line.

To obtain the states of two clusters, we look at the eigenstates of four fermions coupled into a color singlet. Their form is given in Ref. 7; however, they can be reduced to the simpler form (details are given in Refs. 8 and 9)

$$|\Omega\rangle=\frac{1}{\sqrt{4}}\sum_{p_1,p_2,p_3,p_4}\Omega(p_1-Q/4,\ldots,p_4-Q/4)b^\dagger_{p_1}a^\dagger_{p_2}a^\dagger_{p_3}b^\dagger_{p_4}|0\rangle , \tag{27}$$

where the wave function is

$$\Omega(p_1,p_2,p_3,p_4)=\beta\sum_{\nu\in S_4}\frac{(-1)^\nu\delta(p_1+p_2+p_3+p_4)}{(p_1-k_{\nu_1})(p_4-k_{\nu_4})}\left[\frac{k_{\nu_2}}{p_2-k_{\nu_2}}-\delta_\nu\frac{k_{\nu_1}+k_{\nu_2}}{p_1+p_2-k_{\nu_1}-k_{\nu_2}}\right] ; \tag{28}$$

the number δ_ν is just a reminder that if for a given set of numbers k_j, the sum $k_{\nu_1}+k_{\nu_2}$ becomes zero, then the second term inside the large parentheses should be ignored.

VI. STRUCTURE FUNCTIONS

In this section we undertake the main purpose of this investigation, which is the calculation of structure functions of our model nuclear system, with the goal of looking for effects of the quark substructure of the nucleons. The structure functions considered are ground-state matrix elements of various one- or two-body operators, which are, in principle, measured in the scattering of a weakly scattered projectile, e.g., an electron. They are enumerated below.

The main task is the calculation of the structure functions for the model ground state. As we mentioned above, this is impractical to do in configuration space, even though the Bethe ansatz forms are known explicitly for the model, because of the changes of form at each of the boundaries $x_i=x_j$. The momentum representation is more directly useful; we shall use the ground-state function in the form given in (28), which is a simple rational function. The structure functions then reduce to a set of multiple infinite sums over momenta, with constraints for momentum conservation. Although this can be done in principle by direct numerical calculation, it is difficult to obtain good numerical accuracy in reasonable times. Further, it would be quite difficult to extend the calculation to a larger system. Therefore, we have developed a set of finite algorithms to evaluate all the infinite sums analytically.

The algorithms amount to partial fraction expansions that lead to sums that can be evaluated in a simpler form. However, rather than doing a partial fraction expansion first, which generates many terms, and then evaluating the sums, we evaluate sums first, leaving the results in terms of certain functions. We do not attempt to write each function in an algebraic form, but we give a set of rules to evaluate its exact value at any point within its domain. We shall simply present the results of these calculations here. The details of the method are explained elsewhere.[8,9]

Two general features of the structure functions for this model system should be pointed out. First, because of the periodic boundary conditions in length L, the momenta and momentum transfer are discrete. This is an unrealistic feature of our model. There is a compensation, which is that the model is translationally invariant, a nontrivial matter for a bound system. This is a distinct advantage for the calculation of scattering properties. The second property is the power-law behavior of the structure functions for large momentum or momentum transfer. The leading (lowest) powers are easily identified from log plots at large q.

The structure functions studied here are the following.

(1) Quark momentum distribution $N(p)$ given by the probability of finding a quark with momentum p inside the nucleus. With the quark state of the nucleus—(27) and (28)—normalized to 1, the quark momentum distribution is calculated as the expectation value

$$N(p)=\tfrac{1}{4}\langle\Omega|a^\dagger_p a_p+b^\dagger_p b_p|\Omega\rangle . \tag{29}$$

The result for the deuteron is a monotically decreasing function, which decays asymptotically as p^{-4} (Fig. 4).

(2) Elastic form factor $F(q)$ defined as the Fourier transform of the density of quarks in the nucleus. It is related to the elastic-scattering cross section of elementary projectiles such as the electron. In second quantization and momentum representation, we have

$$F(q)=\tfrac{1}{4}\langle\Omega_q|\sum_p(a^\dagger_{p+q}a_p+b^\dagger_{p+q}b_p)|\Omega\rangle , \tag{30}$$

where the states $|\Omega\rangle$ and $|\Omega_q\rangle$ have the same internal structure given by m, but the total momentum of $|\Omega\rangle$ is Q, while that of $|\Omega_q\rangle$ is $Q+q$.

The result is shown as the circles in Fig. 5. At large momentum transfer q, the elastic form factor decays as q^{-6}. Beyond a certain value of q, the result becomes negative (the points where F is negative are indicated by a double circle in Fig. 5). Because of the discrete nature of

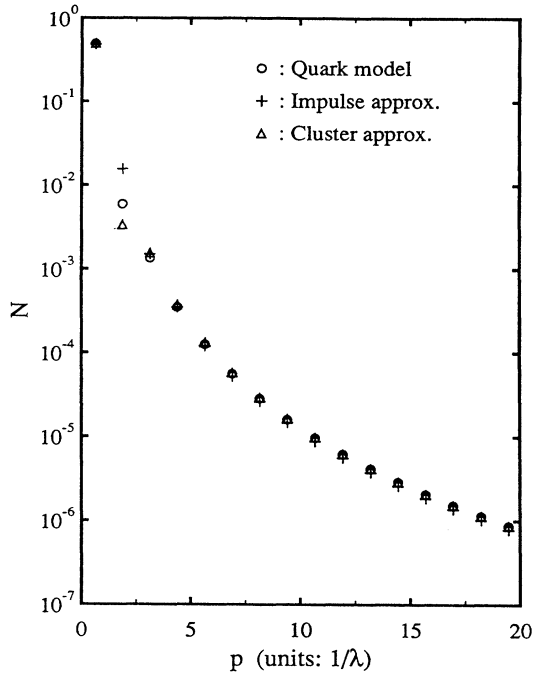

FIG. 4. Quark-momentum distribution in the quark model and cluster and impulse approximations, for $L/\lambda=5$.

the momentum transfer, our model does not give any information between $q=0$ and $q\approx 5/\lambda$.

(3) Quark correlation function, which is the Fourier transform of the probability of finding two quarks at a given distance from each other. In order to discuss the results in a more convenient way, it will be separated into two functions: same color, $C(q)$, and different color, $D(q)$:

$$C(q)\equiv\tfrac{1}{4}\langle\Omega|\sum_{p,r}(a^{\dagger}_{p+q}a^{\dagger}_{r-q}a_r a_p+b^{\dagger}_{p+q}b^{\dagger}_{r-q}b_r b_p)|\Omega\rangle\ , \tag{31}$$

$$D(q)\equiv\tfrac{1}{4}\langle\Omega|\sum_{p,r}a^{\dagger}_{p+q}b^{\dagger}_{r-q}b_r a_p|\Omega\rangle\ . \tag{32}$$

The different-color correlation function has a minimum at $q=2\pi\lambda/L$ and a maximum at $q=4\pi\lambda/L$. As a consequence, the distance between two quarks with different color is most likely close to $L/2$. Beyond the maximum, $D(q)$ decreases monotonically as q^{-2} (see Fig. 6).

The same-color correlation function $C(q)$ is shown in Fig. 7. The probability that two quarks of the same color be at the same point is given by the sum (Fourier series for zero separation)

$$P_c(0)=\frac{1}{L}\sum_q C(q)\ . \tag{33}$$

But according to the exclusion principle, this quantity must vanish and so must the sum of $C(q)$:

$$\sum_q C(q)=0\ . \tag{34}$$

The quark correlation function can be measured indirectly in inelastic-scattering experiments; its separate terms could only be measured with probes sensitive to color, which is not the case in any known process.

(4) The Coulomb sum rule $\mathcal{R}(q)$ is defined as the sum

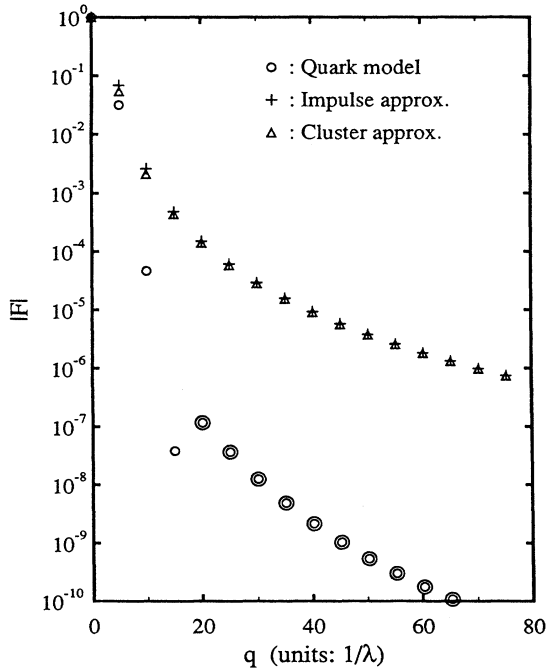

FIG. 5. Elastic form factor for $L/\lambda=5$. The double circles represent the points where the quark model result becomes negative.

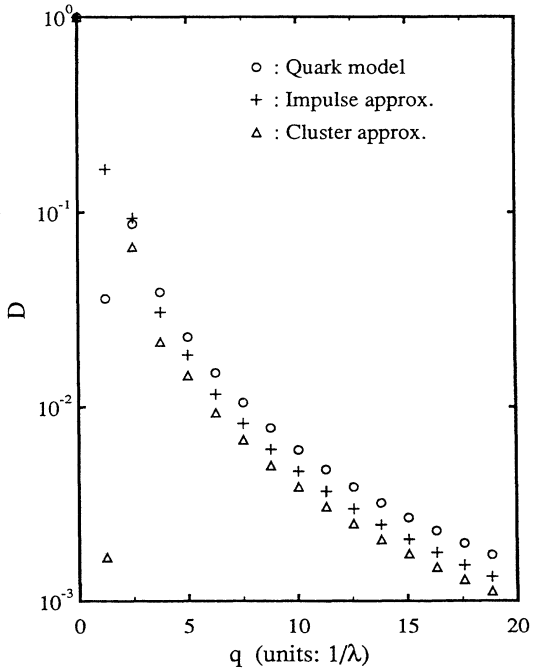

FIG. 6. Different-color correlation function for $L/\lambda=5$.

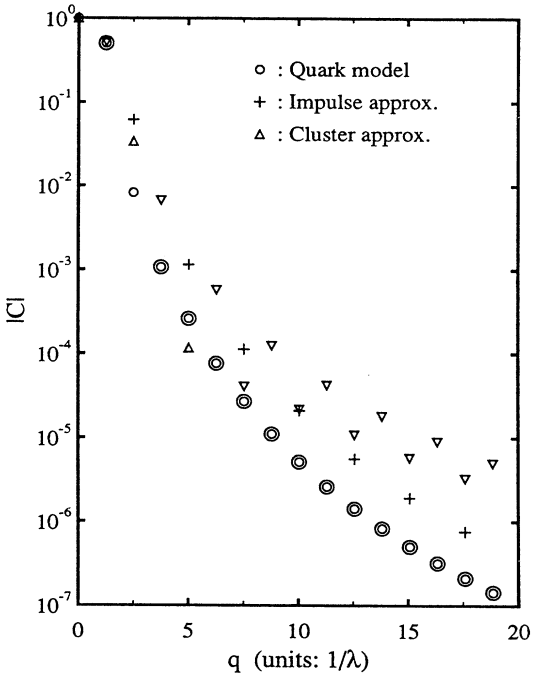

FIG. 7. Same-color correlation function for $L/\lambda=5$. The upside-down triangles and double circles represent those points where C is negative.

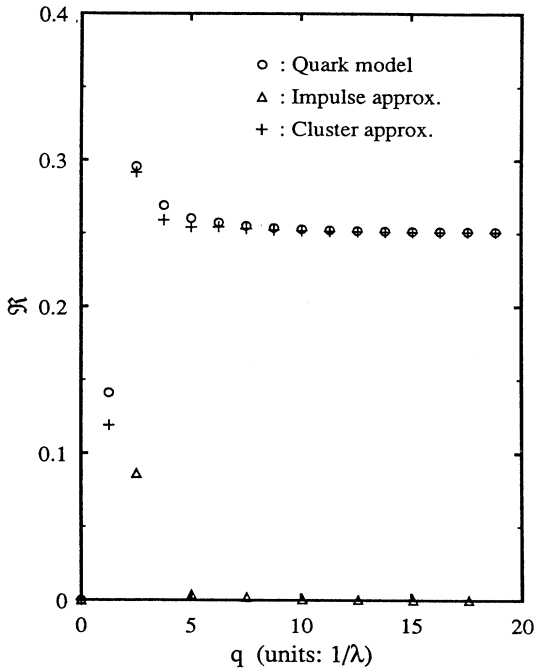

FIG. 8. Coulomb sum rule for $L/\lambda=5$.

of the response function over all possible energy transfers ω, excluding the elastic channel. In terms of the operators a and b, it becomes

$$\mathcal{R}(q)=\sum_{f\neq i}|\langle f|\sum_{p}(a^{\dagger}_{p+q}a_p+b^{\dagger}_{p+q}b_p)|i\rangle|^2 . \tag{35}$$

This function can be written in terms of the correlation functions and the elastic form factor. For the deuteron, $\mathcal{R}(q)$ becomes

$$\mathcal{R}(q)=\tfrac{1}{4}[1+C(q)+2D(q)]-|F(q)|^2 . \tag{36}$$

The result is shown in Fig. 8. The Coulomb sum rule $\mathcal{R}(q)$ saturates at about $5/\lambda$, where it approaches the value $\frac{1}{4}$ (since there are four quarks).

The expectation value of the kinetic energy can be obtained from the momentum distribution. And the expectation value of the potential energy is determined by the different-color correlation function. Since the eigenvalue of the Hamiltonian is a function of the Bethe momentum K —see (22)—we have the following sum rule:

$$\sum_p p^2N(p)-\frac{L}{2\pi^2\lambda}\sum_q D(q)=(\mathrm{Re}K)^2-(\mathrm{Im}K)^2 , \tag{37}$$

which is satisfied by the functions N and D. This sum rule can be used to verify the accuracy of numerical calculations, as well as providing a consistency condition for approximation methods.

The functions considered in this section contain information on both nuclear and quark degrees of freedom. The part that corresponds to nuclear degrees of freedom will be estimated in later sections by introducing approximation schemes based upon an effective theory in terms of nucleon degrees of freedom. The results from the present section will be treated as the "experimental" data to which the approximate results will be compared.

VII. EFFECTIVE THEORY

We now turn to the construction of an effective theory of the nucleus, in which it is regarded as a system of elementary nucleons. The quark observables of the nucleus will be calculated in Sec. VIII in the traditional way: by folding the structure of the nucleus, in the effective theory, with the quark structure of the isolated nucleon (impulse approximation). The effective theory will also be used in Sec. IX to develop a cluster approximation which introduces the effects of quark exchange among nucleons.

Consider a system of bosons, which will represent the *nucleons* in our model, moving inside the one-dimensional finite length L, with attractive delta-function interaction of strength g ($g<0$). The Hamiltonian is then the second-quantized momentum representation of (1), in the case of bosons

$$\hat{H}_e=\frac{B}{2}\left[\sum_p\left[\frac{2\pi\lambda p}{L}\right]^2 A^{\dagger}_p A_p - \frac{2\lambda}{L}\sum_{p_1,p_2,q} A^{\dagger}_{p_1+q}A^{\dagger}_{p_2-q}A_{p_2}A_{p_1}\right] . \tag{38}$$

Here $A^{\dagger}_p$ and A_p are the operators that create and annihilate a boson with dimensionless integer momentum p (we

are using the same units introduced in previous sections). The factor of $\frac{1}{2}$ has been included because the mass of the nucleons in the effective theory is twice the mass of the quark, m, and the constant $B\lambda^2$ is inversely proportional to the mass.

The two-boson eigenstates of this Hamiltonian are determined by a Bethe momentum k as given by (15); however, since the mass of the nucleon is $2m$, the argument in the function f in (15) must be replaced by $k/2$. Furthermore, according to (21), the total momentum of the model deuteron, Q has to be an even number, which is only possible if l is even. Then we restrict the quantum number l to only even values: $l \equiv 2m$; and the Bethe momentum of the two nucleon system is given by

$$k_e = m + f(k_e/2) , \quad m = 1,2,3, \ldots . \tag{39}$$

We do not consider the case $m=0$, because it would correspond to a space-symmetric bound state of four quarks, which in the two-color case is a violation of Pauli exclusion. The momentum k_e gives an excellent approximation to $2\,\mathrm{Re}(K)$, where K is the Bethe momentum (20) for the deuteron, in the quark model. In fact, in Fig. 2(a), $k_e/2$ could not be distinguished from $\mathrm{Re}(K)$. The total energy of the two-boson eigenstate is given by (6) with $N=2$, but divided by 2, because of the factor $\frac{1}{2}$ in the Hamiltonian (38); therefore, the energy of the deuteron in the effective theory is $Bk_e^2(2\pi\lambda/L)^2$. This energy gives a good approximation to the positive term in (22); the negative term will be accounted by the binding energy of the quarks inside the nucleons, as will be explained below.

The restriction to only even total momentum Q is due to the fact that the nucleons are not truly elementary, but clusters of two quarks; the periodic boundary conditions imply that a system of two particles moving together in a cluster must have an even total momentum. Therefore, the allowed momenta for each nucleon in the effective theory are only the even numbers, and the effective eigenstates of the deuteron are

$$|\omega_e\rangle = \frac{1}{\sqrt{2}} \sum_{p_1,p_2} \delta(2p_1+2p_2-Q) \times \omega_e(p_1-p_2) A^\dagger_{2p_1} A^\dagger_{2p_2} |0\rangle , \tag{40}$$

where the wave function is

$$\omega_e(p) = \frac{\beta_e}{p^2-k_e^2} . \tag{41}$$

The internal structure of each nucleon is considered as independent of the presence of other nucleons. In Sec. IV we found the quark state of a nucleon by solving the quark Hamiltonian for a system of two quarks clustered in a color singlet. Each nucleon has a binding energy $E_0 = -2B(2\pi\lambda/L)^2\alpha^2$ [Eq. (17)]. Since α gives a good approximation to the imaginary part of K, the binding energy of the nucleons ($2E_0$) corresponds to the negative term in the energy of the deuteron (22).

Equations (18) and (19), which were coupled, have thus been separated into two independent equations that determine the real and imaginary parts of K [(39) and (15), with $l=0$]. Even though we started with a system of bosons with attractive interaction, by removing a subset of the energy levels, we have ended up with the spectrum of a system with a repulsive effective interaction. To see why the spectrum corresponds to a repulsive potential, let us note that in (39) the solution for a given m is in the interval $(m-\frac{1}{2}, m)$. Thus the mth energy level is within the interval

$$B\left[\frac{2\pi\lambda}{L}\right]^2\left[m-\frac{1}{2}\right]^2 < E_m < B\left[\frac{2\pi\lambda m}{L}\right]^2 . \tag{42}$$

But a system of two noninteracting bosons has energy levels

$$E_m^{(0)} = B\left[\frac{\pi\lambda(m-1)}{L}\right]^2 \quad (m=1,2,\ldots) , \tag{43}$$

which are lower than E_m for any $m=1,2,\ldots$. This means that the *effective interaction* among hadrons is repulsive. The physical origin of the repulsive interaction is the fermionic nature of the constituents of the nucleon, which prevents overlapping of the two hadrons thereby excluding the state with $m=0$.

VIII. NUCLEON IMPULSE APPROXIMATION

In the impulse approximation, the internal structure of the nucleons is assumed to be independent of the presence of other nucleons and equal to the structure of an isolated nucleon. Quark dynamics is needed only to study the system of quarks inside one nucleon. The state of the isolated nucleon, given by (24) and (25), is used to calculate its intrinsic properties: elastic form factor f, quark-momentum distribution n, and quark-quark correlation function d. Independently of this calculation, the nuclear state in the effective theory gives the nuclear properties of the nucleus in terms of nucleonic degrees of freedom: nucleon momentum distribution n_e, elastic form factor f_e, and nucleon-nucleon correlation function c_e. The structure functions of the nucleus in terms of quark degrees of freedom are then calculated as a folding of the two independent sets of functions.

In the model we are using, the momentum p of the isolated nucleon is continuous; namely, p can be any real number. However, in the effective theory the nucleon momentum can only take discrete values. Because of this discrepancy, the structure functions cannot be calculated as a simple folding as usually done in the impulse approximation. To avoid this difficulty, we introduce a *nucleon impulse approximation* in which we assume that the internal structure of the nucleons is the same as that of a single nucleon, inside a length L equal to the nuclear length. The structure functions of such nucleon are then discrete in momentum, and the structure functions of the nucleus can be calculated by simple folding as in the impulse approximation.

To illustrate what we mean by folding of the structure functions, we will derive the same-color correlation function of the deuteron. Two quarks of the same color inside the deuteron can only come from different nucleons.

Suppose that the two nucleons are at the positions R_1 and R_2, and the positions of the two quarks inside their corresponding nucleons are y and z. The probability that the distance between the two quarks be x is then given by the following convolution integral:

$$P_c(x)=\int\int dy\, dz\, \rho(y)\rho(z)p_e(x+y-z)\ , \tag{44}$$

where $\rho(y)$ is the quark density of the nucleon, and $p_e(R_2-R_1)$ is the probability of finding the two nucleons at a distance R_2-R_1 from each other. The same-color correlation function $C(q)$ is the Fourier transform of P_c. In our case the functions ρ and p_e have period L; the same-color correlation function of the deuteron is then

$$C_{\rm IA}(q)=f^2(q)c_e(q)\ , \tag{45}$$

where $f(q)$ is the elastic form factor of the nucleon and c_e is the nucleon-nucleon correlation function, i.e., the Fourier transform of p_e.

The folding can be done in a similar way for the quark-momentum distribution $N(q)$, the elastic form factor $F(q)$, and the different-color correlation function $D(q)$; the results are

$$N_{\rm IA}(p)=\sum_q n(p-q/2)n_e(q)\ , \tag{46}$$

$$F_{\rm IA}(q)=f(q)f_e(q)\ , \tag{47}$$

$$D_{\rm IA}(q)=\tfrac{1}{2}[d(q)+f^2(q)c_e(q)]\ . \tag{48}$$

These functions can all be calculated with some simple algebra.

The Coulomb sum rule is calculated by assuming that the internal structure of the nucleon does not change in proximity to other nucleons, and thus the intrinsic form factor of the nucleon can be factored out of the nuclear inelastic form factor, leading to

$$\mathcal{R}_{\rm IA}(q)=f^2(q)\left[\frac{1+c_e(q)}{2}-f_e^2(q)\right]\ . \tag{49}$$

The results obtained using the impulse approximation are shown in Figs. 4–8. The momentum distribution and different-color correlation function reproduce very well the exact result. This is not surprising, since these two functions are the ones that determine the energy according to the sum rule (37), and the effective theory was constructed in good agreement with the energy spectrum. Therefore, the forms of the functions $N(p)$ and $D(q)$ are constrained by the total energy, through the sum rule (37), for both the quark and impulse models.

By contrast, the impulse approximation $C_{\rm IA}(q)$ to the same-color correlation function is not so close to the quark model function $C(q)$, as is seen in Fig. 7. Part of this disagreement comes from the violation of the Pauli exclusion principle for quarks in the impluse approximation, so that $C_{\rm IA}(q)$ does not satisfy the sum rule (34), which is satisfied by the full quark model. Moreover, the domain of $C_{\rm IA}$ includes only every other term (even integer momenta) in the domain of $C(q)$ from the quark model (Fig. 7) and is always positive, thus clearly violating (34).

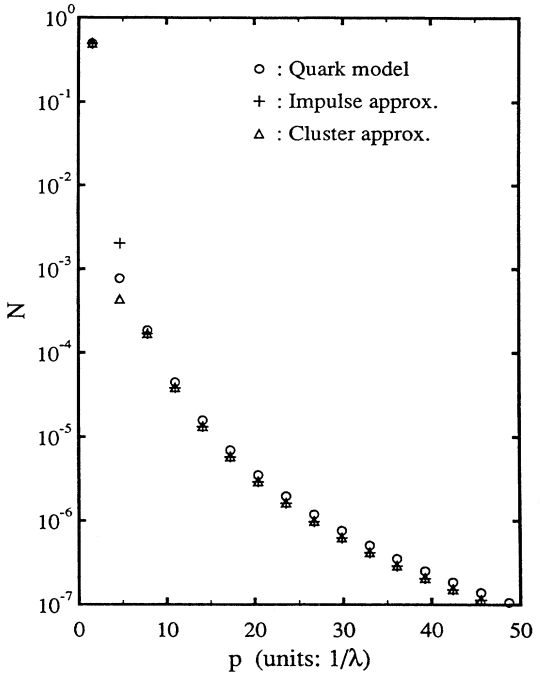

FIG. 9. Quark-momentum distribution in the quark model and cluster and impulse approximations, for $L/\lambda=2$.

The impulse approximation also fails to predict the correct elastic form factor. The asymptotic behavior of the elastic form factor obtained is q^{-4} instead of q^{-6} as in the quark model, presumably because of the factored form of the former approximation, given by (47). Simi-

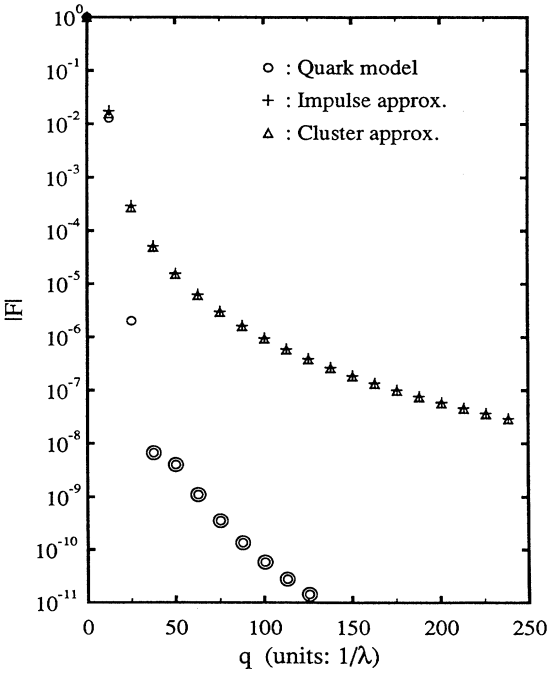

FIG. 10. Elastic form factor for $L/\lambda=2$. The double circles represent the points where the quark model result becomes negative.

larly, the factoring of the (squared) nucleon form factor in the approximate Coulomb sum-rule function (49) leads to a disagreement with the quark model. Since $f(q)\to 0$ asymptotically, $\mathcal{R}_{\rm IA}(q)\to 0$, while $\mathcal{R}(q)\to\frac{1}{4}$, as shown in Fig. 8.

These results are qualitatively similar even if we change our choice of $L/\lambda=5$ to values a little higher or lower. For instance, we show the momentum distribution (Fig. 9) and the elastic form factor (Fig. 10) for the case where the ratio of nucleus to nucleon size is $L/\lambda=2$. The momentum distribution is still well reproduced by the impulse approximation, while the elastic form factor at large momentum transfer shows the same effect found in the case $L/\lambda=5$.

IX. CLUSTER APPROXIMATION

One of the obvious omissions of the nuclear impulse approximation is the violation of the Pauli exclusion principle for quarks. A common method of including this effect is by explicit introduction of the quark variables for each nucleon, with antisymmetry for the quarks.[10] This quark exchange correction can be included in our effective theory if the elementary boson operators $A_p^\dagger$ in the deuteron state (40) are replaced by the quasiboson operators that create a nucleon in terms of quarks. The operator that creates a nucleon with total momentum p out of two quarks, is given by (24) with the wave function (26):

$$\overline{A}{}_p^\dagger \equiv \sum_{p_1,p_2} \delta(p_1+p_2-p)\omega\left[\frac{p_1-p_2}{2}\right] a_{p_1}^\dagger b_{p_2}^\dagger \ . \tag{50}$$

Replacing the bosonic operators by $\overline{A}{}_p^\dagger$ in (40) leads to the four-quark state

$$|\Omega_{\rm CA}\rangle = \tfrac{1}{4} \sum_{(r_i)_{i=1,2,3,4}} \Omega_{\rm CA}(r_1,r_2,r_3,r_4) b_{r_1}^\dagger a_{r_2}^\dagger a_{r_3}^\dagger b_{r_4}^\dagger |0\rangle \ ; \tag{51}$$

the sums over the momenta r_i run over all integral values of $(r_i+\epsilon)$, i.e.,

$$r_i = -\epsilon, \pm 1-\epsilon, \ldots , \tag{52}$$

where ϵ is either 0 or $\frac{1}{2}$, depending on whether the quantum number m is even or odd. Here the wave function is

$$\begin{aligned}\Omega_{\rm CA}(r_1,r_2,r_3,r_4) &= \sqrt{2}\,\delta(r_1+r_2+r_3+r_4) \\ &\times \omega_e\left[\frac{R_1-R_2}{2}\right]\omega\left[\frac{r_1-r_2}{2}\right]\omega\left[\frac{r_3-r_4}{2}\right] ,\end{aligned} \tag{53}$$

where R_1 and R_2 are the momenta of the two clusters: $R_1=r_1+r_2$, $R_2=r_3+r_4$. The state $|\Omega_{\rm CA}\rangle$ is not an eigenstate of the Hamiltonian (23), and it does not satisfy the sum rule (37).

The computation of structure functions can be done directly using the expressions given in Sec. VI. The results can all be written as a sum of direct plus exchange terms:

$$F_{\rm CA}(q)=F_d(q)+F_e(q) \ , \tag{54a}$$

$$C_{\rm CA}(q)=C_d(q)+C_e(q) \ , \tag{54b}$$

$$D_{\rm CA}(q)=D_d(q)+D_e(q) \ , \tag{54c}$$

$$N_{\rm CA}(r)=N_d(r)+N_e(r) \ . \tag{54d}$$

The results are products of two wave functions $\Omega_{\rm CA}$, with certain infinite sums over momenta. The direct terms can be evaluated with some simple algebra, and they all lead to the same results as those of the nucleon impulse approximation, obtained in Sec. VIII. In the exchange terms, the infinite sums over momenta have been computed analytically using the same finite algorithm mentioned in Sec. VI.[8,9]

The norm of the state $|\Omega_{\rm CA}\rangle$ then takes the form

$$\langle\Omega_{\rm CA}|\Omega_{\rm CA}\rangle = 1+E \ , \tag{55}$$

where the direct term gives the normalization 1; the constant E is given by the exchange term and is not equal to zero.

The exchange corrections are in general not very significant, except in the case of the same-color correlation function: With the addition of the exchange term, $C_{\rm CA}(q)$ now sums to zero, as required by Pauli exclusion (34) (see Fig. 7). However, the function obtained is not a good approximation to the exact result. Unlike the case for the impulse function $C_{\rm IA}$, $C_{\rm CA}$ has the correct domain, that is, both even and odd values of the (integer) momentum. However, since the impluse contribution is to even values only, while the exchange terms contribute to both, the result is that $C_{\rm CA}$ fluctuates from point to point.

The failure of the impulse approximation to reproduce the exact elastic form factor is not solved by the introduction of quark exchange in the cluster approximation (Fig. 5). Apparently, Pauli exchange is not a significant correction to the factored form of the impulse approximation; other quark correlations are more important.

The Coulomb sum rule in the cluster approximation is very close to the result from the quark model and gives the correct asymptotic behavior since the cluster approximation state of the deuteron is given in terms of quark degrees of freedom. Figure 11(a) shows the function $S(q)$, defined as

$$S(q)\equiv\frac{2\mathcal{R}(q)}{f^2(q)} \ , \tag{56}$$

where $f(q)$ is the elastic form factor of the nucleon. In the impulse approximation, this function goes asymptotically to unity. However, if one defines the same function in the quark model or cluster approximations [Fig. 11(b)], which are both quark theories, the result would be a function that increases without limit, as pointed out by Horowitz.[14]

The interest in the function $S(q)$ is that it "removes" the structure of the nucleon (in the impulse approximation, only) from the Coulomb sum rule, therefore presumably leaving only the nuclear correlation information. However, this is not appropriate when the full Coulomb sum-rule function $\mathcal{R}(q)$ is used in the numerator. What

is done in actual analysis of (e,e') scattering (e.g., Ref. 15) is to use a finite-energy cutoff in obtaining the sum-rule function from the measured response function, with a theoretical correction for higher nuclear (but not nucleon) excitations. This allows comparison of data with theoretical nuclear predictions of $S(q)$.

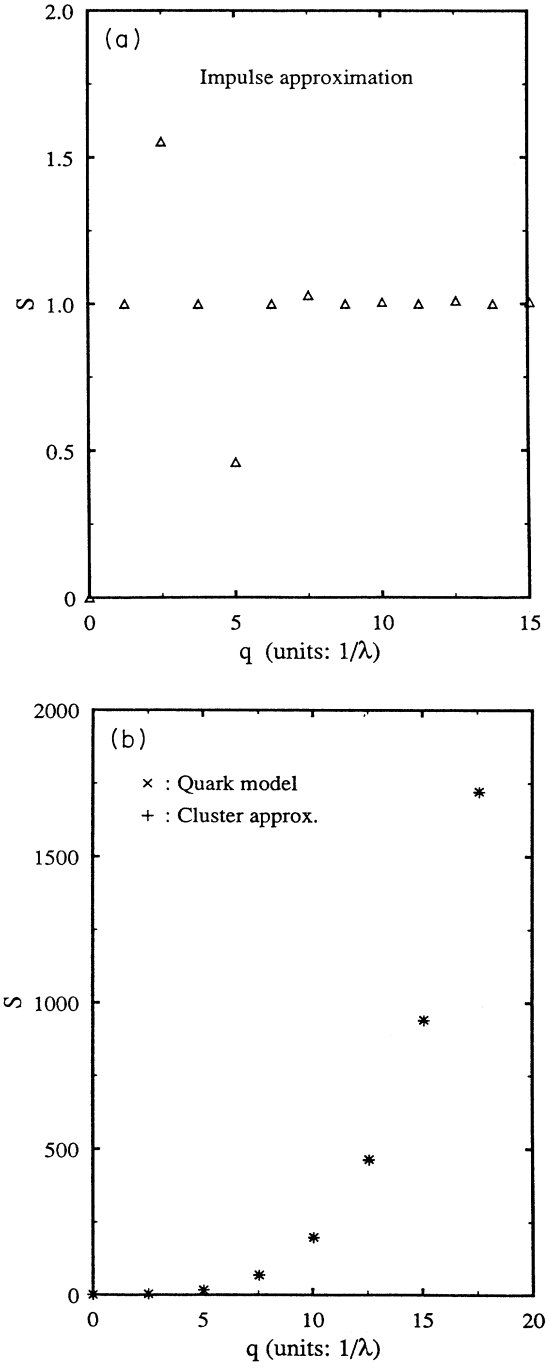

FIG. 11. Nuclear Coulomb sum rule $S(q)$ for $L/\lambda=5$. Part (a) shows the results from the impulse approximation, and (b) is obtained from the quark model and cluster approximation.

X. SUMMARY AND CONCLUSIONS

We have introduced a dynamical model of the nucleus to examine how the internal structure of nucleons might be modified inside the nucleus. Structure functions for elastic and inelastic scattering are used to look for such effects. These functions are calculated using the ground state of our model nucleus in terms of quarks and the contribution from nucleon degrees of freedom is separated by use of an impulse approximation. Once the part due to nucleon degrees of freedom is separated, the remaining part is assigned to the modification of the structure of nucleons.

The model used is a system of quarks with an attractive zero-range interaction. It has the advantage that the one-dimensional version constitutes an integrable system and all eigenstates can be obtained from the Bethe ansatz. The complete spectrum includes bound states in which groups of quarks cluster together. Each cluster represents a nucleon in the model. The nuclear ground state resembles a deuteron, with two clusters of two quarks each. There are also excited states with only one cluster and states with no clusters at all. Thus such a system is complex enough to allow a study of both nucleon (cluster) and quark degrees of freedom in the deuteron.

The complete set of eigenstates is well known and is completely defined by a set of Bethe momenta. We have solved the equations that define the Bethe momenta in the case of two clusters with a finite length L. The ratio of L to λ can be varied. For the deuteron our estimate of this ratio is 5, although we have also studied a range of values of L/λ. We show some results for $L/\lambda=2$ (Figs. 9 and 10) which are qualitatively similar to the results for $L/\lambda=5$, although the nucleus overlap is much larger in the former case.

Since the eigenstates are all known, the kind of observables measured in scattering experiments can be calculated directly, in principle. The problem was made tractable using the representation of wave functions proposed by Sasaki and Kebukawa[6] in momentum space and in second quantization. We have reduced the states given by Kebukawa[7] to a simpler form, and we have developed a finite algorithm to calculate structure functions of the deuteron target in an analytic form. The results have been reported here, and details of the calculations are given elsewhere.[8,9]

The structure functions of the model deuteron contain information on both nuclear and quark degrees of freedom. The impulse approximation, used to identify the contribution from nuclear degrees of freedom, leads to structure functions which are a folding of the internal quark structure of the isolated nucleon with the structure of the nucleus in an effective theory where nucleons are treated as elementary particles. The effective theory was extracted from the model by constructing a nucleon Hamiltonian with an energy spectrum very close to that of the quark Hamiltonian.

It is found that the quark-momentum distribution and the different-color correlation function can both be reproduced well by the impulse approximation. Thus these two observables are not very sensitive to the modification

of nucleons by the nuclear medium. This result, we believe, follows from the sum rule (37): The momentum distribution determines the kinetic energy, while the different-color correlation function gives the potential energy. Since the effective theory was constructed in excellent agreement with the energy spectrum, the sum rule constrains the forms of both functions. By contrast, the Coulomb sum rule and the elastic form factor at large momentum transfer have been found to be the most sensitive to quark effects.

To determine whether the modification of the structure of nucleons is due mainly to quark Pauli exchange, we further developed a cluster approximation to the model, in which the operators that create elementary nucleons in the states of the effective theory are replaced by operators that create clusters of quarks. The resulting state has the correct antisymmetry under exchange of any two quarks in the nucleus. Contrary to what was expected, the results of the impulse approximation appear to be better than those of the cluster approximation. We attribute this striking result to the fact that the cluster approximation attempts to improve on the effective theory, but in the process, no longer corresponds to a model with the same spectrum as in the full quark model. For example, the sum rule (37) does not hold for the cluster approximation. Apparently, the factoring approximation of the impulse approximation is not so serious, at least for $N(q)$ and $D(q)$, and inclusion of Pauli exchange is not an improvement: Other correlations are more important.

It has been suggested by different authors that the nuclear environment modifies some of the physical properties of the nucleon such as its mass or radius. It is clear that a simple swelling of the nucleon inside the nuclear medium does not account for this effect in our model. If we assumed that the nucleon size λ were larger inside the nucleus than in free space, we would obtain a form factor closer to the experimental data. However, even if λ is increased, the asymptotic behavior of $F(q)$ remains q^{-4}. The experimental data decay asymptotically two orders of magnitude faster than that $[F(q) \sim q^{-6}]$. In order to reproduce the exact behavior, we would have to assume a different functional shape for the form factor of the nucleon inside the nucleus. The effect of the periodic boundary conditions, which represents the effect of binding on our nucleons, is small.

It is not clear how dependent our results are on the specific details of our model. For example, using a somewhat different quark model, Kumano and Moniz[11] find that their impulse approximation gives a very close fit to the elastic form factor, but not to the quark-momentum distribution. It is possible that some of this difference is due to the difference of the spectrum of their effective theory from that of the quark model; i.e., their model does not obey a sum rule like (37). On the other hand, our Coulomb sum-rule results are very similar to those given by Horowitz[14] for a model very similar to that of Kumano and Moniz.

Several extensions of the methods of this paper are possible and are under investigation. First, it would be interesting to study the full response function $R(q,\omega)$. In the present model, this can be obtained by calculating directly the inelastic form factors, since there are no continuum states. The three-color version of our deuteron model seems feasible, but the algebra for six quarks is more complicated. Last, it would be interesting to extend these considerations to a denser nuclear system with many nucleons.

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