Reformulation of the variable moment of inertia model in terms of nuclear softness

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The ground-state bands in even-even nuclei are studied by treating the variation of moment of inertia with angular momentum in terms of the relative increase of moment of inertia with angular momentum, called the softness parameter. Such a reformulation of the variable moment of inertia (VMI) model in terms of the various orders of softness of the nuclear softness (NS) model extends the range of validity of the new model, called VMINS, to $2.0 \leq R_{42} \leq 3.33$. This gradual softening of the rigid rotator throughout the Periodic Table requires softness parameter to, at least, second order, with the vibrational energy also becoming important for very soft nuclei.

I. INTRODUCTION

Nuclear softness, first introduced by Morinaga,[1] is now a well accepted concept in the study of yrast (ground-state) bands in even-even nuclei. This is defined through a parameter, called the "softness" parameter $\sigma_1=(1/\mathcal{I}_0)\Delta\mathcal{I}_0/\Delta J$, giving the relative increase of moment of inertia with angular momentum J. Introducing the higher orders of "nuclear softness" (the second and higher derivatives of $\mathcal{I}_J$ with respect to J for $J=0$), one of us developed a model, called the nuclear-softness (NS) model,[2] where the nucleus is treated to belong to the total nuclear-softness regime. In this model, the nucleus is considered simply as a rigid rotator with its moment of inertia varying with angular momentum J. In terms of the various orders of the nuclear softness, this model is applied successfully[2-5] throughout the Periodic Table for all the even-even nuclei, including even those having only two particles or holes with respect to a closed shell. The predictions[2] of the two-parameter NS model (called NS2, with $\mathcal{I}_0$ and first-order softness σ_1 as its parameters) are very much identical but those of three-parameter NS model [the NS3, with an additional order of softness $\sigma_2=(1/2!\mathcal{I}_0)\partial^2\mathcal{I}_0/\partial J^2$] are better than those of the variable moment of inertia (VMI) model of Mariscotti *et al.*[6] and another equally successful shape fluctuation model.[7]

More recently, on the basis of the predictions of the interacting-boson model (IBM1), Klein and his associates[8-10] have proposed two generalizations of the VMI model, namely, the variable anharmonic vibrator model (VAVM) and the generalized VMI (GVMI) model. Both these models use three parameters each and modify the $J(J+1)$ dependence (the rotational energy term) in the VMI model. It may be mentioned here that most of the earlier attempts[11] to modify the VMI expression were focused on its second term (the harmonic potential energy). The predictions of the VAVM are in better agreement with experiments as compared to those of the GVMI model, and the VMI model is a special case of the GVMI model. This work has created a renewed interest[12-14] in the VMI model hypothesis.

In this paper, we extend the NS model concept to the VMI model. In other words, we reformulate the VMI model by treating the variation of the moment of inertia with J in terms of the various orders of nuclear softness alone (referred to as the VMINS model). The interest in this problem is, at least, on two accounts: (i) The VMI model uses two parameters (the ground-state moment of inertia $\mathcal{I}_0$ and the restoring force constant C) which, under equilibrium conditions, are shown[6] to be related to a softness parameter $\sigma=\hbar^2/2C\mathcal{I}_0^3$ (referred to as $\sigma^{\rm VMI}$ in the following). On the other hand, the NS model calculations show[2] that the first-order softness σ_1 is nonzero even for the strongly deformed nuclei, and the second- (and higher-) order nuclear-softness parameters are required for the complete analysis. Thus, the NS model supports the view[8] that one or two additional parameters are required for an extended success of the VMI formalism. We achieve this here through σ_1, σ_2, etc., in the VMINS model, which appear in both the rotational and potential energy terms of the model. (ii) It is important to study whether, for the so-called vibrational or very soft nuclei (large σ_1 values), it is enough to attribute the effects of the variation of moment of inertia with J to the rigid-rotator term alone (as in the NS model), or if the vibrational (harmonic potential energy) term must also be added (as in the VMI model). For this purpose, we compare the predictions of the VMINS model with that of the NS model. In this way, we may learn something on the basic question of whether nuclei are simply rotators with varied degrees of softness, or distinguish themselves as rotators and vibrators.

Sections II and III give, respectively, the VMINS model and its range of validity. The calculations of the ground-state band energies and discussion of parameter systematics are given in Sec. IV. Finally, a summary of our results is presented in Sec. V.

II. THE VMINS MODEL

The VMI model[6] uses the energy expression

$$E_J=\frac{\hbar^2}{2\mathcal{I}_J}J(J+1)+\tfrac{1}{2}C(\mathcal{I}_J-\mathcal{I}_0)^2 \quad (1)$$

and determines the angular momentum dependent moment of inertia $\mathcal{I}_J$, by using the equilibrium condition

$$\frac{\partial E_J}{\partial \mathcal{I}_J}=0 \ . \tag{2}$$

This gives

$$\mathcal{I}_J^3-\mathcal{I}_J^2\mathcal{I}_0-\frac{\hbar^2}{2C}J(J+1)=0 \ , \tag{3}$$

which, on substitution in (1), gives E_J in two parameters $\mathcal{I}_0$ and C.

In the NS model,[2] instead of using the equilibrium condition (2), we make the Taylor series expansion of $\mathcal{I}_J$ about its ground-state value $\mathcal{I}_0$ for $J=0$. This gives us the variation of moment of inertia with J in terms of its ground-state value $\mathcal{I}_0$ and the various orders of "nuclear softness" σ_n:

$$\mathcal{I}_J=\mathcal{I}_0(1+\sigma_1 J+\sigma_2 J^2+\sigma_3 J^3+\cdots) \ , \tag{4}$$

with

$$\sigma_1=\frac{1}{\mathcal{I}_0}\frac{\partial \mathcal{I}_0}{\partial J} \ ,$$
$$\sigma_2=\frac{1}{2!\mathcal{I}_0}\frac{\partial^2 \mathcal{I}_0}{\partial J^2} \ , \tag{5}$$
$$\sigma_3=\frac{1}{3!\mathcal{I}_0}\frac{\partial^3 \mathcal{I}_0}{\partial J^3} \ .$$

Similarly, the Taylor series expansion of $\mathcal{I}_J^{-1}$ about $\mathcal{I}_0^{-1}$ for $J=0$ gives

$$\mathcal{I}_J^{-1}=\mathcal{I}_0^{-1}[1-\sigma_1 J+(\sigma_1^2-\sigma_2)J^2 -(\sigma_1^3-2\sigma_1\sigma_2+\sigma_3)J^3+\cdots] \ . \tag{6}$$

Substituting (4) and (6) in (1), we get

$$E_J=\frac{\hbar^2}{2\mathcal{I}_0}\frac{J(J+1)}{1+\sigma_1 J+\sigma_2 J^2+\sigma_3 J^3+\cdots} +\tfrac{1}{2}C\mathcal{I}_0^2 J^2(\sigma_1+\sigma_2 J+\sigma_3 J^2+\cdots)^2 \ . \tag{7}$$

Notice that here both the $J(J+1)$ and harmonic dependences are modified. We refer to this model as VMINS, since for the restoring force $C=0$, Eq. (7) reduces to the NS model expression of Ref. 2. The parameters of the VMINS model are $\mathcal{I}_0$, C, and σ_1, σ_2, σ_3, etc. If the nuclear softness were allowed to only first order, i.e., σ_2, σ_3, etc., are all equal to zero, we get the three-parameter expression

$$E_J=\frac{\hbar^2}{2\mathcal{I}_0}\frac{J(J+1)}{1+\sigma_1 J}+\tfrac{1}{2}C\mathcal{I}_0^2\sigma_1^2 J^2 \ . \tag{8}$$

In terms of both the first- and second-order nuclear-softness parameters σ_1 and σ_2, Eq. (7) simplifies to the four-parameter equation

$$E_J=\frac{\hbar^2}{2\mathcal{I}_0}\frac{J(J+1)}{1+\sigma_1 J+\sigma_2 J^2}+\tfrac{1}{2}C\mathcal{I}_0^2 J^2(\sigma_1+\sigma_2 J)^2 \ . \tag{9}$$

We refer to these model equations (8) and (9), respectively, as the VMINS3 and VMINS4. The number three or four gives the number of parameters used in the model.

III. THE RANGE OF VALIDITY OF THE VMINS MODEL

First of all, we notice that σ_2, σ_3, etc., are the second-, third-, and higher-order corrections and hence the range of validity of the "nuclear-softness" models can be demonstrated by keeping the nuclear softness to first order alone. Thus, for the VMINS models, we use the VMINS3 model equation (8):

$$E_J=A\frac{J(J+1)}{1+\sigma_1 J}+K\sigma_1^2 J^2 \ , \tag{8'}$$

with

$$A=\frac{\hbar^2}{2\mathcal{I}_0} \ , \tag{10a}$$

$$K=\tfrac{1}{2}C\mathcal{I}_0^2 \ . \tag{10b}$$

In the VMI approach, the range of validity of the above equation can be established in terms of σ_1 or $\mathcal{I}_0$ and C, since the equilibrium condition

$$\frac{\partial E_J}{\partial \sigma_1}=0 \ , \tag{11}$$

for $J=0$, connects the two sets as follows:

$$\sigma_1=\frac{\hbar^2}{2C\mathcal{I}_0^3}(=\sigma^{\mathrm{VMI}}) \ . \tag{12}$$

In the VMINS model, however, σ_1 is an independent parameter, like $\mathcal{I}_0$ and C, and hence is not related to σ^{VMI} in any simple manner. We shall see in the following that for the VMINS model this has an advantage of increasing the range of validity of the VMI approach to the extreme limits of vibrational ($R_{42}=2.0$) and rotational ($R_{42}=3.33$) spectra.

The softness parameter has the limiting values $0\le\sigma_1\le 1$. In the limit of $\sigma_1\to 0$, the energy expression (8') of the VMINS3 model reduces to the rigid-rotation limit

$$E_J(\sigma_1\to 0)=AJ(J+1) \ . \tag{13}$$

In the other limit of $\sigma_1\to 1$, and in the approximation of $K\ll A$ (shown to be true in Sec. V), we get from (8'), the vibrational limit

$$E_J(\sigma_1\to 1)=AJ \ . \tag{14}$$

Thus, defining the energy ratio $R_{42}=E_{4^+}/E_{2^+}$, we get the range of validity of the VMINS3 model, from Eqs. (13) and (14), as

$$2.0\le R_{42}\le 3.33 \ , \tag{15}$$

which is exactly the same[4] as for the NS2 model [NS2 model expression is obtained from Eq. (8') for $C=0$]. Notice that the lower limit here is exactly 2 (the vibrational condition), as compared to 2.23 for the VMI model. Apparently, Eq. (15) means that the two models (NS and VMINS) based completely on "nuclear softness" allow us to correlate the data of ground-state bands in all

TABLE I. Ground-state band energies (in MeV) for a few illustrative nuclei, calculated on various models and compared with experimental data. The dashes refer to the energies used in determining the parameters for each model.

	$E2$	$E4$	$E6$	$E8$	$E10$	$E12$	$E14$	$E16$	$E18$
Expt.^{78}Sr	0.2785	0.7822	1.4946	2.3900	3.4470	4.6570			
VMI	–	–	1.42	2.16	2.99	3.90			
VAVM	–	–	–	2.40	3.48	4.73			
VMINS3	–	–	–	2.4131	3.5370	4.8659			
VMINS4	–	–	–	–	3.4432	4.6299			
NS3	–	–	–	2.4708	3.8344	5.8335			
NS4	–	–	–	–	3.3715	4.2635			
Expt.^{100}Pd	0.6653	1.4158	2.1888	2.9874	3.8688	4.7608	5.7062		
VMI	–	–	2.273	3.216	4.231	5.309	6.443	7.628	8.859
VAVM	–	–	–	2.980	3.785	4.603	5.431	6.269	7.115
VMINS3	–	–	–	2.9745	3.7698	4.5737	5.3854	6.2048	7.0315
VMINS4	–	–	–	–	3.8238	4.7125	5.6693	6.7103	7.8524
NS3	–	–	–	2.9745	3.7701	4.5744	5.3870	6.2075	7.0360
NS4	–	–	–	–	3.8248	4.7186	5.6912	6.7710	7.9958
Expt.^{118}Xe	0.3372	0.8099	1.3964	2.0729	2.8140	3.5880			
VMI	–	–	1.372	2.004	2.693	3.431	4.213	5.035	5.892
VAVM	–	–	–	2.074	2.826	3.641	4.510	5.427	6.388
VMINS3	–	–	–	2.0935	2.9003	3.8163	4.8414	5.9756	7.2186
VMINS4	–	–	–	–	2.8180	3.6131	4.4426	5.2940	6.1584
NS3	–	–	–	2.1212	3.0312	4.2027	5.7644	7.9477	11.2133
NS4	–	–	–	–	2.7879	3.4640	4.0183	4.3922	4.5706
Expt.^{126}Ba	0.2561	0.7113	1.3327	2.0889	2.9423	3.7475	4.4197	5.2451	(6.1947)
VMI	–	–	1.292	1.968	2.720				
VAVM	–	–	–	2.087	2.948				
VMINS3	–	–	–	2.1133	3.0508	4.1440	5.3926		
VMINS4	–	–	–	–	2.9506	3.8907	4.8841		
NS3	–	–	–	2.1415	3.1945	4.5941	6.5243		
NS4	–	–	–	–	2.9082	3.6767	4.2742		
Expt.^{152}Sm	0.1218	0.3665	0.7069	1.1254	1.6093	2.1489	2.7363		
VMI	–	–	0.6979	1.0960	1.5485	2.0474	2.5871	3.1631	3.7722
VAVM	–	–	–	1.1201	1.5912	2.1105	2.6714	3.2689	3.8995
VMINS3	–	–	–	1.1329	1.6398	2.2250	2.8872	3.6255	4.4394
VMINS4	–	–	–	–	1.6064	2.1359	2.7012	3.2903	3.8924
NS3	–	–	–	1.1288	1.6251	2.1933	2.8341	3.5507	4.3487
NS4	–	–	–	–	1.6065	2.1343	2.6913	3.2581	3.8148
Expt.^{154}Gd	0.1231	0.3710	0.7177	1.1445	1.6372	2.1850	2.7780	3.4051	4.0168
VMI	–	–	0.7074	1.1118	1.5717	2.0791	2.6280	3.2141	3.8450
VAVM	–	–	–	1.1397	1.6219	2.1543	2.7299	3.3438	3.9920
VMINS3	–	–	–	1.1539	1.6754	2.2800	2.9665	3.7342	4.5826
VMINS4	–	–	–	–	1.6345	2.1725	2.7442	3.3367	3.9380
NS3	–	–	–	1.1505	1.6640	2.2582	2.9364	3.7054	4.5751
NS4	–	–	–	–	1.6318	2.1562	2.6907	3.2063	3.6760
Expt.^{162}Dy	0.0807	0.2657	0.5485	0.9213	1.3751	1.9030	2.4940	3.1430	3.8360
VMI	–	–	0.5484	0.9209	1.3749	1.9031	2.4988	3.1564	3.8711
VAVM	–	–	–	0.9193	1.3678	1.8853	2.4642	3.0983	3.7826
VMINS3	–	–	–	0.9278	1.4030	1.9738	2.6402	3.4022	4.2596
VMINS4	–	–	–	–	1.3761	1.9052	2.5014	3.1572	3.8658
NS3	–	–	–	0.9202	1.3696	1.8845	2.4523	3.0603	3.6965
NS4	–	–	–	–	1.3761	1.9064	2.5079	3.1791	3.9219
Expt.^{164}Er	0.0914	0.2995	0.6144	1.0246	1.5179	2.0828	2.7026	3.4112	4.1212
VMI	–	–	0.6142	1.0244	1.5194	2.0904	2.7296	3.4310	4.1892
VAVM	–	–	–	1.0223	1.5106	2.0688	2.6884	3.3631	4.0874
VMINS3	–	–	–	1.0336	1.5561	2.1814	2.9093	3.7397	4.6725
VMINS4	–	–	–	–	1.5188	2.0861	2.7161	3.3986	4.1241
NS3	–	–	–	1.0231	1.5102	2.0597	2.6553	3.2821	3.9263
NS4	–	–	–	–	1.5191	2.0890	2.7286	3.4362	4.2140

TABLE I. (*Continued*).

	$E2$	$E4$	$E6$	$E8$	$E10$	$E12$	$E14$	$E16$	$E18$
Expt.^{174}Yb	0.0765	0.2531	0.5260	0.8895	1.3362	1.8606	2.4565	3.1170	3.8360
VMI	–	–	0.5258	0.8894	1.3378	1.8651	2.4657	3.1346	3.8670
VAVM	–	–	–	0.8884	1.3329	1.8519	2.4388	3.0877	3.7934
VMINS3	–	–	–	0.8945	1.3583	1.9174	2.5717	3.3212	4.1659
VMINS4	–	–	–	–	1.3378	1.8654	2.4666	3.1363	3.8689
NS3	–	–	–	0.8895	1.3365	1.8586	2.4465	3.0908	3.7817
NS4	–	–	–	–	1.3362	1.8576	2.4442	3.0858	3.7721
Expt.^{172}Hf	0.0953	0.3093	0.6281	1.0375	1.5213	2.0648	2.6543	3.2775	3.9199
VMI	–	–	0.6276	1.0358	1.5220	2.0766	2.6921	3.3625	4.0829
VAVM	–	–	–	1.0343	1.5137	2.0557	2.6521	3.2970	3.9854
VMINS3	–	–	–	1.0471	1.5642	2.1785	2.8895	3.6969	4.6004
VMINS4	–	–	–	–	1.5238	2.0740	2.6759	3.3176	3.9881
NS3	–	–	–	1.0343	1.5090	2.0335	2.5902	3.1638	3.7415
NS4	–	–	–	–	1.5268	2.0914	2.7335	3.4635	4.3016
Expt.^{174}W	0.1119	0.3550	0.7040	1.1370	1.6350	2.1860	2.7800	3.3920	3.9730
VMI	–	–	0.70	1.14	1.65	2.22	2.84	3.51	
VAVM	–	–	–	1.14	1.63	2.19	2.79	3.43	
VMINS3	–	–	–	1.1462	1.6745	2.2847	2.9740	3.7406	4.5833
VMINS4	–	–	–	–	1.6338	2.1757	2.7455	3.3275	3.9076
NS3	–	–	–	1.1324	1.6151	2.1301	2.6593	3.1888	3.7079
NS4	–	–	–	–	1.6399	2.2079	2.8452	3.5662	4.3965
Expt.^{192}Os	0.2058	0.5803	1.0886	1.7081	2.4185	(3.212)			
VMI	–	–	1.0629	1.6268	2.2571	2.9440	3.6806	4.4617	5.2834
VAVM	–	–	–	1.700	2.393	3.154	3.975	4.848	5.768
VMINS3	–	–	–	1.7218	2.4764	3.3511	4.3449	5.4575	6.6885
VMINS4	–	–	–	–	2.4189	3.2026	4.0423	4.9223	5.8283
NS3	–	–	–	1.7347	2.5441	3.5635	4.8695	6.5888	8.9419
NS4	–	–	–	–	2.4005	3.1048	3.7460	4.2555	4.5930
Expt.^{232}Th	0.0494	0.1621	0.3331	0.5569	0.8270	1.1374	1.4833	1.8595	2.2634
VMI	–	–	0.3334	0.5576	0.8292	1.1434	1.4963	1.8843	2.3046
VAVM	–	–	–	0.555	0.821	1.125	1.463	1.832	2.227
VMINS3	–	–	–	0.5610	0.8453	1.1857	1.5822	2.0347	2.5431
VMINS4	–	–	–	–	0.8282	1.1420	1.4933	1.8772	2.2893
NS3	–	–	–	0.5551	0.8197	1.1176	1.4397	1.7774	2.1231
NS4	–	–	–	–	0.8299	1.1516	1.5260	1.9627	2.4787
Expt.^{234}U	0.0435	0.1433	0.2960	0.4966	0.7404	1.0233	1.3401	1.6873	2.0623
VMI	–	–	0.2961	0.4978	0.7440	1.0308	1.3547	1.7127	2.1021
VAVM	–	–	–	0.4960	0.7381	1.0173	1.3296	1.6716	2.0407
VMINS3	–	–	–	0.5008	0.7575	1.0658	1.4258	1.8374	2.3006
VMINS4	–	–	–	–	0.7401	1.0218	1.3370	1.6812	2.0503
NS3	–	–	–	0.4966	0.7389	1.0161	1.3211	1.6468	1.9866
NS4	–	–	–	–	0.7389	1.0160	1.3207	1.6459	1.9850
Expt.^{248}Cm	0.0434	0.1440	0.2986	0.5055	0.7613	1.0621	1.4036	1.7809	2.1891
VMI	–	–	0.300	0.510	0.771	1.080	1.435	1.833	2.271
VAVM	–	–	–	0.502	0.749	1.035	1.356	1.708	2.088
VMINS3	–	–	–	0.5065	0.7675	1.0815	1.4483	1.8681	2.3407
VMINS4	–	–	–	–	0.7632	1.0703	1.4255	1.8273	2.2745
NS3	–	–	–	0.5016	0.7454	1.0215	1.3207	1.6346	1.9550
NS4	–	–	–	–	0.7687	1.1021	1.5369	2.1372	3.0448

the even-even nuclei, without distinguishing for the much accepted rotational or vibrational nature of the nuclei. Then, the comparison of the results of two models (carried out in Sec. IV) will tell us whether the higher-order softness effects in the rotational picture of the NS model are enough to describe also the very soft nuclei with vibrational spectra, or if one must include to rotational energy the harmonic vibrational energy (as in VMINS) in order to be able to unify the vibrational and rotational pictures in one single model equation.

IV. CALCULATION AND DISCUSSION OF RESULTS

A. Ground-state band energies

We have made our calculations for all the even-even nuclei (about 130 cases) whose yrast bands are observed experimentally up to the 8^+ state or more. The softness parameters in Eq. (7) are kept to only first and second orders, thereby using only the VMINS3 and VMINS4 model equations (8) and (9), respectively, with three ($\mathcal{I}_0$, C, and σ_1) and four ($\mathcal{I}_0$, C, σ_1, and σ_2) parameters. Using the point of view[9,10] that bands are built from the "ground state up," the parameters are determined by fitting exactly the states up to 6^+ or 8^+, respectively, for three or four parameters.

Table I gives our calculated energies for the VMINS3 and VMINS4 models, in comparison with experimental data,[15] for a few representative nuclei. A complete tabulation of the calculated energies for all the 130 nuclei will be published elsewhere.[16] Since the NS model gives, at least, the first "bending" of $\mathcal{I}(\omega^2)$ plots,[3] we have included here the states with $J \leq 18^+$ which are below the point where the second bending normally begins. For some ten nuclei, calculations are also made[16] for J up to 28^+ but they are not included here in Table I. The results of calculations for the VMI, VAVM, and the NS3 and NS4 models are given here for comparison. The predictions of GVMI and NS2 models are not included in this table, since VAVM gives better results than GVMI (Ref. 10) and the NS2 model predictions are almost identical to the VMI model predictions.[2] A careful comparison shows that the predictions of our VMINS3 model are an improvement over the VMI model only for the soft nuclei like ^{78}Sr, ^{100}Pd, ^{118}Xe, ^{126}Ba, and ^{192}Os with large σ_1 values (see Table II). For the strongly deformed nuclei like ^{154}Gd, ^{162}Dy, ^{164}Er, ^{174}Yb, ^{172}Hf, and ^{174}W the VMINS3 is rather poor as compared to the VMI model. However, for strongly deformed nuclei we notice that the NS3 model (as well as the VAVM) gives results in much better agreement with experiments than the VMINS3. This means that since the first-order softness σ_1 is nonzero even for strongly deformed nuclei (see Table II), it will be more realistic to investigate the predictions of the VMINS4 model which contains softness to second order ($\mathcal{I}_0, C, \sigma_1, \sigma_2$ parameters).

The VMINS4 model calculations should be compared with the NS4 model calculations since it also has four parameters ($\mathcal{I}_0, \sigma_1, \sigma_2, \sigma_3$). Following Bonatsos and Klein,[9] this comparison is made in terms of the average of nu-

TABLE II. The parameters of the various nuclear-softness models.

Nucleus	$\mathcal{I}_0$ (MeV^{-1})				σ_1				σ_2		σ_3	$C \times 10^5$ (MeV^3)	
	VMINS3	VMINS4	NS3	NS4	VMINS3	VMINS4	NS3	NS4	NS3	NS4	NS4	VMINS3	VMINS4
^{78}Sr	7.24	5.20	8.01	7.52	0.6758	1.4155	0.1954	0.2682	−0.0116	−0.0288	0.001 36	213.66	120.56
^{100}Pd	1.93	1.99	1.93	1.98	0.6728	0.6429	0.6696	0.6279	−0.0015	0.0051	−0.000 55	103.03	554.01
^{118}Xe	4.33	3.66	4.69	4.29	0.7241	1.0815	0.4895	0.6192	−0.0202	−0.0452	0.001 92	276.30	245.45
^{126}Ba	8.69	6.82	8.72	8.30	0.4644	1.0089	0.1899	0.2464	−0.0093	−0.0225	0.001 06	236.78	120.09
^{152}Sm	25.00	24.52	21.75	21.65	0.2074	0.4711	0.0690	0.0735	−0.0013	−0.0025	0.000 09	68.73	22.86
^{154}Gd	24.85	23.33	21.53	21.36	0.2242	0.5699	0.0693	0.0769	−0.0016	−0.0035	0.000 16	63.99	18.53
^{162}Dy	48.49	31.80	36.94	36.99	0.4382	1.4112	0.0021	0.0009	0.0007	0.0010	−0.000 03	5.28	1.42
^{164}Er	43.74	27.29	32.48	32.53	0.3508	1.4876	0.0036	0.0021	0.0009	0.0013	−0.000 03	10.85	1.99
^{174}Yb	48.41	31.72	39.08	39.08	0.5717	1.5344	0.0010	0.0011	0.0004	0.0004	0.000 00	3.11	1.14
^{172}Hf	41.77	30.38	30.93	31.03	0.2567	1.1238	0.0070	0.0038	0.0011	0.0019	−0.000 07	20.79	2.88
^{174}W	31.65	29.43	25.75	25.88	0.1239	0.7697	0.0176	0.0130	0.0015	0.0027	−0.000 10	116.84	7.33
^{192}Os	11.84	10.79	11.33	11.01	0.3623	0.6196	0.1566	0.1881	−0.0066	−0.0141	0.000 61	159.92	93.83
^{232}Th	81.47	58.78	60.29	60.50	0.3582	1.1412	0.0021	−0.0011	0.0009	0.0018	−0.000 07	1.64	0.38
^{234}U	90.27	46.49	68.61	68.61	0.4383	2.0723	0.0014	0.0014	0.0007	0.0007	0.000 00	0.82	0.17
^{248}Cm	92.43	86.78	69.37	69.93	0.4554	0.6147	−0.0038	−0.0111	0.0010	0.0030	−0.000 17	0.75	0.49

TABLE III. The average numerical deviation $\overline{\Delta E}$ between the experimental and calculated energies for $J \geq 8^+$, for various models. Here n gives the number of energies used in the calculations.

Nucleus	VMI	VAVM	VMINS3	VMINS4	NS3	NS4	n
^{78}Sr	0.607 00	0.053 00	0.149 45	0.015 45	0.781 95	0.234 50	2
^{100}Pd	0.549 07	0.172 27	0.202 30	0.043 40	0.201 43	0.033 73	3
^{118}Xe	0.139 00	0.032 50	0.157 30	0.014 55	0.415 95	0.075 05	2
^{126}Ba	0.222 30	0.005 70	0.108 50	0.008 30	0.252 20	0.034 10	1[a]
^{152}Sm	0.103 83	0.040 47	0.085 83	0.017 00	0.052 67	0.020 80	3
^{154}Gd	0.136 84	0.036 04	0.243 32	0.039 24	0.223 40	0.132 22	5
^{162}Dy	0.010 72	0.030 58	0.185 54	0.010 92	0.057 58	0.028 06	5
^{164}Er	0.024 78	0.023 48	0.244 66	0.006 64	0.080 42	0.030 24	5
^{174}Yb	0.012 78	0.020 32	0.145 64	0.013 74	0.018 56	0.022 08	5
^{172}Hf	0.059 66	0.020 78	0.298 34	0.028 32	0.079 96	0.135 80	5
^{174}W	0.056 75	0.014 25	0.170 20	0.027 63	0.099 93	0.066 55	4[a]
^{192}Os	0.214 70	0.041 75	0.098 50	0.004 90	0.238 55	0.062 60	2
^{232}Th	0.017 44	0.020 52	0.124 08	0.011 88	0.058 62	0.075 66	5
^{234}U	0.018 18	0.011 22	0.106 74	0.004 60	0.028 78	0.029 38	5
^{248}Cm	0.038 60	0.052 20	0.061 82	0.032 76	0.103 96	0.278 54	5

[a]For these two cases, we have taken the value of n for J values up to first bending only since VMI is shown (Ref. 10) to be good only up to this point.

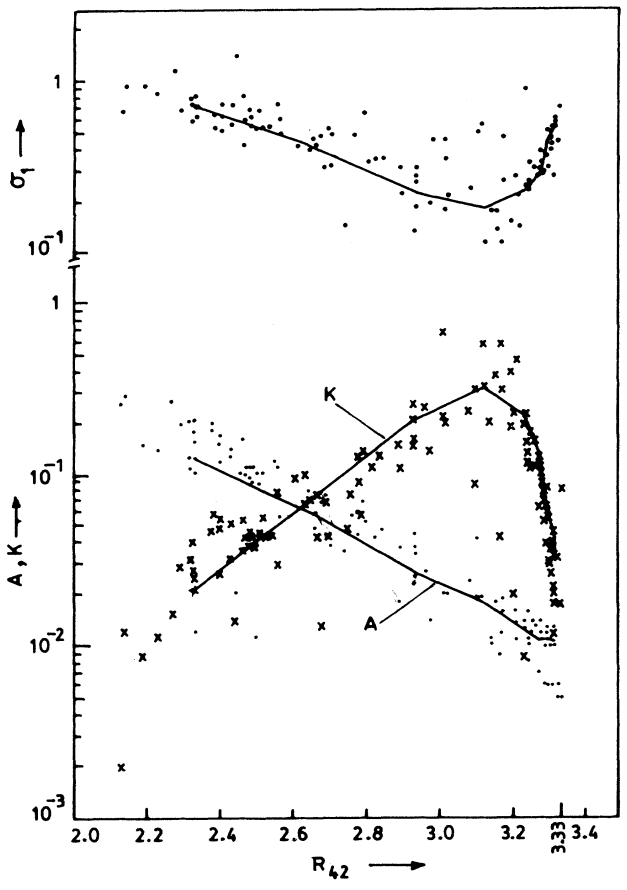

FIG. 1. The first-order softness parameter σ_1, the ground-state moment of inertia parameter A (in units of $\hbar^2$), and the restoring force constant parameter K as a function of energy ratio $R_{42}=E_{4+}/E_{2+}$ for all the even-even nuclei, using the VMINS3 model [Eq. (8′)]. The solid lines join the points for Yb isotopes.

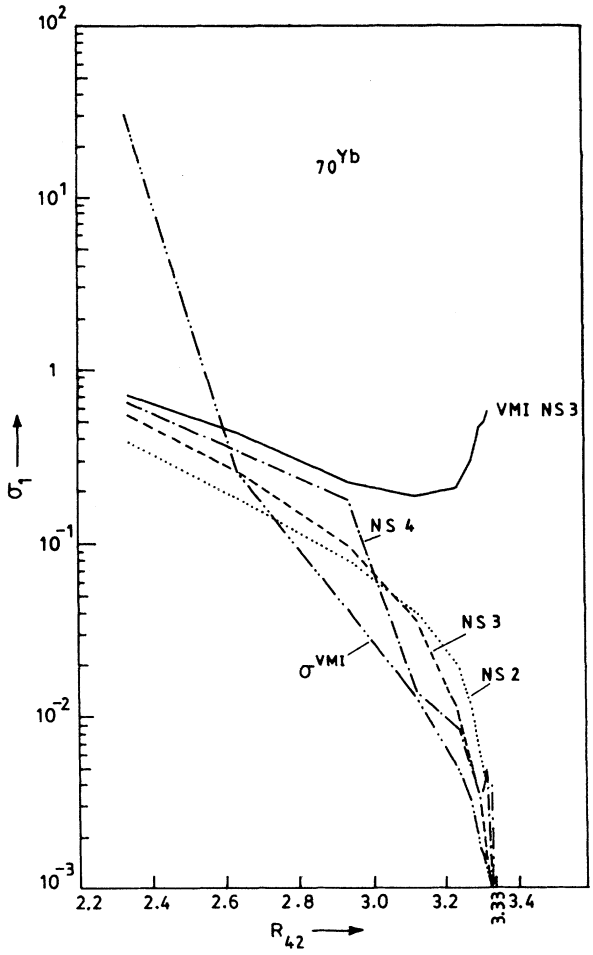

FIG. 2. The softness parameter σ_1 as a function of $R_{42}=E_{4+}/E_{2+}$ of the Yb isotopes for various models.

merical deviations between the experimental and calculated energies (denoted as ΔE), in Table III. We notice that though the NS4 model also shows a very good improvement over both the NS3 and VMINS3 models, the VMINS4 gives the least average deviation with experiments for all the nuclei. This result means that the ground-state band energies of even-even nuclei throughout the Periodic Table can be interpreted on a simple model based on nuclear softness, without distinguishing between their rotational or vibrational characters. The softness must be included, at least, up to second order and, though the *major* softness effects are contained in the rotational energy (NS model), the harmonic vibrational energy (VMINS model) helps in improving the comparisons between the calculations and experiments, in particular for very soft nuclei. Hence, our calculations stress that in nuclei there is no phase change from a deformed rotor to a harmonic vibrator but a gradual softening of the rotor takes place throughout the Periodic Table. Once the rotor becomes very soft, it is natural to add the vibrational effects to the rotational energy.

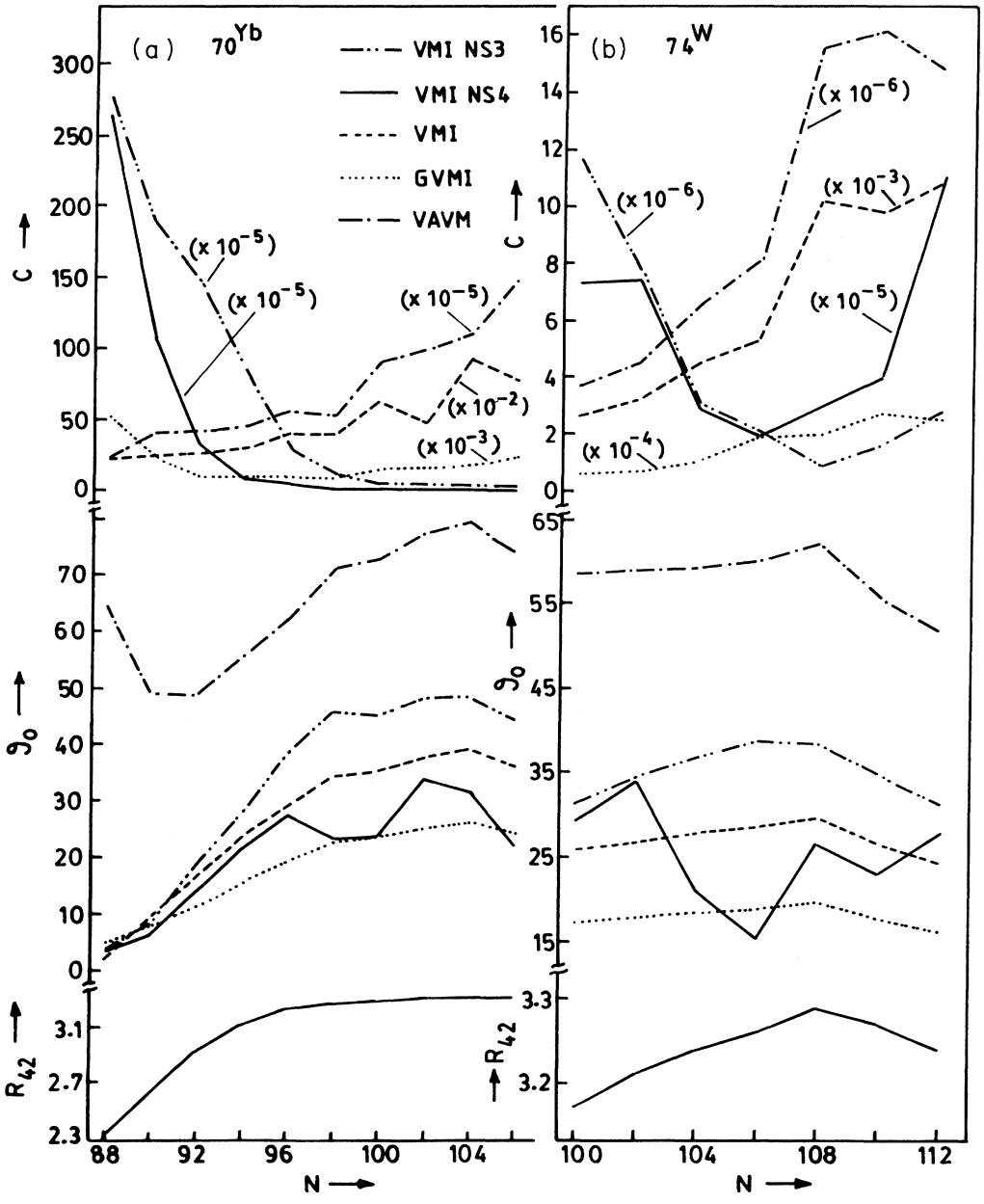

FIG. 3. The energy ratio R_{42} and the ground-state moment of inertia ϑ_0 and the restoring force constant C of various models as a function of the neutron number N for (a) Yb and (b) W isotopes. For C, the multiplicative factors of the different models are also shown.

B. The parameter systematics

Figure 1 gives a plot of the three parameters of the VMINS3 model as a function of energy ratio $R_{42}=E_{4^+}/E_{2^+}$, for all the even-even nuclei throughout the Periodic Table (dots and crosses, whose numerical values are found in Ref. 16). Here σ_1 is the first-order softness parameter and A and K are, respectively, the ground-state moment of inertia ($A=\hbar^2/2\mathcal{J}_0$) and the restoring force constant ($K=\frac{1}{2}C\mathcal{J}_0^2$). The solid lines, showing the trend of these three parameters, are obtained by joining the points for Yb isotopes. It is interesting to find that, except for only two nuclei (^{76}Kr and ^{100}Ru), the σ_1 values lie in the range of 0 and 1. Secondly, for nuclei with $\sigma_1 \rightarrow 1$, $K \ll A$. Both these results are used to establish the range of validity [Eq. (15)] of the VMINS3 model.

Figure 2 shows a comparison of the softness parameter σ_1 of VMINS3 with other models, for the illustrative case of Yb isotopes. We notice that whereas the limiting values of softness parameter σ_1 are 0 to 1 for the nuclear-softness models (NS and VMINS), they are 0 to ∞ for the VMI model. This is what limits the range of validity of the VMI model below $R_{42}=2.23$. For the physical range ($0 \leq \sigma_1 < 1$), the σ^{VMI} values are shown to lie close to the NS model values rather than to the VMINS. This is contrary to expectations since the NS model contains only the rotational energy. Apparently, this result stresses that, to first-order softness effects, the addition of vibrational energy in the VMI model does not contribute much, which makes the NS model a better formulation since, in the NS model, softness can be included to all orders and the first-order σ_1 values remain within physical limits of 0 and 1 for the limiting values 2.0 and 3.33 of R_{42}. Combining this result with that from Table III, the vibrational energy, however, becomes important for softer nuclei where higher-order nuclear-softness effects are also needed.

Finally, in Fig. 3 we have plotted the energy ratio R_{42} and the other parameters $\mathcal{J}_0$ and C of the various models as a function of neutron number N, for illustrative Yb and W isotopes. We first notice that the two parameters in all the models follow approximately the variation (increase or decrease) of R_{42} with N. For the moment of inertia parameter $\mathcal{J}_0$, we notice that, except for the VAVM model, all other model values lie within a factor of 2. The VAVM values of $\mathcal{J}_0$ are considerably higher. On the other hand, the restoring force constant C in different models differ both in the magnitude and in their variation with N. The two model approaches (the VMI and the VMINS) give almost reverse dependences.

V. SUMMARY

We have reformulated the VMI model of Mariscotti *et al.*[6] in terms of the nuclear softness, introduced to various orders in the NS model of one of us.[2] This model, called the VMINS model, is now applicable to all the nuclei with $2.0 \leq R_{42} < 3.33$. The softness is found to be essential, at least, up to second order. This establishes a gradual softening of the rigid rotator throughout the Periodic Table, rather than a phase change from a rigid rotator to a soft vibrator. The softness effects are shown to be contained mainly in the rotational energy but for very soft nuclei, the vibrational energy also contributes significantly.

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