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Cluster spectroscopic factor in the vibron model

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We propose a series expansion similar to that of the Hamiltonian to obtain cluster spectroscopic factors in the vibron model. The reason for this prescription is given by the link between the vibron model and the microscopic cluster model in the harmonic-oscillator limit. The $^{16}O+\alpha$ and $^{12}C+\alpha$ systems are discussed as illustrative examples within the dynamic symmetry approach.

I. INTRODUCTION

The vibron model,[1,2] being an algebraic model of dipole collectivity, may be applied both to chemical molecules[2] and to molecular states.[3–6] Among the nuclear molecular states we can find many resonances, and in their theoretical description the calculation of the decay widths is a natural requirement. The vibron model is a model of bound states [its group structure is given by the compact group of U(4)] so it is not obvious how to deal with resonances in this framework. It seems to be a serious drawback from the viewpoint of its application to nuclear cluster states.[7] In this paper we suggest a simple way to calculate cluster spectroscopic factor in the vibron model.

In what follows, in Sec. II we first summarize a few basic features of this model and its relation to the microscopic cluster model in the harmonic-oscillator limit. Based on this link we propose a purely phenomenologic way to obtain spectroscopic factors in Sec. III. Then parameters corresponding to the harmonic-oscillator limit of a few core-plus-alpha-particle systems are determined (Sec. IV). Finally, experimental data of well-known cluster bands of ^{20}Ne and ^{16}O are analyzed in Sec. V. All these calculations are performed within the U(3) dynamical symmetry approach.

II. SOME RELEVANT FEATURES OF THE VIBRON MODEL

A. Phenomenologic calculations

The vibron model belongs to the family of the interacting boson models. In these models the spectrum of the collective motion is determined by the interaction of a finite set of bosons with conserved particle number. A few single-particle states having a definite angular momentum and parity (l^{π}) are available for the bosons.[8] In case of the vibron model, there are four of them, with $l^{\pi}=0^{+}$ and 1^{-}. The bosons sitting in these states are called σ and π bosons, respectively. The number conservation guarantees a compact algebraic structure, in fact the dynamic group is U(n), where n is equal to the number of single-particle states. In the vibron model it is U(4). The representations of physical importance are the completely symmetric ones, because of the boson statistics, and their dimensions are determined by the total and conserved number of bosons. Due to the compactness these dimensions are always finite.

The basis states for the model calculations are provided by nested group chains which start with the U(4) dynamical group and end with the O(3) angular momentum group. The representation labels of the groups appearing in the chain serve as quantum numbers of the basis states. In the vibron model the following two group chains are available with the corresponding subgroups and quantum numbers:

$$\begin{array}{ccccc} U(4) & \supset & O(4) & \supset & O(3) \\ N_t & & \omega & & L\ ; \end{array} \tag{1a}$$

$$\begin{array}{ccccc} U(4) & \supset & U(3) & \supset & O(3) \\ N_t & & n_{\pi} & & L\ . \end{array} \tag{1b}$$

Here the physical content of the indices is the following. N_t is the total number of bosons, L is the angular momentum, ω is related to the vibrational quantum number, and n_{π} is the number of the π bosons. The two group chains give two sets of basis states which are, in principle, equally good for calculations. For practical reasons one of them may be more convenient for a specific physical system, however, it is worth emphasizing, that the choice of the basis set does not mean any restriction from the viewpoint of the shape of the rotating-vibrating system.

The physical operators of the vibron model are obtained as series expansions in terms of particle-number-conserving bilinear products of boson creation and annihilation operators. The expansion usually ends with second-order, i.e., two-body terms. Since the angular moment is always a good quantum number of the states under consideration, the boson operators are coupled to spherical tensors, according to the scalar, vector, or higher-order tensor character of the physical quantity in question. Specifically, the energy is obtained as an expression in terms of O(3) scalars.

The general description of a system in terms of the vibron model involves the following steps. First, the matrix elements of the physical quantities are determined between states of one of the basis sets. The basis can be either (1a) or (1b). Then, the energy eigenvalue problem is solved by a numerical diagonalization of the Hamiltonian matrix. This provides us with wave-function components too. Knowing the wave function in terms of the basis states one can calculate the matrix elements of other physical operators between different states. So, for the general description of a physical quantity we need to define only its matrix elements between basis states of any of the two sets of (1). This point will be recalled in Sec. III.

In addition to providing us with basis states, the group chains (1a) and (1b) have another important feature. They define special relations between the expansion coefficients of the physical operators. When these relations hold, the eigenvalue problem can be solved analytically. Then it is said that the system has a dynamical symmetry. Since the matrix elements can be obtained analytically, these limiting cases are very convenient for analyzing the experimental data. In addition, they usually give a deeper insight into the nature of a rather abstract algebraic description. In the vibron model the O(4) dynamical symmetry (1a) describes a system with a permanent dipole deformation in the equilibrium state. This limit is called the rigid molecule limit and it proved to be important in chemical applications. The U(3) dynamical symmetry describes the rotational-vibrational motion of the system around a spherical equilibrium shape. It is called the soft molecular limit, and it attracted more attention in nuclear physical applications.

When the simple description in terms of the closed formulas of the dynamical symmetry is not satisfactory, it means that the system is not close to that dynamical symmetry and one should perform the more complicated numerical calculations.

From the viewpoint of its application to nuclear molecular states, the vibron model with U(4) group structure can be considered as a simple cluster model, which takes into account the relative motion of two structureless clusters. In this respect it is is similar to another phenomenologic approach based on local potentials.[9] The model space of the simple vibron model is further discussed in Sec. II B, where its relation to the microscopic cluster model is considered. This relation also gives a physical reason for the choice between the two possible basis sets when the model is used in nuclear physics.

The extension of the model space by including other degrees of freedom, like the collective excitations of one of the clusters[10] and unpaired fermions[11,12] are also possible. In these cases the group structure is more complex. In what follows, however, we shall concentrate mainly on the simple vibron model which has the U(4) dynamical group.

B. Relation to the microscopic cluster model in the harmonic-oscillator limit

The U(3) dynamical symmetry treatment of the two-cluster relative motion corresponds to a description based on an anharmonic potential.[2,13] When we go to the harmonic-oscillator limit of this dynamical symmetry then we arrive at the description of the relative motion in terms of the harmonic-oscillator cluster model. So, in the harmonic-oscillator limit the vibron model and the microscopic cluster model are related to each other in a very simple way: the vibron model description means that the internal degrees of freedom of the two clusters are freezed in, only their relative motion is allowed.

Together with the internal degrees of freedom in the vibron model we also neglect the antisymmetrization. However, the link between the cluster model and the shell model in the harmonic-oscillator limit gives a simple prescription for the exclusion of the Pauli forbidden states from the model space. In the harmonic-oscillator limit the Hamiltonians and the wave functions of the shell model and the microscopic cluster model can be transformed into each other.[14] Since the Hamiltonians of the two models have the same eigenvalues, and in the harmonic-oscillator shell-model description the Pauli allowed states carry a certain number of excitation quanta, the relative motion in the cluster model description has to carry the same number of excitation quanta, when the two clusters are in their ground states. So there is a lower limit for the excitation quanta of the Pauli allowed states (Q), and it is given by the Wildermuth condition:

$$Q \geq \sum_{i=1}^{k} 2n_i + l_i \ , \tag{2}$$

where k is the number of nucleons of the lighter cluster, n_i and l_i are the node number and angular momentum quantum numbers of the ith nucleon in the harmonic-oscillator shell-model description. This condition results in a truncation of the model space from the low-energy side, by excluding the Pauli forbidden states. In the harmonic-oscillator approximation the distinction between the forbidden and allowed states can be done exactly, while in the more realistic anharmonic case it works only approximately, and the rest of the antisymmetrization effect is supposed to be taken care of by the phenomenologic two-cluster interaction. This is so both in the local potential model,[9] and in the vibron model.[5,6] In the harmonic-oscillator limit of the vibron model the number of excitation quanta is given by n_π. The ground state is a condensate of σ bosons and each π boson carriers one excitation quantum. Consequently, in this algebraic description Q of Eq. (2) stands for the lowest allowed value of n_π. The physical content of the π bosons has an important consequence for the calculation of the spectroscopic factor, as discussed in Sec. III. On the microscopic level a π boson is known to be a phonon of the relative motion of the two clusters,[2,13] and on the microscopic level it is a shell-model excitation quantum (approximately in the anharmonic case, and exactly in the harmonic case).

Since the relation of the vibron model to microscopic nuclear models manifests itself via the U(3) dynamical symmetry, the basis of (1a) means a natural choice in cluster studies. Due to this fact we shall use this basis in the considerations of Sec. III, too. However, as mentioned in the previous subsection, this does not mean any restriction from the viewpoint of the nuclear deformation.

The relation of the vibron model to the microscopic models as discussed here, resembles that of the interacting boson model (IBM) and the microscopic description. In both cases the model space of the phenomenologic algebraic model is obtained as a truncation of the microscopic model space. For the IBM of the quadrupole collectivity, the truncated space is obtained by keeping the *s* and *d* states of the nucleon pairs in the valence shell, and neglecting the rest.[15,16] For the vibron model of the dipole collectivity, the truncated space is obtained by keeping the relative motion degrees of freedom of the two-cluster systems, and neglecting the rest. Both procedures result in a one-to-one correspondence between the states of the phenomenologic model space and those of the truncated microscopic model space.

In the case of the quadrupole collectivity the relation between the IBM interaction and the microscopic interaction were established by equating the phenomenologic and microscopic matrix elements between the corresponding states.[15] Having the mapping between the vibron model states and the microscopic cluster model and shell-model states the same kind of connection can be built up for the Hamiltonians. This has not been done yet, and it can be the subject for a separate study. Here we are interested in the role of the spectroscopic factor in such a procedure, and in Sec. IV we come back to this question.

III. THE CLUSTER SPECTROSCOPIC FACTOR

From the technical point of view there is a strict similarity between the IBM and the vibron model. In the IBM the particle emission is treated by calculating matrix element of the boson annihilation operator. Since, however, in the vibron model the bosons do not correspond to nucleon pairs, the similarity does not help in this respect, therefore some other method has to be found.

In the emission process the unitarity is not conserved, so the possibility of including nonunitary representations by extending the group structure has been studied for a one-dimensional model problem.[17] It is not clear, however, how to deal with real three-dimensional problems in the same manner. In addition, by changing the group structure, one has a different model, and the question whether the resonances can be described within the framework of the vibron model still remains open.

In the following subsections first we propose a simple phenomenologic way to calculate cluster spectroscopic factor in the vibron model, then by considering the microscopic definition of the spectroscopic factor we find arguments in favor of this prescription.

A. Phenomenologic calculations in the vibron model

Let us denote by Γ_{xL} the width of the resonance which is observed in the emission of the particle x in the Lth partial wave. Then Γ_{xL} contains two quantities:[18]

$$\Gamma_{xL}=2P_{xL}\gamma_{xL}^2 \,, \tag{3}$$

where P_{xL} is the penetration factor, depending on the reaction channel, and γ_{xL}^2 is the reduced width, containing the information on the nuclear structure. For resonances the spectroscopic factor is considered to be

$$\Theta_{xl}^2=\frac{\gamma_{xL}^2}{\gamma_{xL}^2(W)} \,, \tag{4}$$

where $\gamma_{xL}^2(W)$ is the Wigner limit:

$$\gamma_{xL}^2(W)=\frac{3}{2}\frac{\hbar^2}{\mu a^2} \,. \tag{5}$$

Here μ is the reduced mass, and a is the channel radius.

The first point which we would like to emphasize is that since the vibron model is a model of structure, it is more natural to define a spectroscopic factor than resonance width in this model. Having the spectroscopic factor, one can obtain the resonance width via Eqs. (3)–(5) if it is necessary.

Now we recall the general method of the interacting boson models for the construction of a physical operator, i.e., the series expansion in terms of number-conserving boson operators with the proper tensorial character. The spectroscopic factor is an O(3) scalar, so the expansion is exactly the same as that of the Hamiltonian.[2] When written explicitly up to two-body terms, it reads:

$$\begin{aligned}S=\;&a_0+a_1^{(1)}[\sigma^\dagger\times\tilde\sigma]_0^{(0)}+a_2^{(1)}[\pi^\dagger\times\tilde\pi]_0^{(0)}+a_1^{(2)}[[\pi^\dagger\times\pi^\dagger]^{(0)}\times[\tilde\pi\times\tilde\pi]^{(0)}]_0^{(0)}\\&+a_2^{(2)}[[\pi^\dagger\times\pi^\dagger]^{(2)}\times[\tilde\pi\times\tilde\pi]^{(2)}]_0^{(0)}+a_3^{(2)}[[\pi^\dagger\times\pi^\dagger]^{(0)}\times[\tilde\sigma\times\tilde\sigma]^{(0)}+[\sigma^\dagger\times\sigma^\dagger]^{(0)}\times[\tilde\pi\times\tilde\pi]^{(0)}]_0^{(0)}\\&+a_4^{(2)}[[\sigma^\dagger\times\sigma^\dagger]^{(0)}\times[\tilde\sigma\times\tilde\sigma]^{(0)}]_0^{(0)}+a_5^{(2)}[[\pi^\dagger\times\sigma^\dagger]^{(1)}\times[\tilde\pi\times\tilde\sigma]^{(1)}]_0^{(0)}+\cdots\,.\end{aligned} \tag{6}$$

The matrix elements of this operator can again be calculated in the same way as it is done in the case of the energy. For reasons mentioned before, the U(3) basis is more suitable to be used then the O(4) one. Specifically, when the U(3) dynamical symmetry holds, some of the coefficients vanish, and the matrix elements are diagonal in basis (1b):

$$S_{n_\pi L}=\alpha_0+\alpha_1 n_\pi+\alpha_2 n_\pi^2+\beta_2 L(L+1)+\cdots \,, \tag{7}$$

where the coefficients α_i and β_i are some linear combinations of those in Eq. (6). The importance of different terms in Eq. (6) and in Eq. (7) is selected by the specific physical systems under study. In general, the coefficients of these equations should be fitted to experimental data. The question whether or not they should be treated independently from the similar coefficients of the energy expression is discussed in Sec. V.

When the U(3) dynamical symmetry is not a good approximation of the cluster system, the spectroscopic factor of a state which is a linear combination of the U(3) basis states can be obtained from numerical calculations, as mentioned in Sec. II, very much in the same way, as, e.g., the electromagnetic transition rates are calculated.

When one of the clusters is not a closed shell nucleus, i.e., other degrees of freedom are present in addition to the relative motion, as in the case of the nuclear vibron model,[10] or the vibron-fermion model,[11,12] additional terms appear in the spectroscopic factor, due to the coupling. Again, their importance is determined by the problem under considerations; a general expectation is that those terms play the main role which contain contribution from the relative motion.

B. Microscopic cluster model in the harmonic-oscillator approximation

Let us consider the dissociation of nucleus A into nucleus A' plus x. Each nucleus has spin zero in this example. Let Φ_A, $\Phi_{A'}$, and Φ_x be their wave functions antisymmetrized with respect to the interchange of the $1,\ldots,A$ nucleons for Φ_A, those of $1,\ldots,A'$, for $\Phi_{A'}$, and $A'+1,\ldots,A'+x$ for Φ_x. By Φ_{NLM} we denote an orthonormal complete set of functions in the relative coordinates of A' and x, where N, L, and M stand for the number of radial nodes, the angular momentum, and its z component, respectively. Since we consider the harmonic-oscillator cluster model, here Φ_{NLM} is an oscillator function.

The A_{NLM} spectroscopic amplitude is defined[19,20] as

$$A_{NL}(A,A')=\langle \Phi_{A'}\Phi_x\Phi_{NLM}|\mathcal{A}|\Phi_A\rangle \ , \tag{8}$$

where $\mathcal{A}$ is an antisymmetrizer between nucleons in A' and x, and the integration goes over all the coordinates of the A nucleons. So, the spectroscopic amplitude is an overlap between two totally antisymmetrized wave functions of the nucleon system A, one of them ($|\Phi_A\rangle$) describing a definite state of nucleus A, the other one ($\langle\Phi_A{}'\Phi_x\Phi_{NLM}|\mathcal{A}$) describing its separation into two parts. The S_{xL} spectroscopic factor is given by

$$S_{xL}=A_L^2=\sum_N A_{NL}^2 \ . \tag{9}$$

Different models may give different values for A_{NL} (and for S_{xL}) by substituting the model wave functions into Φ_A. The microscopic harmonic-oscillator cluster model wave function is

$$\Phi_A=N_{NL}(A,A')\mathcal{A}[\Phi_{A'}\Phi_x\Phi_{NLM}] \ , \tag{10}$$

where N_{NL} is a normalization factor, and $\mathcal{A}$ is, again, an antisymmetrizer between the nucleons in A' and x. By combining (8) and (10) we obtain the spectroscopic factor given by the microscopic harmonic-oscillator cluster model:

$$A_{NL}(A,A')=\langle\Phi_{A'}\Phi_x\Phi_{NLM}|\mathcal{A}N_{NL}\mathcal{A}|\Phi_{A'}\Phi_x\Phi_{NLM}\rangle \ . \tag{11}$$

These formulas explain why it is necessary to introduce the spectroscopic factor operator S of Eq. (6) in the vibron model. With respect of Eq. (11) the relation of the microscopic cluster model and the vibron model in the harmonic-oscillator limit means the following. The wave function Φ_{NLM} of the relative motion corresponds to the vibron model wave function, $\Phi_{A'}$ and Φ_x are internal wave functions of the two clusters. In the simple case of the structureless clusters they do not appear, while in case of a deformed core, for instance, $\Phi_{A'}$ can be an IBM wave function. What is missing from the phenomenologic treatment of the vibron model is the operator:

$$\mathbf{A}_{NL}=\mathcal{A}N_{NL}\mathcal{A} \ . \tag{12}$$

In the vibron model $N_{NL}=1$, and there is no antisymmetrization operator $\mathcal{A}$.

In the microscopic model the antisymmetrization has a twofold effect. First, it excludes the Pauli forbidden states, and second, it gives a nonunity value of $A_{NL}(A,A')$ for the Pauli-allowed states. Therefore in the phenomenologic treatment, in addition to the basis truncation, which takes into account the first effect, as described in Sec. II B, we have to handle the second effect, too. This is done by the introduction of the spectroscopic factor operator S of Eq. (6). Considering the content of Eq. (12) and the relation of these two models, Eq. (6) should be considered as a phenomenologic approximation to the A_{NL}^2 operator. Consequently, the coefficients of S can be obtained not only from a fit to the experimental data, but also from the microscopic model. We consider this problem in the next section.

IV. CORE-PLUS-ALPHA-PARTICLE SYSTEMS IN HARMONIC-OSCILLATOR APPROXIMATION

The simplest two-cluster systems are those in which both nuclei have closed-shell structure, like ^{4}He, ^{16}O, and ^{40}Ca. In the SU(3) shell model they have $(\lambda,\mu)=(0,0)$ quantum numbers, so they are also called SU(3) scalars. When they described within the harmonic-oscillator approximation, the spectroscopic factors can be calculated analytically. According to the terminology of the microscopic cluster model they are obtained as eigenvalues of the norm kernel operator,[21] the zero eigenvalue stands for a forbidden state.

As a first step of the calculation of spectroscopic factor in the vibron model, it is desirable to consider these examples, i.e., to determine the parameters of Eq. (7) required by the harmonic-oscillator spectroscopic factor of these cluster systems.

As a function of the number of oscillator excitation quanta (n) of the relative motion, this microscopic cluster model spectroscopic factor shows a smooth behavior, and it is independent of the angular momentum content. (So, for the sake of simplicity, we can denote the spectroscopic factor by S_α.) For small n values $S_\alpha=0$, corresponding to the forbiddenness of the cluster configuration. For large values of n $S_\alpha=1$, and there is a smooth transition in between these two straight lines. For the $^{16}\mathrm{O}+\alpha$ system, it is shown by the crosses in Fig. 1. In case of the two other systems we have considered, i.e., $^4\mathrm{He}+\alpha$ and $^{40}\mathrm{Ca}+\alpha$, $S_\alpha(n)$ has a fairly similar shape.

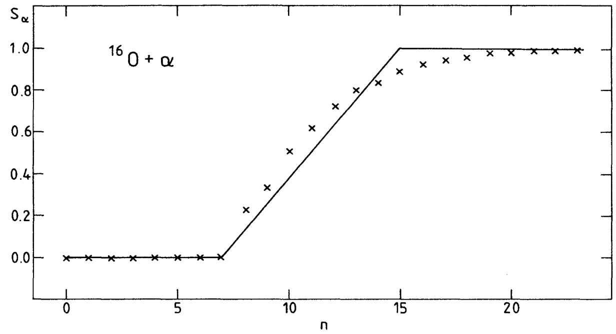

FIG. 1. Alpha-particle spectroscopic factors of ^{20}Ne in the harmonic-oscillator microscopic cluster model (crosses), and their approximation in the vibron model.

When determining the parameters of Eq. (7) corresponding to the harmonic-oscillator spectroscopic factors of these systems we have followed a simple procedure. For small values of n (called n_π in the vibron model) the cluster configuration is forbidden. This fact requires a spectroscopic factor equal to zero below a threshold in n_π, which is obtained from the Wildermuth condition.

As for the parameters in Eq. (7) first we realize that there is no need for L dependence. Then we note that the harmonic-oscillator limit of the vibron model is represented by a Hamiltonian containing only zeroth- and first-order terms in n_π. We suppose the same form for $S_\alpha(n_\pi)$ and since α_0 is determined by the exclusion principle, there is only one parameter to fit, namely, α_1. According to the physical meaning of the spectroscopic factor, above the n_π value where it reaches the unity, we continue with this constant value. (In Fig. 1 the vibron model spectroscopic factor is shown by the continuous line.) Table I contains the parameters obtained for three systems in this kind of fitting procedure.

Not only the n_π threshold, but the α_1 parameter, as well as shows a monotonous dependence on the mass number of the heavier nucleus.

The number of two-cluster systems consisting of closed-shell nuclei is rather limited. However, one can also inquire, whether the values of these parameters obtained for systems with non SU(3) scalar clusters fit into this trend with respect to the mass number. As an illustration we consider here the $^{12}C+\alpha$ system.

When describing this system in terms of the vibron model, there are two possibilities. One can either apply the simple U(4) model which means that core excitations are not allowed, or one can use the U(6)×U(4) nuclear vibron model[10] to take into account some exited states of the ^{12}C. In this latter case the model space is larger, and there are more terms in the physical operators due to the coupling to the internal degrees of freedom. Similarly to the Hamiltonian, for example, the spectroscopic factor contains quadratic or even higher-order terms, so we have to introduce the second-order Casimir operators of the coupled SU(3) or O(3) groups in the series expansion of S_α. Obviously, in order to make a comparison with the systematics suggested by Table I, we should follow the first procedure.

TABLE I. Parameters of the vibron model spectroscopic factor corresponding to the harmonic-oscillator description of the indicated systems.

System	$n_\pi \geq$	α_1
$\alpha+\alpha$	4	0.750
$^{16}O+\alpha$	8	0.137
$^{40}Ca+\alpha$	12	0.086

In the microscopic treatment of the $^{12}C+\alpha$ problem the description based on the $^{12}C(g.s.)+\alpha$ configuration is called the weak-coupling scheme. [In the strong-coupling scheme the wave function of a cluster state has contribution from the $^{12}C(0^+,2^+,4^+)+\alpha$ configuration.] The weak-coupling scheme can be applied to excited rotational bands (above 6.05 MeV) in ^{16}O; it may not be good for the ground state of ^{16}O, since the nonorthogonality between the $^{12}C(0^+)+\alpha$ channel and the $^{12}C(2^+,4^+)+\alpha$ channels. To obtain α_0 and α_1 of Eq. (7) we used the weak-coupling spectroscopic factors. They are obtained as

$$\begin{aligned} S_n(J^\pi) &= S_n[{}^{16}O(J^\pi)\rightarrow{}^{12}C(0^+)+\alpha] \\ &= \sum_{(\lambda,\mu)\kappa} \langle (0,4)0;(n,0)J\|(\lambda,\mu)\kappa J\rangle^2 \mu^{n(\lambda,\mu)} , \end{aligned} \tag{13}$$

where n is the number of excitation quanta in the relative motion, (0,4) is the SU(3) shell-model configuration of the ^{12}C, the double bar symbol stands for the SU(3)⊃O(3) Wigner coefficients,[22] and $\mu^{n(\lambda,\mu)}$ is the eigenvalue of the norm kernel operator.[23] The fit to the data with $n<20$, $J^\pi=0^+,1^-,2^+,3^-,4^+$ gives $\alpha_0=-0.373$ ($n_\pi \geq 3.63$) and $\alpha_1=0.103$. These values are consistent with Table I.

The way that we determined the coefficients of Eq. (7) from the microscopic model spectroscopic factors here can be considered as a simple example of the mapping procedure discussed in Sec. II B. The difference between this and those usual in the IBM is that instead of equating the phenomenologic matrix elements to one specific microscopic matrix element, here by the fitting procedure, we have taken an average of the microscopic values. Also, we restricted ourselves to the harmonic-oscillator interaction. When one considers a more realistic interaction, the mapping includes quadratic terms as well. According to the present proposition, in that case too, the mapping should be done not only for the energy, but for the spectroscopic factor as well.

V. APPLICATION TO EXPERIMENTAL DATA

As a first example we consider three bands of ^{20}Ne, which is one of the best known cluster nuclei.[24] It has been studied in terms of various models, and one of their joint conclusions is that in several bands of this nucleus the cluster configurations have a large overlap with the wave function. The $0_1^+,0^-$ bands as well as the 0_4^+ are known to have a structure close to the core-plus-alpha-particle configuration, and as an illustrative example we take these bands. The energy spectrum of the $^{16}O+\alpha$ system has been discussed within the vibron model, too.[5,6] Here we concentrate on the spectroscopic factor, and on the question whether or not the relative weights of operators appearing in these two physical quantities are similar. For this purpose we keep the number of free parameters as small as possible, and no special attention is paid here to questions like the strength of the rotation-vibration coupling, the behavior of other bands, etc.

The experimental data we have used are given in Table II.

Both the energy spectrum and the spectroscopic factors have been fitted with an expression containing one- and two-body terms:

$$E=\beta L(L+1)+\gamma n_\pi+\delta n_\pi^2+\epsilon \ , \tag{14a}$$

$$S_\alpha=\beta_2 L(L+1)+\alpha_1 n_\pi+\alpha_2 n_\pi^2+\alpha_0 \ . \tag{14b}$$

In accordance with the Wildermuth condition $n_\pi=8$, 9, 10, were assigned to the 0_1^+, 0^-, 0_4^+, bands, respectively. The best agreement was obtained with the parameter sets as follows: $\beta=0.141$ MeV, $\gamma=18.99$ MeV, $\delta=-0.874$ MeV, $\epsilon=-94.6$ MeV, and $\beta_2=-0.0054$, $\alpha_1=10.73$, $\alpha_2=-0.585$, $\alpha_0=-48.1$. The result is shown in Fig. 2. as Th(a) in comparison with the experimental spectrum.

One of the consequences of obtaining the physical quantities in the interacting boson models by series expansions is that the number of free parameters can be large. When the same two or more operators appear in the expression of two different quantities, it is possible and advantageous to use them with the same relative weights. This question has been discussed in relation with the quadrupole operator appearing in the Hamiltonian and in the electric transition operator $T(E2)$ of the IBM-1.[29] That parametrization, called consistent-Q formalism, reduces the number of parameters by one.

TABLE II. Experimental data for the 0_1^+, 0^-, and 0_4^+ bands in ^{20}Ne. When no other sources are indicated, are taken from Ref. 25.

K^π	J^π	E (MeV)	Θ_α^2	Comment
0_1^+	0^+	0.0	0.15	a
	2^+	1.63	0.11	a
	4^+	4.25	0.12	a
	6^+	8.78	0.085	
	8^+	11.95	0.01	
0^-	1^-	5.79	1.03	b
	3^-	7.16	0.87	b
	5^-	10.26	0.90	b
	7^-	13.69	0.84	b
	9^-	17.43	0.48	b
0_4^+	0^+	~8.7	0.70	
	2^+	~8.8	0.95	
	4^+	10.80	0.33	
	6^+	12.58	0.14	c
	8^+	17.30	0.12	c

[a] Spectroscopic factor from Ref. 26.

[b] Spectroscopic factor from Ref. 27.

[c] Reference 28. In light of more recent works, the K^π assignment to these states is uncertain, so we took them with a weight of 50%.

Here we can raise a similar question addressed to the relative weights of the zeroth-, first-, and second-order n_π operators in Eq. (14). So, we investigated how good agreement can be achieved when the number of parameters in (14b) is reduced by two, via imposing the constraints

$$\frac{\alpha_0}{\alpha_1}=\frac{\epsilon}{\gamma}, \quad \frac{\alpha_2}{\alpha_1}=\frac{\delta}{\gamma} \ . \tag{15}$$

The agreement turns out to be still reasonable, the spectrum obtained this way is shown in Fig. 2 as Th(b). The β, γ, δ, and ϵ parameters are the same as before, while for the two independent parameters of S_α we obtained $\beta_2=-0.0018$ and $\alpha_1=2.018$.

The anharmonic, i.e., the quadratic, terms in (14) are not negligible at all; for several states they have the same order of magnitude as the linear terms. So the harmonic-oscillator approximation is not realistic.

A similar conclusion can be drawn for the $^{12}C+\alpha$ system as well. In this case we considered the positive- and negative-parity alpha-cluster bands, sometimes called the inversion doublets[24] (Table III), and we have applied Eq. (14) of the simple vibron model without the n_π^2 terms. To exclude the partially forbidden shells, the $n_\pi=8,9$ assignment was made to the $K^\pi=0^+$, 0^- bands, respectively. The spectrum of Fig. 3 was calculated with the parameters $\beta=0.218$ MeV, $\gamma=2.67$ MeV, $\epsilon=-15.30$ MeV; $\beta_2=-0.0089$, $\alpha_1=0.224$, $(\alpha_0/\alpha_1)=(\epsilon/\gamma)$, $\delta=\alpha_2=0$. Here, again the splitting due to the $L(L+1)$ terms is considerable, and the α_1 value is larger by a factor of two than that obtained in the previous section for this system. (Its comparison to that of ^{20}Ne does not tell us too much, because of the presence of the n_π^2 term there.)

In addition to the states of Table III, many other reso-

TABLE III. Experimental data for the $^{12}C+\alpha$ system. The energies are taken from Ref. 30.

K^π	J^π	E (MeV)	Θ_α^2	Comment-
0^+	0^+	6.049	0.42	a
	2^+	6.917	1.10	a
	4^+	10.356	0.28	b
	6^+	16.275	0.28	b,c
0^-	1^-	9.585	0.50	b
	3^-	11.60	0.56	b
	5^-	14.66	0.39	b
	7^-	20.857	0.19	b

[a] The spectroscopic factors are obtained from the $\Theta_\alpha^2/\Theta_\alpha^2(2^+)$ ratios of Table 16.13 in Ref. 30 and from the $\Theta_\alpha^2(4^+)$ value.

[b] Θ_α^2 is calculated with the channel radius parameter $r_0=1.4$ fm, and $\gamma_\alpha^2(W)=706.98$ keV, following the choice of most of the authors cited in Ref. 31.

[c] In Ref. 30 the K^π assignment to this state is uncertain, so we took it in the fitting procedure with a weight of 50%.

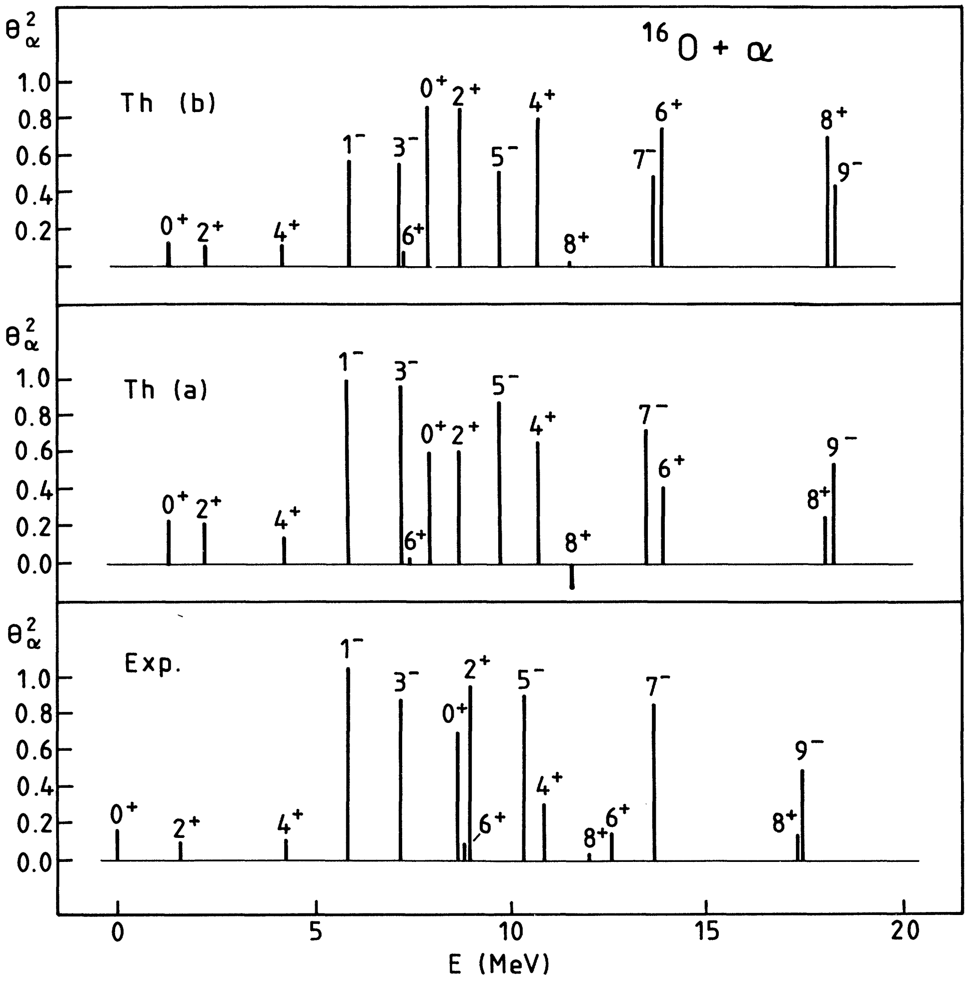

FIG. 2. Energy spectrum and alpha-particle spectroscopic factors of cluster states in ^{20}Ne. Experimental data for the 0_1^+, 0^-, and 0_4^+ bands, and vibron model calculation with independent (a) and consistent (b) parametrization.

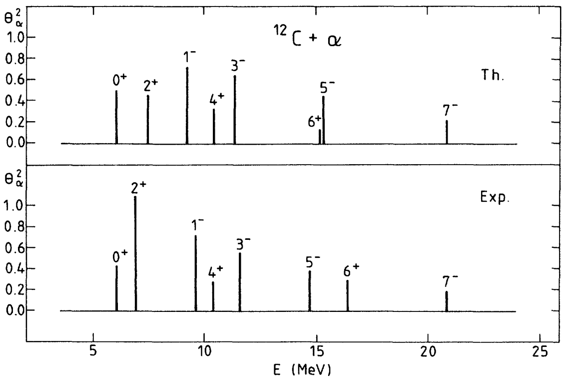

FIG. 3. Energy spectrum and alpha-particle spectroscopic factors of cluster states in ^{16}O. Experimental data for the 0^+ and 0^- bands, and vibron model calculation with consistent parametrization.

nances are known to be populated in alpha scattering with considerable reduced widths.[30,31] It turns out that the alpha-particle strength distribution is consistent with the picture of the fragmentation of a single underlying cluster state in the partial waves with $L = 4$, 6, 1, 3, 5, 7, and the parameters of these simple states can be determined from the strength distribution.[32] Substituting the states with these angular momenta in Table III by the simple states of the strength distribution one obtains another set of data which can be considered as parameters of two cluster bands. The parameters of Eq. (14) obtained from these data proved to be very close to those required by the states in Table III.

VI. SUMMARY AND CONCLUSIONS

In this paper we have proposed a series expansion similar to that of the Hamiltonian to obtain cluster spectroscopic factor in the vibron model. The explanation for this prescription is given by the definition of the spectroscopic factor in the harmonic-oscillator microscopic cluster model and the relation between this description and the limiting case of the vibron model.

In the harmonic-oscillator approximation the interacting boson approach to clusterization and the microscopic cluster model are related to each other in a simple way: the vibron model corresponds to a description based on the truncated model space of the microscopic treatment. This relation resembles that of the models of quadrupole collectivity, where the microscopic collective model is equivalent to the interacting boson approximation in the harmonic-oscillator limit.[16]

Due to the facts that the vibron model wave functions are (i) wave functions of bound states, (ii) which are normalized to unity, and (iii) the effect of the antisymmetrization is accounted for by the basis truncation method only partly, there is a need for taking into account the action of the $A_{NL} = \mathcal{A} N_{NL} \mathcal{A}$ operator explicitly. So, in a sense, we can say that similarly to the truncation of the U(3) basis found previously,[5,6] the necessity of the introduction of the spectroscopic factor as discussed here, can be considered as a consequence of the Pauli principle.

We have determined the parameters of the vibron model spectroscopic factor corresponding to the harmonic-oscillator approximation for a few core-plus-alpha-particle systems. Within the U(3) dynamical symmetry approach we also discussed the spectroscopic factors of some well-known cluster bands in ^{20}Ne and ^{16}O. These examples support the idea of simplifying the parametrization by applying the polynomials in n_π with the same relative coefficients in S_α, as they are obtained from the energy spectrum.

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