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Collective boson realization for the nuclear Sp(6) model

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We demonstrate the construction of an exact *sd*-boson realization of the Sp(6) fermion dynamical symmetry model (Ginocchio model) from an application of the generalized Dyson mapping and subsequent truncation. This realization is an illustration of the validity of the procedure initiated by Geyer and Hahne and analyzed in full generality by Kim and Vincent. We discuss the connection of the Sp(6) model with the SU(3) limit of the interacting-boson model and the implications for the interpretation of its parameters.

I. INTRODUCTION

Models which address collective nuclear properties in terms of a specific and manageable algebraic structure have recently had a major impact on nuclear structure physics, both as a result of their ability to correlate known data and to predict new nuclear properties. Although there have been precursors which also relied on algebraic structure, the major new drive behind the implementation of algebraic structure has come from the interacting-boson model (IBM), pioneered by Arima and Iachello.[1,2] Although many variants of this phenomenological model (with various degrees of microscopic input) have since been used, they all share one tenuous link to the underlying shell model, namely, the loose association made between bosons and correlated fermion pairs. On the phenomenological level this association is quantified by normally taking the number of bosons for a specific nucleus to be half the number of valence nucleons.[1,2]

In the wake of successful phenomenology there inevitably follow attempts at clarification in terms of a more fundamental microscopy. In this particular case the implied link between the phenomenology and the underlying shell model has given rise to and facilitated two related developments. On the one hand, there have been numerous contributions aimed at making the association between bosons and fermions more precise on the general bifermion operator level (as opposed to association on the number operator level only) or, alternatively, on the level of association between boson and fermion states.[3,4] At the same time there have been efforts, initiated by Ginocchio,[5] to construct a model on the fermion level with the same physical input of dominance by S and D pairs and with an algebraic structure which could be exploited as in the case of the IBM. These efforts have lead to Ginocchio's SO(8) and Sp(6) models[5] and their refinement, the fermion dynamical symmetry model (FDSM).[6,7]

From the phenomenological point of view Ginocchio's construction of a fermion model with the above properties at first did not seem to be successful, because the $\mathrm{Sp}(6)\supset\mathrm{SU}(3)$ branch of the model, which represents its rotational limit, does not contain the required and phenomenologically well established SU(3) representation associated with the ground-state rotational band.[5] Nevertheless, many instructive formal studies, concerned with the fermion-boson association referred to above, have been carried out within the framework of this model, especially within its SO(8) branch.[4,8–14] Moreover, in the form of the fermion dynamical symmetry model[6,7] (FDSM) the above unwelcome restriction on allowed SU(3) representations has been circumvented by essentially a reinterpretation of parameters, namely, by restricting "active" nucleons to the normal parity levels and considering the abnormal parity levels in a given major shell as a "sink for excess nucleons."[6,7] This simple, but significant, measure has lead to numerous successful applications of the FDSM on a phenomenological level and to the interpretation of some novel nuclear structure properties.[15,16]

In the present paper we construct an exact boson analog of the Sp(6) branch of the FDSM in terms of s and d bosons only. This construction is presented as an illustration of the method introduced by Geyer and Hahne for the SO(8) branch of the FDSM (Ginocchio model in the original context)[8] and vindicated in full generality by Kim and Vincent.[17] Hereby the connection to the traditional SU(3) limit of the IBM becomes quite transparent, as do the implications of interpretation of shell-model parameters.

It should be pointed out that an *sd*-boson realization of the Sp(6) algebra has been constructed and discussed by Bonatsos and Klein[18] (see also the comment by Wybourne[19]), albeit from the point of view of a boson realization of p-shell fermion operators. Furthermore, their construction is based on enforcing of commutation relations within a given algebra. We view the present construction and discussion as complementary to that in Ref. 18.

The paper is organized as follows. Section II contains a résume of the generalized Dyson mapping and the considerations that allow the construction of a boson realization of a particular collective subalgebra of the complete fermion shell-model algebra. In Sec. III we briefly introduce the Sp(6) branch of the FDSM and in Sec. IV we proceed to construct the promised *sd*-boson realization of this algebra. In Sec. V we discuss some implications of this model and the present construction for the interpretation of shell-model parameters that play a role in the IBM. We conclude in Sec. VI with a summary and some final remarks. In the Appendix we demonstrate the inadequacy of a class of similarity transformations to Hermitize the general Sp(6) *sd*-boson Hamiltonian.

II. THE GENERALIZED DYSON-MALEEV BOSON MAPPING

For the case of an even fermion system with only one distinct type of fermion, the generalized Dyson-Maleev (GDM) mapping of the bifermion operators is defined by[20] (we use the notation of Ref. 8 with the usual summation convention)

$$\begin{aligned} b^{\alpha\beta} &\equiv c^{\alpha}c^{\beta} \to R^{\alpha\beta} \equiv B^{\alpha\beta} - B^{\alpha\theta}B^{\beta\rho}B_{\theta\rho} \ (= -R^{\beta\alpha}) \ , \\ b_{\alpha\beta} &\equiv c_{\beta}c_{\alpha} \to R_{\alpha\beta} \equiv B_{\alpha\beta} \ (= -R_{\beta\alpha}) \ , \\ b^{\alpha}_{\beta} &\equiv c^{\alpha}c_{\beta} \to R^{\alpha}_{\beta} \equiv B^{\alpha\theta}B_{\beta\theta} \ . \end{aligned} \tag{1}$$

Here the operators c^{α} and c_{α} denote the usual fermion creation and annihilation operators, respectively. The index α denotes the quantum numbers j_{α}, m_{α}, i.e., $\alpha \equiv (j_{\alpha}, m_{\alpha})$.

The ideal boson creation (annihilation) operators $B^{\alpha\beta}$ ($B_{\alpha\beta}$) satisfy

$$\begin{aligned} & B^{\alpha\beta} = (B_{\alpha\beta})^{\dagger}, \quad B^{\alpha\beta} = -B^{\beta\alpha} \ , \\ & [B_{\alpha\beta}, B^{\mu\nu}] = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha} \ , \\ & [B_{\alpha\beta}, B_{\mu\nu}] = [B^{\alpha\beta}, B^{\mu\nu}] = 0 \ , \end{aligned} \tag{2}$$

while the ideal boson vacuum state $|0)$ is defined by the requirement $B_{\mu\nu}|0) = 0$.

It is well known (see, e.g., the paper by Okubo[21]) that the bifermion operators close under the SO($2n$) algebra, where n denotes the number of single-particle states. By construction the mapped operators $R^{\alpha\beta}$, $R_{\alpha\beta}$, and R^{α}_{β} constitute an exact realization of this SO($2n$) algebra.

Clearly any shell-model problem for an even number of fermions can be formulated in terms of the SO($2n$) algebra spanned by the bifermion operators. However, when the dynamics dictate the existence of a collective subspace which is decoupled from the rest of the fermion space, the degrees of freedom associated with collective motion can be expressed in terms of only those generators which span this "collective" subalgebra of the complete bifermion algebra. For a restricted class of models a collective subspace may be defined by the Tamm-Dancoff approximation (TDA). However, in transitional and deformed nuclei the important ground-state correlations are not taken into account by the TDA and a more general method is required to identify a collective subspace. In this respect Klein[22] and Klein and Vallières[23] have discussed criteria, based on variational considerations, for identifying and obtaining a collective subspace. A more recent contribution in this direction is presented by Kuchta.[24] The notion of collectivity is incorporated in the boson description by introducing coherent bifermion operators through the following linear combinations of the bifermion operators:

$$\begin{aligned} b^{\sigma} &\equiv \chi^{\sigma}_{\alpha\beta} b^{\alpha\beta} \ , \\ b_{\sigma} &\equiv \chi^{\alpha\beta}_{\sigma} b_{\alpha\beta} \ , \\ \rho_{\sigma} &\equiv \xi^{\beta}_{\sigma\alpha} b^{\alpha}_{\beta} \ , \end{aligned} \tag{3}$$

with

$$\begin{aligned} \chi^{\sigma}_{\alpha\beta} &\equiv (\chi^{\alpha\beta}_{\sigma})^{*} \ , \\ \chi^{\sigma}_{\alpha\beta} &\equiv -\chi^{\sigma}_{\beta\alpha} \ . \end{aligned} \tag{4}$$

Here $\{\chi^{\sigma}_{\alpha\beta}\}$ forms a complete set of two-particle wave functions which includes the wave functions associated with the collective subspace. The set $\{\chi^{\sigma}_{\alpha\beta}\}$ can be chosen as the two-particle TDA wave functions, or a set of wave functions determined from variational considerations, as already alluded to. Whatever their origin, we adopt a normalization for the wave functions which is expressed by

$$\begin{aligned} \chi^{\sigma}_{\alpha\beta}\chi^{\alpha\beta}_{\eta} &= 2\delta^{\sigma}_{\eta} \ , \\ \chi^{\sigma}_{\alpha\beta}\chi^{\rho\theta}_{\sigma} &= \delta^{\rho}_{\alpha}\delta^{\theta}_{\beta} - \delta^{\rho}_{\beta}\delta^{\theta}_{\alpha} \ . \end{aligned} \tag{5}$$

The coherent transition operators ρ_{σ} can be expressed[17,25] as Hermitian linearly independent combinations of the commutators $[b^{\sigma}, b_{\sigma'}]$, which determine the functions $\xi^{\alpha}_{\sigma\beta}$ in terms of the functions $\chi^{\sigma}_{\alpha\beta}$.

In further discussions we distinguish between the *truly* collective modes (denoted by c_i, $i = 1, 2, \ldots$) and the noncollective modes (denoted by n_j, $j = 1, 2, \ldots$). For instance, in the TDA solutions of some schematic models, the (single) collective TDA state is referred to as the truly collective mode ($\sigma = c$) while the other states are referred to as the noncollective modes ($\sigma = n_j$).

Collective-type bosons are introduced by the transformation

$$\begin{aligned} B^{\sigma} &\equiv \tfrac{1}{2}\chi^{\sigma}_{\alpha\beta}B^{\alpha\beta} \ , \\ B_{\sigma} &\equiv \tfrac{1}{2}\chi^{\alpha\beta}_{\sigma}B_{\alpha\beta} \equiv (B^{\sigma})^{\dagger} \ , \end{aligned} \tag{6}$$

with $\chi^{\sigma}_{\alpha\beta}$ taken from Eqs. (3). The orthogonality and completeness relations (5) ensure that the collective-type bosons satisfy the boson algebra

$$\begin{aligned} & [B_{\sigma}, B^{\eta}] = \delta^{\eta}_{\sigma} \ , \\ & [B_{\sigma}, B_{\eta}] = [B^{\sigma}, B^{\eta}] = 0 \ . \end{aligned} \tag{7}$$

The collective subalgebra of the complete bifermion algebra (denoted by A^{CF}) is spanned by a subset of the coherent bifermion operators (3), which is singled out by the collective index $\sigma = c$. In general, c may represent more than one collective mode, but in order to keep the notation simple, we use c rather than c_i.

The GDM mapping (1), followed by a transformation to collective-type bosons (6), may now be used to obtain an exact boson realization (denoted by A^{D}) of the collective algebra A^{CF}, namely,

$$\begin{aligned} b^{c} &\to (b^{c})^{\mathrm{D}} = B^{c} - \chi^{c}_{\alpha\beta}\chi^{\alpha\theta}_{\sigma_1}\chi^{\beta\rho}_{\sigma_2}\chi^{\sigma_3}_{\theta\rho}B^{\sigma_1}B^{\sigma_2}B_{\sigma_3} \ , \\ b_{c} &\to (b_{c})^{\mathrm{D}} = B_{c} \ , \\ \rho_{c} &\to (\rho_{c})^{\mathrm{D}} = \xi^{\alpha}_{c\beta}\chi^{\alpha\theta}_{\sigma_1}\chi^{\sigma_2}_{\beta\theta}B^{\sigma_1}B_{\sigma_2} \ . \end{aligned} \tag{8}$$

It is clear that not only truly collective bosons, but also noncollective bosons, are required in the Dyson images of the collective bifermion operators. It is therefore not trivial to determine *a priori* if the truncation to truly collective bosons will retain the isomorphism between $A^{\rm CF}$ and $A^{\rm D}$. That this is the case when $A^{\rm CF}\equiv {\rm SO}(8)$, has been proved and demonstrated by Geyer and Hahne.[8] Subsequently, Kim and Vincent[17,25] have shown that this result holds much more generally. Their demonstration rests on properties of the collective algebra $A^{\rm CF}$, the most significant of which is the self-conjugacy of $A^{\rm CF}$. This condition requires that $\chi^{\sigma}_{\alpha\beta}=(\chi^{\alpha\beta}_{\sigma})^*$. The truly collective boson realization of $A^{\rm CF}$, obtained by truncating to truly collective bosons in $A^{\rm D}$, will be denoted by $A^{\rm TC}$. We note that all those requirements identified by Kim and Vincent which conspire to validate the above truncation are mostly satisfied by the generators of shell-model algebras. (See Ref. 26 for an example where self-conjugacy does not hold.)

It is furthermore noteworthy that isomorphism between algebraic structures alone does not guarantee equivalence, in particular not as far as physical consequences are concerned. In the context of boson mappings one also requires that the boson space carries the proper spinor representations.[27] On a more pedestrian level this is guaranteed if the boson images of operators, which annihilate the fermion vacuum, annihilate the chosen boson vacuum— a situation fulfilled for Dyson-type mappings. On a more formal level, Kim and Vincent[17,25] also prove this equivalence between the realizations $A^{\rm CF}$ and $A^{\rm TC}$.

We now proceed to implement the program outlined in this section in the case of the FDSM (Ginocchio) Sp(6) collective subalgebra.

III. THE FDSM (GINOCCHIO) Sp(6) MODEL

Ginocchio originally introduced two fermion models with SO(8) and Sp(6) symmetries,[5] respectively. The FDSM is essentially the same model, but with different roles assigned to nucleons in normal and abnormal parity levels.[6,7] (See also the subsequent discussion.) The construction of these models is based on the preselection of a set of single-nucleon angular momenta **j** which can be separated into a pseudo-orbital angular momentum **k** and a pseudospin **i**, i.e., $\mathbf{j}=\mathbf{k}+\mathbf{i}$, with either k or i fixed. (k takes on integer values and i half-integer values.) The single-nucleon creation and annihilation operators in this k-i coupling basis are related to the normal shell-model operators by

$$a^{km_k,im_i}=\sum_{jm}\langle km_k im_i|jm\rangle c^{jm}\ ,\qquad a_{km_k,im_i}=(a^{km_k,im_i})^{\dagger}\ , \tag{9}$$

with covariant components of the annihilation operators given by

$$\tilde a_{km_k,im_i}=(-1)^{k+i-m_k-m_i}a_{k-m_k,i-m_i}\ . \tag{10}$$

We define a special class of fermion pair creation operators within the k-i basis by

$$F^{JM}(KI;ki)\equiv\frac{1}{\sqrt{2}}[a^{k,i}a^{k,i}]^{(KI)JM}\ , \tag{11}$$

with

$$(a^{k,i}a^{k,i})^{(KI)JM}=\sum\langle km_k km_k'|KM_K\rangle\langle im_i im_i'|IM_I\rangle\times\langle KM_K IM_I|JM\rangle a^{km_k,im_i}a^{km_k',im_i'}\ . \tag{12}$$

Alternatively one has

$$F^{JM}(KI;ki)=\frac{1}{\sqrt{2}}\hat K\hat I\sum_{j_1j_2}\hat j_1\hat j_2\begin{Bmatrix}k & i & j_1\\ k & i & j_2\\ K & I & J\end{Bmatrix}[c^{j_1}c^{j_2}]^{JM}\ , \tag{13}$$

with $\hat J=\sqrt{2J+1}$. The partitioning of j in terms of k and i is now explicitly contained in the Wigner 9-j symbol.

The Pauli principle restricts $K+I$ to even values only. If we set $k=1$ then the only angular momenta allowed for $I=0$ in Eqs. (12) and (13) are $J=K=0,2$. This choice of k and I defines the Sp(6) model. The pairs with $J=0$ and $J=2$ are called S and D pairs, respectively, and play the role of collective pairs with respect to a general Sp(6) Hamiltonian. In other words, when the model Hamiltonian is written in terms of Sp(6) generators, the states with S and D pairs only, decouple from states which contain pairs of the type (13) with $I\neq 0$. This property can be traced to the orthogonality of the Wigner 9-j symbols which appear in Eq. (13).

The SO(8) model, with $i=\frac{3}{2}$ and $K=0$ in (13), presents an alternative way to obtain collective S and D pairs. In both cases, the repeated application of the pair creation operators (11) with $J=0$ and $J=2$ onto the vacuum state $|0\rangle$ yields a collective S-D subspace.

Together with the pair creation operators $F^{JM}(KI;ki)$, we also introduce the operators

$$P^J_M(KI;ki)=-\hat k\hat I[a^{k,i}\tilde a_{k,i}]^{(KI)J}_M=-\hat k\hat\imath\hat K\hat I\sum_{j_1j_2}\hat j_1\hat j_2\begin{Bmatrix}k & i & j_1\\ k & i & j_2\\ K & I & J\end{Bmatrix}[c^{j_1}\tilde c_{j_2}]^J_M\ , \tag{14}$$

with $\tilde c_{jm}=(-1)^{j-m}c_{j-m}$. Following Helmers,[28] the Sp(6) generators are now identified as

$$
\begin{aligned}
&S^{\dagger}\equiv F^{00}(00;1i)=\frac{1}{2\sqrt{\Omega}}\sum_{jm}(-1)^{j-m}c^{jm}c^{j-m}\ ,\\
&D^{\mu}\equiv F^{2\mu}(20;1i)=-\frac{1}{2}\left[\frac{3}{\Omega}\right]^{1/2}\sum_{j_1j_2}(-1)^{j_2+i}\hat{j}_1\hat{j}_2\begin{Bmatrix} j_1 & j_2 & 2\\ 1 & 1 & i\end{Bmatrix}[c^{j_1}c^{j_2}]^{2\mu}\ ,\\
&P^r_{\mu}\equiv P^r_{\mu}(r0;1i)=\sqrt{3}\sum_{j_1j_2}(-1)^{j_2+i+r}\hat{j}_1\hat{j}_2\begin{Bmatrix} j_1 & j_2 & r\\ 1 & 1 & i\end{Bmatrix}[c^{j_1}\tilde{c}_{j_2}]^r_{\mu}\quad (r=0,1,2)\ ,\\
&S\equiv(S^{\dagger})^{\dagger}\ ,\\
&D_{\mu}\equiv(D^{\mu})^{\dagger}\ .
\end{aligned}
\tag{15}
$$

Here Ω is the usual measure of the size of the fermion space,

$$2\Omega\equiv\sum_j(2j+1)=3(2i+1)\ . \tag{16}$$

The operators (15) close under the Sp(6) algebra

$$
\begin{aligned}
&[S^{\dagger},D^{\mu}]=[S,D_{\mu}]=0\ ,\\
&[S^{\dagger},S]=\frac{1}{\Omega}(c^{jm}c_{jm}-\Omega)\equiv\frac{2}{\Omega}S_0\ ,\\
&[D_{\mu},D^{\nu}]=-\frac{2}{\Omega}\delta^{\nu}_{\mu}S_0-\frac{5}{\Omega}\sqrt{3}\sum_{t=1,2}(-1)^{\mu}\langle 2-\mu 2\nu|t\nu-\mu\rangle\begin{Bmatrix}2 & 2 & t\\ 1 & 1 & 1\end{Bmatrix}P^t_{\nu-\mu}\ ,\\
&[D^{\mu},S]=\frac{1}{\Omega}P^2_{\mu}\ ,\\
&[P^r_{\mu},S^{\dagger}]=2\delta^r_0\delta^0_{\mu}S^{\dagger}+2\delta^r_2D^{\mu}\ ,\\
&[P^r_{\mu},D^{\nu}]=2(-1)^{\mu}\delta^r_2\delta^{\nu}_{-\mu}S^{\dagger}+2\sqrt{15}\hat{r}\langle r\mu 2\nu|2\mu+\nu\rangle\begin{Bmatrix}2 & 2 & r\\ 1 & 1 & 1\end{Bmatrix}D^{\mu+\nu}\ ,\\
&[P^r_{\mu},P^s_{\nu}]=\sqrt{3}\hat{r}\hat{s}\sum_{t=0,1,2}[(-1)^t-(-1)^{r+s}]\langle r\mu s\nu|t\mu+\nu\rangle\begin{Bmatrix}r & s & t\\ 1 & 1 & 1\end{Bmatrix}P^t_{\mu+\nu}\ .
\end{aligned}
\tag{17}
$$

Note that in our definition of the Sp(6) generators a different normalization than the one given in Ref. 7 is used. This difference is reflected in Eqs. (17) above as a slight modification of the commutation relations given in Ref. 7.

The operators $S^{\dagger}$ and D^{μ} denote the monopole (S) and quadrupole (D) pair creation operators, respectively. The S-D subspace is spanned by the states

$$|NN_d\gamma;JM\rangle=(S^{\dagger})^{N-N_d}(D^{\dagger})^{N_d}_{\gamma JM}|0\rangle\ , \tag{18}$$

where N denotes the total number of pairs and N_d the total number of D pairs. The S and D pairs in the states (18) are coupled to angular momentum J and projection M. Any additional quantum numbers necessary for a complete labeling are represented by γ.

The closure of the algebra (17) implies that any Hamiltonian written in terms of the Sp(6) generators (15) will leave the S-D subspace invariant. The S-D subspace therefore represents a collective subspace as discussed in the preceding section. The schematic shell-model Hamiltonian

$$H_{\mathrm{Sp}(6)}=G_0S^{\dagger}S+G_2D^{\dagger}\cdot\tilde{D}+b_2P^2\cdot P^2+b_1P^1\cdot P^1 \tag{19}$$

represents a general example of such a Hamiltonian. The first two terms are the monopole and quadrupole pairing interactions, respectively, followed by a quadrupole-quadrupole and a dipole interaction.

The two-particle wave functions

$$
\chi^{J\sigma}_{j_1j_2}\equiv\chi^{J(KI)}_{j_1j_2}\equiv\sqrt{2}\hat{j}_1\hat{j}_2\hat{K}\hat{I}\begin{Bmatrix}k & i & j_1\\ k & i & j_2\\ K & I & J\end{Bmatrix}
=\chi^{j_1j_2}_{J\sigma}\equiv(\chi^{J\sigma}_{j_1j_2})^*\ , \tag{20}
$$

with $I=0$, appear in the Sp(6) generators (15), which indicates that these generators are in fact collective bifermion operators. The wave functions with $I=0$ are therefore, in conjunction with our discussions in the preceding section, termed truly collective, while those with $I\neq 0$ are termed noncollective. In the notation of the preceding section we therefore have that $A^{\mathrm{CF}}\equiv\mathrm{Sp}(6)$. Furthermore, the Sp(6) algebra satisfies all the requirements identified by Kim and Vincent[17] for the truncated Dyson boson images of the generators (15) to constitute a realization A^{TC} of the Sp(6) algebra.

IV. COLLECTIVE BOSON REALIZATION OF THE Sp(6) MODEL

A. The collective boson framework

Since the Sp(6) generators (15) are given in the coupled angular momentum representation, the collective framework of the preceeding sections must be modified in order to accommodate bosons with good angular momentum. This modification simply requires the introduction of the coupled boson operators

$$B^{(j_1j_2)JM} \equiv \sum_{m_1m_2} \langle j_1m_1j_2m_2|JM\rangle B^{j_1m_1j_2m_2} ,\quad B_{(j_1j_2)JM} \equiv (B^{(j_1j_2)JM})^\dagger , \tag{21}$$

and then collective-type operators, as in Eqs. (6), by

$$B^{JM\sigma} \equiv \tfrac{1}{2} \sum_{j_1j_2} \chi^{J\sigma}_{j_1j_2} B^{(j_1j_2)JM} ,\quad B_{JM\sigma} \equiv (B^{JM\sigma})^\dagger . \tag{22}$$

Here $\{\chi^{J\sigma}_{j_1j_2}\}$ is the complete set of two-particle wave functions defined by expression (20). In view of the previous discussions, $\sigma\equiv(KI)$ represent the collective index, with $c\equiv c_J\equiv(J,0)$ the truly collective index for the Sp(6) algebra.

The orthogonality relation and symmetry properties of the 9-j symbol can be used to obtain the normalization and completeness relations of the wave functions as

$$\sum_{j_1j_2} \chi^{J\sigma'}_{j_1j_2}\chi^{j_1j_2}_{J\sigma} = 2\delta^{\sigma'}_{\sigma} , \tag{23}$$

$$\sum_{\sigma} \chi^{J\sigma}_{j_1j_2}\chi^{j_3j_4}_{J\sigma} = \delta^{j_1}_{j_3}\delta^{j_2}_{j_4} + (-1)^{\phi}\delta^{j_1}_{j_4}\delta^{j_2}_{j_3} , \tag{24}$$

with $\phi=2k+2i+j_1+j_2+I+K+J$. The above normalization ensures the boson commutation relation

$$[B_{JM\sigma},B^{J'M'\sigma'}] = \delta^{J'}_{J}\delta^{M'}_{M}\delta^{\sigma'}_{\sigma} . \tag{25}$$

The wave functions also satisfy the symmetry relation

$$\chi^{J\sigma}_{j_1j_2} = (-1)^{\phi}\chi^{J\sigma}_{j_2j_1} . \tag{26}$$

Symmetry of the wave functions, together with symmetry properties of the Clebsch-Gordan coefficients, imply

$$B^{JM\sigma} = -(-1)^{2k+2i+I+K}B^{JM\sigma} , \tag{27}$$

which again shows that $I+K$ must be even.

The s- and d-boson operators are identified as the truly collective boson operators

$$s^\dagger \equiv B^{00c} ,\quad s \equiv B_{00c} , \tag{28}$$

$$d^{\mu} \equiv d^\dagger_{\mu} \equiv B^{2\mu c} ,\quad d_{\mu} \equiv B_{2\mu c} . \tag{29}$$

For subsequent use we note that inversion of expression (22) by means of the completeness relation leads to

$$B^{j_1m_1j_2m_2} = \sum_{JM\sigma} \chi^{j_1j_2}_{JM\sigma}\langle j_1m_1j_2m_2|JM\rangle B^{JM\sigma} . \tag{30}$$

B. The mapping procedure

The GDM mapping of the Sp(6) generators and the subsequent truncation to the truly collective indices are now discussed. From

$$\chi^{Jc}_{j_1j_2} = -(-1)^{j_2+i+J}\left[\frac{3}{\Omega}\right]^{1/2}\hat{j}_1\hat{j}_2\begin{Bmatrix} j_1 & j_2 & J \\ 1 & 1 & i \end{Bmatrix} , \tag{31}$$

we may rewrite D^{μ} in expression (15) as

$$D^{\mu} = \tfrac{1}{2} \sum_{j_1j_2m_1m_2} \chi^{2c}_{j_1j_2}\langle j_1m_1j_2m_2|2\mu\rangle c^{j_1m_1}c^{j_2m_2} . \tag{32}$$

The GDM mapping (1), applied to the operator D^{μ}, gives

$$D^{\mu}\to(D^{\mu})^{\rm D} = \tfrac{1}{2} \sum_{j_1j_2m_1m_2} \chi^{2c}_{j_1j_2}\langle j_1m_1j_2m_2|2\mu\rangle B^{j_1m_1j_2m_2} - \sum_{\substack{j_1\cdots j_4 \\ m_1\cdots m_4}} \chi^{2c}_{j_1j_2}\langle j_1m_1j_2m_2|2\mu\rangle \times B^{j_1m_1j_3m_3}B^{j_2m_2j_4m_4}B_{j_3m_3j_4m_4} , \tag{33}$$

and from the transformation (30) we obtain

$$(D^{\mu})^{\rm D} = B^{2\mu c} - \tfrac{1}{2}\sum \chi^{2c}_{j_1j_2}\chi^{j_1j_3}_{J_1\sigma_1}\chi^{j_2j_4}_{J_2\sigma_2}\chi^{J_3\sigma_3}_{j_3j_4}\langle j_1m_1j_2m_2|2\mu\rangle\langle j_1m_1j_3m_3|J_1M_1\rangle\langle j_2m_2j_4m_4|J_2M_2\rangle \times\langle j_3m_3j_4m_4|J_3M_3\rangle B^{J_1M_1\sigma_1}B^{J_2M_2\sigma_2}B_{J_3M_3\sigma_3} . \tag{34}$$

The angular momentum coupling of the boson operators introduces two additional Clebsch-Gordan coefficients which, together with the four appearing in Eq. (34), can be combined into a 9-j symbol. This gives

$$(D^{\mu})^{\rm D} = d^{\mu} - \tfrac{1}{2}\sum (-1)^{J_4}\hat{J}_1\hat{J}_2\hat{J}_3\hat{J}_4 \begin{Bmatrix} J_1 & j_1 & j_3 \\ J_2 & j_2 & j_4 \\ J_4 & 2 & J_3 \end{Bmatrix} \chi^{2c}_{j_1j_2}\chi^{j_1j_3}_{J_1\sigma_1}\chi^{j_2j_4}_{J_2\sigma_2}\chi^{J_3\sigma_3}_{j_3j_4}[[B^{J_1\sigma_1}B^{J_2\sigma_2}]^{J_4}\tilde{B}_{J_3\sigma_3}]^{2\mu} . \tag{35}$$

Here $\tilde{B}_{JM\sigma}$ are the covariant components of the collective-type boson annihilation operators, namely,

$$\tilde{B}_{JM\sigma} \equiv (-1)^{J-M}B_{J-M\sigma} . \tag{36}$$

The above expression for $(D^{\mu})^{\mathrm{D}}$ represents the most general realization of D^{μ} in terms of collective-type bosons. These bosons are characterized by a collective index σ, including collective indices other than c.

A truly collective boson realization of D^{μ} in terms of s and d bosons only, can now be obtained from (35) by simply truncating to the truly collective indices, i.e., by restricting the indices σ in Eq. (35) to c. From the expression (31) we obtain this truncated form of $(D^{\mu})^{\mathrm{D}}$ as

$$(D^{\mu})^{\mathrm{TC}}=d^{\mu}+\tfrac{9}{2}\sum(-1)^{j_1+j_4+J_4}\left[\frac{1}{\Omega}\hat{j}_1\hat{j}_2\hat{j}_3\hat{j}_4\right]^2\hat{J}_1\hat{J}_2\hat{J}_3\hat{J}_4\begin{Bmatrix} j_1 & j_2 & 2\\ 1 & 1 & i\end{Bmatrix}\begin{Bmatrix} j_1 & j_3 & J_1\\ 1 & 1 & i\end{Bmatrix}$$
$$\times\begin{Bmatrix} j_2 & j_4 & J_2\\ 1 & 1 & i\end{Bmatrix}\begin{Bmatrix} j_3 & j_4 & J_3\\ 1 & 1 & i\end{Bmatrix}\begin{Bmatrix} J_1 & j_3 & j_1\\ J_2 & j_4 & j_2\\ J_4 & J_3 & 2\end{Bmatrix}[[B^{J_1c}B^{J_2c}]^{J_4}\tilde{B}_{J_3c}]^{2\mu}\ . \tag{37}$$

We carry out the summation over j_3 and j_4 by using a sum rule for four 6-j symbols and a 9-j symbol.[29] Expression (37) then simplifies to

$$(D^{\mu})^{\mathrm{TC}}=d^{\mu}-\tfrac{9}{2}\sum\left[\frac{1}{\Omega}\hat{j}_1\hat{j}_2\right]^2\hat{J}_1\hat{J}_2\hat{J}_3\hat{J}_4\begin{Bmatrix} j_1 & j_2 & 2\\ 1 & 1 & i\end{Bmatrix}^2\begin{Bmatrix} J_4 & J_1 & J_2\\ J_3 & 1 & 1\\ 2 & 1 & 1\end{Bmatrix}[[B^{J_1c}B^{J_cc}]^{J_4}\tilde{B}_{J_3c}]^{2\mu}\ . \tag{38}$$

In obtaining the above equation we have also used the fact that J_1, J_2, and J_3 must be even, since $K+I$ is restricted to even values. Using orthogonality of the 6-j symbols and the definition (16) of Ω we finally obtain

$$(D^{\mu})^{\mathrm{TC}}=d^{\mu}-\frac{3}{\Omega}\sum_{J_1J_2J_3J_4}\hat{J}_1\hat{J}_2\hat{J}_3\hat{J}_4\begin{Bmatrix} J_4 & J_1 & J_2\\ J_3 & 1 & 1\\ 2 & 1 & 1\end{Bmatrix}[[B^{J_1c}B^{J_2c}]^{J_4}\tilde{B}_{J_3c}]^{2\mu}\ . \tag{39}$$

The triangle conditions of the 9-j symbol above restrict the J values of the boson operators to $J=0,2$. The truncation to the truly collective indices therefore automatically ensures that only s and d bosons enter the expression.

The GDM mapping of the operator D_{μ} is straightforward. From Eqs. (15) and (32) we have

$$D_{\mu}=\tfrac{1}{2}\sum_{j_1j_2m_1m_2}\chi_{2c}^{j_1j_2}\langle j_1m_1j_2m_2|2\mu\rangle c_{j_2m_2}c_{j_1m_1}\ . \tag{40}$$

Within the collective framework, the GDM mapping of D_{μ} is simply

$$D_{\mu}\to(D_{\mu})^{\mathrm{D}}=(D_{\mu})^{\mathrm{TC}}=B_{2\mu c}=d_{\mu}\ . \tag{41}$$

By using the same procedures as described above for the operators D^{μ} and D_{μ}, the following truly collective realizations for $S^{\dagger}$, S, P^{r}_{μ}, and S_0 are obtained:

$$(S^{\dagger})^{\mathrm{TC}}=s^{\dagger}-\frac{1}{\Omega}\sqrt{3}\sum_{J_1J_2J_3}\hat{J}_1\hat{J}_2\hat{J}_3\begin{Bmatrix} J_1 & J_2 & J_3\\ 1 & 1 & 1\end{Bmatrix}[[B^{J_1c}B^{J_2c}]^{J_3}\tilde{B}_{J_3c}]^{(0)}\ , \tag{42}$$

$$(S)^{\mathrm{TC}}=s\ , \tag{43}$$

$$(P^{r}_{\mu})^{\mathrm{TC}}=2\sqrt{3}\sum_{J_1J_2}(-1)^{r}\hat{J}_1\hat{J}_2\begin{Bmatrix} J_1 & J_2 & r\\ 1 & 1 & 1\end{Bmatrix}[B^{J_1c}\tilde{B}_{J_2c}]^{r}_{\mu}\ , \tag{44}$$

$$(S_0)^{\mathrm{TC}}=\sum_{J=0,2}\sum_{M}B^{JMc}B_{JMc}-\tfrac{1}{2}\Omega\ . \tag{45}$$

As in the case of the operators $(D^{\mu})^{\mathrm{TC}}$ and $(D_{\mu})^{\mathrm{TC}}$ only s- and d-boson operators appear in the above realization of the operators $S^{\dagger}$, S, P^{r}_{μ}, and S_0—a direct consequence of the truncation to the truly collective indices. This truly collective boson realization of the Sp(6) generators, written explicitly in terms of s and d bosons, is then

$$(S_0)^{\mathrm{TC}}=s^{\dagger}s+\sum_{\mu}d^{\mu}d_{\mu}-\tfrac{1}{2}\Omega\equiv\hat{n}_s+\hat{n}_d-\tfrac{1}{2}\Omega\ , \tag{46}$$

$$(S^\dagger)^{\rm TC}=s^\dagger-\frac{1}{\Omega}(\hat{n}_s-1)s^\dagger-\frac{2}{\Omega}\hat{n}_d s^\dagger-\frac{1}{\Omega}[d^\dagger\cdot d^\dagger]s-\frac{1}{2\Omega}\sqrt{35}[[d^\dagger d^\dagger]^2\tilde{d}]^{(0)}\ , \tag{47}$$

$$(S)^{\rm TC}=s\ , \tag{48}$$

$$(D^\mu)^{\rm TC}=d^\mu-\frac{1}{\Omega}s^\dagger s^\dagger\tilde{d}_\mu-\frac{2}{\Omega}\hat{n}_s d^\mu-\frac{1}{\Omega}\sqrt{7}s^\dagger[d^\dagger\tilde{d}]^{2\mu}-\frac{1}{2\Omega}\sqrt{7}[d^\dagger d^\dagger]^{2\mu}s-\frac{15}{\Omega}\sqrt{5}\sum_{L=0,2,4}\hat{L}\begin{Bmatrix}1&1&2\\1&1&2\\2&2&L\end{Bmatrix}[[d^\dagger d^\dagger]^L\tilde{d}]^{2\mu}\ , \tag{49}$$

$$(D_\mu)^{\rm TC}=d_\mu\ , \tag{50}$$

$$(P^{(0)})^{\rm TC}=2(\hat{n}_s+\hat{n}_d)\ , \tag{51}$$

$$(P^1_\mu)^{\rm TC}=\sqrt{15}(d^\dagger d)^1_\mu\ , \tag{52}$$

$$(P^2_\mu)^{\rm TC}=2(d^\mu s+s^\dagger\tilde{d}_\mu)+\sqrt{7}[d^\dagger\tilde{d}]^2_\mu\ . \tag{53}$$

Note that up to a normalization factor the quadrupole operator $(P^2_\mu)^{\rm TC}$ is identical to that IBM quadrupole operator in the SU(3) limit which is associated with a positive quadrupole moment (oblate deformation). We show below that a similarity transformation on the above boson images can be found such that the quadrupole operator corresponds to prolate deformation, i.e., has a relative negative sign between the $(d^\dagger d)^2_\mu$ term and the other two terms. (In the IBM context the minus sign is associated with a prolate shape, the positive sign with an oblate shape.)

As described above, the realization (46)–(53) follows from a truncation to the collective indices defined after expressions (22). Based on the self-conjugacy of the Sp(6) algebra (which can be verified in a trivial manner), the general results of Kim and Vincent,[17] however, guarantee that this drastic measure will result in an exact realization for Sp(6), namely, the one given explicitly in expressions (46)–(53). [Confirmation of this result by explicitly establishing the various commutation relations (17) from the realization (46)–(53) is given in Ref. 25. Although this leads to some instructive intermediate relations, we refrain from discussing these algebraic manipulations here.]

In Sec. V of Ref. 18 an exact *sd*-boson realization is obtained for the fermion pair and multipole operators in the neutron (proton) *p* shell, which close under commutation to form the Sp(6) algebra.[19] The Ginocchio Sp(6) model is in fact restricted to the *p* shell if in the expressions (15) $i=\frac{1}{2}$. For this choice of i the *sd*-boson realization given by Bonatsos and Klein[18] must clearly be isomorphic to the collective *sd*-boson realization $A^{\rm TC}={\rm Sp}(6)$ given above. In fact, the two realizations are identical up to a factor $\sqrt{3}$ in $(P^r_\mu)^{\rm TC}$ due to a difference in the normalization of P^r_μ.

In Sec. VII of Ref. 18 an alternative *sd*-boson realization of the Sp(6) algebra is discussed, differing from the one above in that the terms $(d^\mu s+s^\dagger\tilde{d}_\mu)$ and $(d^\dagger\tilde{d})^2_\mu$ in the quadrupole operator appear with opposite signs. We can obtain this boson realization by performing a similarity transformation on the collective *sd*-boson realization $A^{\rm TC}$. The required similarity transformation is accomplished by the unitary operator $T\equiv e^{i\pi\hat{n}_s}$. We note that $TsT^{-1}=-s$ and $Ts^\dagger T^{-1}=-s^\dagger$, while T commutes with d-boson operators. It is therefore clear that T induces the change of relative sign in the quadrupole operator referred to above. In fact, the two *sd*-boson realizations, obtained in Secs. V and VII of Ref. 18 by the commutator method, are simply related by the similarity transformation T.

We finally refer to the work of Chen *et al.*,[30] which discusses the relationship between the Sp(6) branch and the IBM for the SU(3) limit. [The mapping of the operators (46)–(50) does not enter that discussion.] They point out that the choice of the positive sign in Eq. (53) (oblate deformation) leads to a simple one-to-one correspondence between the basis vectors of the $(\lambda,\mu)\equiv(20)$ representation of SU(3) when expressed in the FDSM and IBM frameworks, respectively. For the choice of a negative relative sign in the IBM quadrupole operator, a different (permuted) correspondence exists where some of the simplicity is lost. [See Eqs. (5.12) and (5.13) of Ref. 30.]

V. Sp(6) AND IBM-LIKE STRUCTURES

The collective *sd*-boson realization of the operators $S^\dagger$, S, D^ν, D_ν, and P^r_μ can be used to obtain a mapping of the shell-model Hamiltonian $H_{{\rm Sp}(6)}$ given in Eq. (19). The boson Hamiltonian thus obtained is given by

$$
\begin{aligned}
(H_{\mathrm{Sp}(6)})^{\mathrm{TC}} = & \frac{G_0}{\Omega}[\Omega\hat{n}_s - \hat{n}_s(\hat{n}_s-1) - 2\hat{n}_d\hat{n}_s] + \frac{G_2}{\Omega}(\Omega\hat{n}_d - 2\hat{n}_d\hat{n}_s) \\
& -\frac{1}{\Omega}\sqrt{5}\{G_0[[d^\dagger d^\dagger]^0[ss]^0]^{(0)} + G_2[[s^\dagger s^\dagger]^0[\tilde{d}\tilde{d}]^0]^{(0)}\} \\
& -\frac{1}{\Omega}\sqrt{35}\{\tfrac{1}{2}(G_0+G_2)[[d^\dagger d^\dagger]^2[\tilde{d}s]^2]^{(0)} + G_2[[d^\dagger s^\dagger]^2[\tilde{d}\tilde{d}]^2]^{(0)}\} \\
& -\frac{75}{\Omega}\sum_{L=0,2,4}\hat{L}\begin{Bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & L \end{Bmatrix}[[d^\dagger d^\dagger]^L[\tilde{d}\tilde{d}]^L]^{(0)} + \sum_{r=1,2} b_r (P^r)^{\mathrm{TC}}\cdot(P^r)^{\mathrm{TC}} . \qquad (54)
\end{aligned}
$$

This is a one- plus two-body *sd*-boson Hamiltonian which is Hermitian only when $G_0 = G_2$. In the Appendix it is shown that no simple similarity transformation (of a certain class) exists which can transform $(H_{\mathrm{Sp}(6)})^{\mathrm{TC}}$ into a Hermitian form while retaining the one- plus two-body operator nature of $(H_{\mathrm{Sp}(6)})^{\mathrm{TC}}$. This contrasts with the situation for SO(8) where such a simple similarity transformation can in fact be constructed.[12]

The general non-Hermiticity of $(H_{\mathrm{Sp}(6)})^{\mathrm{TC}}$ does of course not invalidate the proof by Kim and Vincent[17] of equivalence between the collective bifermion realization and the truly collective boson realization as applied to the Sp(6) algebra. Consequently, the diagonalization of $(H_{\mathrm{Sp}(6)})^{\mathrm{TC}}$ in the basis.

$$
|N, N_d, \gamma; J, M) \equiv [(S^\dagger)^{\mathrm{TC}}]^{N-N_d}[(D^\dagger)^{\mathrm{TC}}]^{N_d}_{\gamma JM}|0) \qquad (55)
$$

will necessarily produce real eigenvalues. [These states are the truly collective boson images of the states (18).]

Furthermore, in the case of $G_0 = G_2$, the Hamiltonian $H_{\mathrm{Sp}(6)}$ has SU(3) symmetry,[7] i.e., it can be written in terms of SU(3) Casimir invariants. Since all the SU(3) generators map (up to normalization) onto the standard IBM SU(3) generators (46), (52), and (53), $(H_{\mathrm{Sp}(6)})^{\mathrm{TC}}$ reduces to a Hermitian SU(3) IBM Hamiltonian with IBM strengths parametrized by Ω. On face value it therefore appears that the Sp(6) model provides direct microscopic support for the SU(3) limit of the IBM. However, based on the observation[5] that the SU(3) irreducible representations $(\lambda, 0)$ are not allowed when $\lambda > \frac{2}{3}\Omega$ or $N_d > \frac{1}{3}\Omega, N_d$ being the number of D pairs, Ginocchio[5] rejected the Sp(6) model as a microscopic foundation of the SU(3) limit of the IBM.

In the IBM, for nuclei with SU(3) dynamical symmetry, the states in the highest SU(3) representation lie lowest in energy. When the nucleon pair number $N > \frac{1}{3}\Omega$, the highest SU(3) representations and thus the ground-state rotational band are not allowed in the Sp(6) model. However, many deformed nuclei with rotational spectra are in fact found near midshell, i.e., $N \approx \frac{1}{2}\Omega$. As a consequence, the Sp(6) model, in its original form, is not compatible with the SU(3) limit of the phenomenological IBM.

The rejection of the Sp(6) as basis for the SU(3) limit of the IBM therefore seems to be justified. However, this conclusion depends inherently on the interpretation of the parameters N and Ω. In the Ginocchio Sp(6) model, N is interpreted as the total number of nucleon pairs in the valence shell, while Ω is seen as the pair degeneracy of the full valence shell. This interpretation corresponds to the usual IBM interpretation. However, the fermion dynamical symmetry model (FDSM) of Wu *et al.*[6,7] adopts a different viewpoint of the Sp(6) model, in particular as far as the interpretation of N and Ω is concerned.

In the FDSM the total number of nucleon valence pairs, N, is taken to be the sum of pairs in the normal parity levels (denoted by N_n) and the pairs in the abnormal parity levels (denoted by N_a), with corresponding pair degeneracies Ω_n and Ω_a. The only difference between the Ginocchio Sp(6) model and the Sp(6) limit of the FDSM is that the allowed SU(3) irreducible representations are differently determined. In the FDSM they are determined by N_n, in the Ginocchio model by N. Furthermore, nucleons in the abnormal parity levels are essentially spectators as far as the dynamics is concerned. Similar to, but different from, the Ginocchio model, is that the highest SU(3) representation $(2N_n, 0)$ is not allowed in the FDSM when $N_n > \frac{1}{3}\Omega_n$. However, since $N = N_n + N_a$, the highest SU(3) representation $(2N_n, 0)$ may still be realized even if $N \approx \frac{1}{2}\Omega$. Therefore, the abnormal parity states, which act as a "sink" for excess nucleons in the normal parity states, provide a mechanism in the FDSM for allowing the rotational ground band in nuclei near the midshell (i.e., $N \approx \frac{1}{2}\Omega$).

We note that the truly collective boson realization of the Sp(6) algebra, parametrized in terms of N and Ω, may simply be introduced on a phenomenological basis. The parameter N can then be treated on the phenomenological level as a label for the SU(3) representations. When it comes to the physical interpretation of these parameters, the interpretation provided by the FDSM seems to be appropriate. We have already mentioned that this interpretation associates the (microscopically linked) parameters with the *normal* parity levels. The usual tendency in IBM analyses has been to take all levels into account whenever parameters such as the total boson number or degeneracy are involved, but this need not necessarily be part and parcel of the model.

Geyer *et al.*[13,14] have in fact demonstrated that the SO(7) symmetry, first analyzed in the framework of the FDSM (Ref. 15) and shown to be appropriate for the Pd-Ru region, can be accommodated within the framework of IBM dynamical symmetries. The parameters that correspond to the Pd-Ru data are, however, clearly those suggested by the FDSM.

From the present work, it follows that the same point of view may be adopted for the SU(3) limit of the IBM, namely, to link allowed representations to the number of valence nucleons in the normal parity states only. It should be emphasized, however, that the restrictions on the allowed representations reflected in $N_n < \Omega_n/3$ can only be inferred from the full Sp(6) algebra. On the fermion level this can be deduced[5] by considering the stretched state $(D^2)^{N_d}|0\rangle$, which is a state in the highest SU(3) irreducible representation $(2N_d,0)$, and using m-scheme counting in the k-i basis.

On the boson level the same information is of course guaranteed to be retained by the fact that we are dealing with an exact mapping of Sp(6) in the expressions (46)–(53). Explicitly this can be seen in the simplest way by rewriting $(D^\mu)^{\rm TC}$ in expression (49) as (see also Ref. 33)

$$\Omega(D^\mu)^{\rm TC}=d^\mu(\Omega-2n_d-2n_s)+(d^\nu\tilde{d}^{\,\nu}-s^\dagger s^\dagger)\tilde{d}_\mu -\sqrt{7}s^\dagger[d^\dagger d]^{2\mu}-\frac{\sqrt{7}}{2}[d^\dagger d^\dagger]^{2\mu}s +\tfrac{7}{4}[[d^\dagger d^\dagger]^2\tilde{d}]^{2\mu}-\tfrac{7}{2}[d^\dagger[d^\dagger\tilde{d}]^2]^{2\mu} \tag{56}$$

and to consider the state $[(D^2)^{\rm TC}]^{n_d}|0)$. For $n_d=2$ we simply have

$$[(D^2)^{\rm TC}]^2|0)=(D^2)^{\rm TC}d^2|0)\ . \tag{57}$$

A simple analysis of expression (56) shows that only the first and last terms will contribute to the state (57). In the other terms either the Clebsch-Gordan coefficient $\langle 2\lambda 2-2|22\rangle$ $(=0)$ appears, or n_s gives zero because no s bosons are created. Inserting the numerical values for the remaining Clebsch-Gordan coefficients one finds

$$[(D^2)^{\rm TC}]^2|0)=\Omega^{-1}(\Omega-3)(d^2)^2|0)\ . \tag{58}$$

For stretched states $[(D^2)^{\rm TC}]^{n_d}|0)$ with larger values of n_d the above considerations simply repeat themselves and one finds in general

$$[(D^2)^{\rm TC}]^{n_d}|0)=\Omega^{-(n_d-1)}(\Omega-3)\times(\Omega-6)\cdots[\Omega-3(n_d-1)](d^2)^{n_d}|0)\ , \tag{59}$$

which explicitly demonstrates that the stretched state $[(D^2)^{\rm TC}]^{n_d}|0)$ will vanish when $\Omega/3<n_d$. [Remember that $\Omega=3(i+\frac{1}{2})$ takes on the values 3, 6, 9, . . .]. One now simply uses Ginocchio's original argument[5] that this result implies that *all* the states in the associated $(2n_d,0)$ representation of SU(3) vanish.

It should be emphasized, however, that this information will not be available to the IBM practitioner who only has the SU(3) generators at his disposal. The appearance of SU(3) Pauli effects are specifically linked to the $\mathrm{Sp}(6)\supset\mathrm{SU}(3)$ embedding and their role has been vividly demonstrated by Wu *et al.*[16]

At the same time it should be pointed out that the FDSM view of S and D pairs can lead to difficulties. Halse[31,32] has, e.g., demonstrated that in the *sd* shell the low-lying states of a realistic interaction are almost orthogonal to the FDSM candidate states. Whereas one of course still has to interpret such a result against the background of effective interactions, it does suggest, as pointed out by Halse, that a successful phenomenology (in this case the FDSM) may not in itself imply that the important degrees of freedom have been isolated. (See Ref. 31 for further discussion.)

Since the IBM can be viewed as being on a "more" phenomenological level than the FDSM, since it is less committed to a particular microscopic structure, it can, however, still adopt the interpretation of parameters as suggested by the FDSM, independently from the FDSM structure as such. Further work, however, is needed to determine where and when this interpretation of parameters in the IBM will be appropriate.

VI. CONCLUSION

We have demonstrated how the generalized Dyson mapping and subsequent truncation to a set of collective degrees of freedom can be utilized to construct an exact *sd*-boson realization of the Sp(6) branch of the FDSM. This serves as a further explicit demonstration of a general result by Kim and Vincent.[17] The SU(3) limit of the FDSM is thereby compared with, and shown to be identical to, the SU(3) limit of the IBM as far as SU(3) generators are concerned, although in the FDSM the allowed representations are determined differently as a result of the $\mathrm{Sp}(6)\supset\mathrm{SU}(3)$ embedding. We suggest that the interpretation of shell-model linked parameters in the phenomenological IBM can be taken over from the FDSM, with hindsight of course even independently from other implications of the FDSM. To what extent this should be done universally in the IBM, requires further investigations.

Note added. After completion of this work we became aware of the work of Próchniak and Szpikowski,[33] who construct the mapping (46)–(53) using the generalized coherent-state method introduced by Dobaczewski.[34] They do not, however, discuss any connection with the FDSM.

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APPENDIX: A CLASS OF SIMILARITY TRANSFORMATIONS

We illustrate that the simplest generalization of the similarity transformation employed by Kim and Vincent[12] for the general SO(8) Hamiltonian cannot transform the Hamiltonian $(H_{\mathrm{Sp}(6)})^{\rm TC}$ of Eq. (54) into a Hermitian one- plus two-body form.

For this purpose we consider the most general transformation $f(n_s,n_d)$ which is a function of the s- and d-boson number operators only. It suffices to consider the effect of this transformation on the non-Hermitian part of $(H_{\mathrm{Sp}(6)})^{\rm TC}$ in Eq. (54), namely,

$$(H^n_{\mathrm{Sp}(6)})^{\mathrm{TC}} = -\frac{1}{\Omega}\sqrt{5}\{G_0[(d^\dagger d^\dagger)^0(ss)^0]^{(0)} + G_2[(s^\dagger s^\dagger)^0(\tilde{d}\tilde{d})^0]^{(0)}\} - \frac{1}{\Omega}\sqrt{35}\{\tfrac{1}{2}(G_0+G_2)[(d^\dagger d^\dagger)^2(\tilde{d}s)^2]^{(0)} + G_2[(d^\dagger s^\dagger)^2(\tilde{d}\tilde{d})^2]^{(0)}\} . \tag{A1}$$

It is clear that the requirement, that $f(n_s,n_d)(H^n_{\mathrm{Sp}(6)})^{\mathrm{TC}} f^{-1}(n_s,n_d)$ be Hermitian, leads to two equations which f has to fulfill simultaneously. Straightforward manipulation gives

$$G_0 f^2(n_s,n_d) = G_2 f^2(n_s+2,n_d-2) \tag{A2}$$

and

$$\tfrac{1}{2}(G_0+G_2) f^2(n_s,n_d) = G_2 f^2(n_s+1,n_d-1) . \tag{A3}$$

Equation (A3) can be rewritten as

$$(x+1) f^2(n_s,n_d) = 2f^2(n_s+1,n_d-1) \tag{A4}$$

$$= 2x f^2(n_s-1,n_d+1) , \tag{A5}$$

where $x = G_0/G_2$ and where Eq. (A2) has been used in the last step.

Next, rewrite Eq. (A4) in the form

$$(x+1) f^2(n_s-1,n_d+1) = 2f^2(n_s,n_d) , \tag{A6}$$

and combine this result with Eq. (A5). This finally yields

$$\frac{2x}{x+1} = \frac{x+1}{2} , \tag{A7}$$

with $x=1$ as only solution.

However, when this condition holds, $(H_{\mathrm{Sp}(6)})^{\mathrm{TC}}$ is already Hermitian. The simplest class of similarity transformations which depend on n_s and n_d only, can therefore not transform $(H_{\mathrm{Sp}(6)})^{\mathrm{TC}}$ into a one- plus two-body Hermitian form. Whether any other transformation with this property exists, remains an open question.

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