PHYSICAL REVIEW C VOLUME 42, NUMBER 2 AUGUST 1990

Role of Levinson's theorem in neutron-deuteron quartet S-wave scattering

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(Received 4 April 1990)

The real part of the phase shift for elastic neutron-deuteron scattering in the quartet S-wave channel, as calculated with the exact three-body theory, assumes at threshold the value π if normalized to zero at infinity; that is, it does not comply with the expectations raised by a naive application of Levinson's theorem since no bound state exists in this channel. A description of this situation on an equivalent two-body level via a potential, constructed by means of the Marchenko inverse scattering theory, necessitates the introduction of a fictitious bound state. This predominantly attractive, equivalent local potential can be related via supersymmetry to a strictly phase equivalent partner potential. The latter is unique and purely repulsive, a behavior already exhibited by the underlying exact effective neutron-deuteron interaction. At the origin it possesses a singularity of the centrifugal barrier type which admits of the required zero-energy phase shift value of π by means of a modified version of Levinson's theorem. Hence, the unphysical bound state of the attractive equivalent local potential plays a role in three-body scattering theory analogous to the one of a Pauli-forbidden state in the context of the resonating group method.

The exact formulation of the nonrelativistic three-body theory[1–5] made feasible the first calculations of scattering of neutrons off deuterons which correctly took into account the compositeness, and the resulting possibility of dissociation, of the latter. Only a few years after the first numerical calculations[6] had been published it was pointed out by Sloan[7] that the (real part of the) elastic neutron-deuteron (*nd*) scattering phase shift in the quartet channel with total orbital angular momentum $l=0$ apparently assumed the value π at threshold if the conventional normalization to zero at infinity is adhered to, i.e., ${}^4\delta_0(0)=\pi$. Such a finding is—at first sight—surprising because in this channel, as a consequence of the Pauli principle, the effective interaction between the neutron and the deuteron derived from the three-body theory acts in a repulsive way, precluding the existence of a bound state. Thus, the quartet S-wave phase shift appears to be in contradiction to what one might expect from a (naive) application of Levinson's theorem.[8] The latter states that for two-body scattering with a real, regular,[9] energy-independent, local potential $V(r)$, the phase shift $\delta_l(0)=n_l\pi$, where n_l equals the number of bound states in the particular angular momentum state l considered (in the absence of a zero-energy bound state for $l=0$).

Of course, as Sloan pointed out, a straightforward application of the two-body Levinson theorem to the scattering of composite particles may not be allowed even if the latter is formulated as an effective two-particle theory.[5] For the exact effective interaction between the neutron and the deuteron occurring in such a formulation is nonlocal, energy dependent, and becomes complex as soon as the deuteron breakup channel opens up.

This situation is reminiscent of the one frequently encountered in the resonanting group model (RGM) approach to composite-particle scattering. There, threshold phase shift values of $n_l\pi$ are often encountered where n_l is larger than the number of true bound states existing in the system considered. In such cases, in an equivalent two-body description, deep local target-projectile potentials are found which give rise to identical phase shifts as the nonlocal effective RGM interactions. Thus, they are phase but not binding energy equivalent since, in addition to the physical bound states, they also support unphysical ones, the so-called Pauli-forbidden states (PFS). The latter are related to the redundant states of the RGM Hamiltonian. Since, according to a generalization of Levinson's theorem to the scattering of composite particles in the framework of the RGM by Swan,[10,11] both true and Pauli-forbidden bound states contribute to the value of n_l, agreement with the observed multiplicity of π of the threshold phase shift is restored.

Therefore, guided by these facts, Sloan suggested that the result ${}^4\delta_0(0)=\pi$ might be due to the existence of a PFS in the quartet S-wave channel. His argument was, however, not convincing since the notion of a PFS is given a clear meaning only in the resonating group and related methods (see, e.g., Ref. 12), but is alien to the composite-particle scattering theory based on integral equations.

On the other hand phenomenological, shallow, local, two-body target-projectile potentials exist describing the same scattering data as the deep potentials discussed above but supporting the physical bound states only. However, as has been pointed out in Ref. 13 for the case of $\alpha+{}^{16}O$ scattering, the requirement that they satisfy the (generalized) Levinson theorem forces them to be irregular at the origin with centrifugal barrier-type singularity there. In fact, a connection is suggested in Ref. 11, between the strength of the short-distance singularity of these potentials which are phase equivalent to the multiparticle RGM interaction, and the number of redundant states of the corresponding RGM Hamiltonian.

Recently, in an application of the ideas of supersymmetric (SS) quantum mechanics Baye[14] has explicitly established the relation between these two classes of two-body potentials. When removing the PFS from the deep potential via SS transformations he ended up with the corresponding shallow potential, while retaining in this process strict phase equivalence, and hence also the multiplicity of π at the threshold. This fact is in agreement with another generalization of Levinson's theorem.[11,15]

On the three-body level the above-mentioned threshold value of the *nd* quartet *S*-wave phase shift is not yet understood. In the present paper we, therefore, try to shed some light on this situation by reverting to an equivalent two-body description, in analogy to the procedure discussed above for the RGM case. Thereby, the basic idea is similar to the one pursued in Ref. 14. We demonstrate that the *nd* quartet *S*-wave phase shifts, calculated from the rigorous three-body theory, can be reproduced by a predominantly attractive, local two-body potential supporting one unphysical bound state, as well as by a purely repulsive local potential. Their interrelation is established by application of SS quantum mechanics. The strict phase equivalence can, however, be maintained only at the expense of the occurrence of a $1/\rho^2$ singularity in the purely repulsive supersymmetric partner where ρ is the distance between the neutron and the (center of mass of the) deuteron.

For the present investigation we choose a very simple and therefore frequently used but, nevertheless, qualitatively and also quantitatively quite successful model based on separable nucleon-nucleon forces. They are chosen to be spin dependent but, for simplicity, to act in nucleon-nucleon *S* waves only. This has the consequence that the total spin and the total orbital angular momentum are separately conserved. Since we are at present interested only in the scattering in the quartet state in which the spins of the three nucleons are parallel (i.e., the total spin equals $\frac{3}{2}$) it suffices to specify the $^3S_1$ nucleon-nucleon potential. The latter is taken to be of Yamaguchi form, with the strength and range parameter chosen so as to reproduce the experimental deuteron binding energy and the triplet scattering length.

For such a nuclear-force model the three-body equations can be solved without any approximation. The *nd* quartet *S*-wave phase shift calculated in this way is real below the deuteron breakup threshold (at 3.339 MeV incoming neutron laboratory energy), and complex above as a consequence of the opening of the dissociation channel. Its real and imaginary parts are depicted in Fig. 1. Inspection reveals that, in fact, $\mathrm{Re}[^4\delta_0(0)]=\pi$. Since we are attempting here to elucidate the consequences for the equivalent local potential of this value of $\mathrm{Re}[^4\delta_0(0)]$, despite the absence of a bound state, it is sufficient to consider in the following only the real part of the phase shift, or equivalently to "unitarize" the corresponding *S*-matrix element. Because of the weakness of the absorption, reflected in the smallness of the imaginary part of the phase shift, this has negligible consequences.

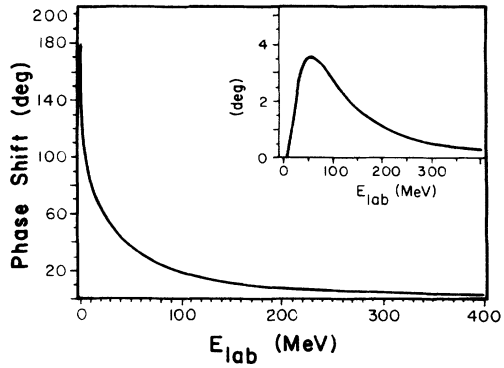

FIG. 1. Real part of the neutron-deuteron quartet *S*-wave phase shift as function of the neutron laboratory energy. Inset shows the corresponding imaginary part.

In order to construct a deep, energy independent but l dependent, local two-body potential which reproduces the real part of the phase shift from threshold up to infinity we employ the Marchenko inverse scattering formalism valid for regular potentials.[9,16-18] The value of π of $\mathrm{Re}[^4\delta_0(0)]$ necessitates the introduction of one (fictitious) bound state in order that the usual form of Levinson's theorem be satisfied. However, since the value $E^{(0)}$ of its binding energy is not fixed by any physical principle we arrive in this way at a whole family of equivalent local potentials depending parametrically on $E^{(0)}$. We denote them by $V(\rho;E^{(0)})$. Here and in the following we suppress the index $l=0$ characterizing the orbital angular momentum whenever no confusion can arise.

The inversion procedure is greatly simplified by employing a rational (Bargmann type) parametrization of the (unitarized) *S* function

$$S(q):=\frac{^4S_0(q)}{|^4S_0(q)|}=\frac{(q+i\kappa^{(0)})}{(q-i\kappa^{(0)})}\prod_{m=1}^{N}\frac{(q+a_m)}{(q-a_m)}, \qquad (1)$$

N odd, with $\kappa^{(0)}>0$ related to the binding energy via $E^{(0)}=-\hbar^2\kappa^{(0)2}/2\mu$. q is the neutron-deuteron relative momentum canonically conjugate to ρ, and μ the corresponding reduced mass. The parameters a_m can be determined by fitting the form (1) to the numerically given S function. The consequence of such a choice is that the kernel of the Marchenko fundamental equation becomes degenerate and, therefore, analytic solutions can be obtained.

If the Marchenko equation is to yield a unique equivalent local potential, in addition to the binding energy the value of the bound-state normalization constant A also has to be specified. We choose[9,17]

$$A=A(E^{(0)})=iF(-i\kappa^{(0)})/\dot{F}(i\kappa^{(0)}), \qquad (2)$$

where $F(q)$ is the Jost function, and $\dot{F}(q)=dF(q)/dq$.

Application of this approach leads, for each chosen value of the binding-energy parameter $E^{(0)}$, to a unique, deep, local two-body potential $V(\rho;E^{(0)})$ which exactly reproduces $\mathrm{Re}[^4\delta_0(q)]$, $0\le q<\infty$. In Fig. 2 we show

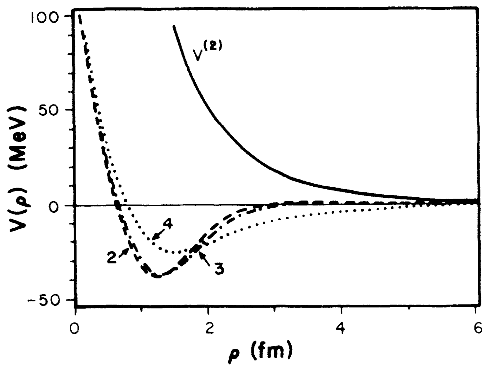

FIG. 2. Equivalent local potentials, all of which exactly reproduce the real part of the neutron-deuteron quartet S-wave phase shift, as a function of the distance ρ between the neutron and the deuteron. The three with an attractive part are selected members of a one-parameter family, each supporting a bound state at an energy, the absolute value of which is indicated. $V^{(2)}$ is the unique supersymmetric partner of this whole family.

several members of this family related to binding-energies $E^{(0)}$ of -2, -3, and -4 MeV. They are all attractive for intermediate neutron-deuteron separation, with the minimal range attained approximately at $E^{(0)} \approx -3$ MeV. We mention in this connection that choosing bound-state normalization constants different from the value (2) has been found to increase the range of the associated potential. A more detailed discussion of the dependence of the equivalent local potentials, and the corresponding wave functions, on the binding-energy parameter and the bound-state normalization constant will be given elsewhere.

To a deep potential supporting one bound state the corresponding "shallow" potential can be constructed by the methods of SS quantum mechanics.[19-22] As shown in Ref. 14 the application of two consecutive SS transformations is required for this purpose. The first one eliminates the bound state of the original potential, here denoted by $V^{(0)}(\rho)$, but also changes the phase shift. Phase equivalence is then restored by a second SS transformation leading to the desired potential $V^{(2)}(\rho)$. The exact phase equivalence of $V^{(0)}(\rho)$ and $V^{(2)}(\rho)$ implies that the phase-shift value at threshold is the same for both potentials, despite the fact that the bound state supported by $V^{(0)}(\rho)$ is absent in $V^{(2)}(\rho)$. The fact, however, does not imply for $V^{(2)}(\rho)$ a failure of Levinson's theorem. The reason becomes clear when looking at the short-distance behavior of $V^{(2)}(\rho)$ which is found to be (for $l=0$)

$$V^{(2)}(\rho) \underset{\rho\to 0}{\sim} V^{(0)}(\rho) + \frac{6}{\rho^2}\frac{\hbar^2}{2\mu} . \tag{3}$$

Thus, the singularity at the origin excludes $V^{(2)}(\rho)$ from the Levinson class of potentials. However, the value of $\mathrm{Re}[{}^4\delta_0(0)]$ produced by it is in agreement with what one should find for potentials with a singularity of the form and strength as displayed in Eq. (3), according to the generalized Levinson theorem.[15]

Identifying now any one of the previously discussed phase-equivalent potentials $V(\rho;E^{(0)})$ with $V^{(0)}(\rho)$ we find the corresponding SS related singular potential $V^{(2)}(\rho;E^{(0)})$. Here we again indicate explicitly the dependence on the value of the binding-energy parameter $E^{(0)}$. It turns out, however, that all members of the family of potentials $V^{(2)}(\rho;E^{(0)})$ coincide, independent of the actual value of $E^{(0)}$,

$$V^{(2)}(\rho;E^{(0)}) =: V^{(2)}(\rho) , \quad \text{for all } E^{(0)} . \tag{4}$$

In fact, such a remarkable coincidence is found even when the bound-state normalization constant is chosen different from the value given by Eq. (2). This unique potential $V^{(2)}(\rho)$ is also shown in Fig. 2. It can be seen to be purely repulsive, in agreement with the expectation raised by the repulsive nature of the effective *nd* interaction mentioned above.

Thus, we have demonstrated that apparently the unphysical bound state, whose introduction was required for the applicability of the inversion procedure to the *nd* quartet S-wave phase shift, plays an analogous role in the framework of the exact three-body theory as a PFS in the RGM. Namely, its existence enforces a deep attraction in the equivalent local two-body potential. Eliminating it by means of SS transformations leads to a—in our example purely repulsive—potential having no unphysical bound states. But, in order to preserve phase equivalence and hence also the phase-shift value at threshold, the latter has to develop a repulsive singularity for vanishing intercluster distances; this being another manifestation of the repulsion induced by the Pauli principle. These features appear to be independent of the original formulation of the theory, either in terms of exact three-body integral equations or of the RGM. Hence, they can arise only by the mapping of a multiparticle scattering theory onto an equivalent two-body formalism. However, there is one important difference. Namely, at present we are not aware of any physical criterion for selecting a unique value of the binding energy of this fictitious bound state; in contrast to the situation in the RGM, any value is acceptable. One might hope that in the future a deeper understanding of its origin on the three-body level may eventually uncover some prescription for fixing this quantity. We finally mention that a first attempt in this direction has been made by Kukulin *et al.*[23]

One of the authors (E.O.A.) gratefully acknowledges the hospitality extended to him while visiting the Physics Department of the University of South Africa.

[1]L. D. Faddeev, Zh. Eksp. Teor. Fiz. **39**, 1459 (1960) [Sov. Phys. JETP **12**, 1014 (1961)].
[2]R. D. Amado, Phys. Rev. **132**, 485 (1963).
[3]A. G. Sitenko and V. Kharchenko, Nucl. Phys. **49**, 15 (1963).
[4]A. N. Mitra and V. S. Bhasin, Phys. Rev. **131**, 1265 (1963); A. N. Mitra, *ibid.* **139**, B1472 (1965).
[5]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. **B2**, 167 (1967).
[6]R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. B **140**, 1291 (1965).
[7]I. H. Sloan, Phys. Lett. **34B**, 243 (1971).
[8]N. Levinson, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **25**, 9 (1949).
[9]R. G. Newton, *Scattering of Waves and Particles,* 2nd ed. (Springer, New York, 1982).
[10]P. Swan, Proc. R. Soc. London Ser. A **228**, 10 (1955).
[11]P. Swan, Ann. Phys. (N.Y.) **48**, 455 (1968).
[12]H. Horiuchi, Prog. Theor. Phys. **69**, 886 (1983), and references therein.
[13]F. Michel and G. Reidemeister, J. Phys. G **11**, 835 (1985).
[14]D. Baye, Phys. Rev. Lett. **58**, 2738 (1987).
[15]P. Swan, Nucl. Phys. **46**, 669 (1963).
[16]Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of the Scattering Theory* (Gordon and Breach, New York, 1963).
[17]K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory,* 2nd ed. (Springer, New York, 1989).
[18]V. de Alfaro and T. Regge, *Potential Scattering* (Wiley, New York, 1965).
[19]E. Witten, Nucl. Phys. **B188**, 513 (1981).
[20]A. A. Adrianov, N. V. Borisov, and M. V. Ioffe, Phys. Lett. **105A**, 19 (1984).
[21]C. V. Sukumar, J. Phys. A **18**, 2917 (1985).
[22]C. V. Sukumar, J. Phys. A **18**, 2937 (1985).
[23]V. I. Kukulin, V. G. Neudatchin, I. T. Obukhovski, and Yu. F. Smirnov, in *Clustering Phenomena in Nuclei,* edited by K. Wildermuth and P. Kramer (Vieweg, Braunschweig, 1983), Vol. 3.