# Relativistic treatment of the long wavelength limit for the photon and the multipole amplitudes of $^2H(\gamma,n)^1H$

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We derive expressions for the low energy behavior of the multipole amplitudes of $^2H(\gamma,n)^1H$. Our results follow from the relativistic treatment of the long wavelength limit for the photon. We enforce unitarity, point out the incompatibility of this with gauge invariance, and suggest a possible solution. We discuss in some detail the forward differential cross section.

## I. INTRODUCTION

The gauge invariance property of the electromagnetic interactions demands that the currents entering a photonuclear process be conserved. Nonrelativistically (NR) this is implemented in the low photon energy domain (long wavelength limit) through the use of Siegert's theorem.[1] Improvements of this theorem have been performed so as to extend its regime of applicability but maintaining however its NR nature.[2]

The limitation of the approach can be appreciated in the low energy forward deuteron photodisintegration. Relativistic corrections to the electromagnetic current have been calculated and shown to be essential in understanding data at energies as low as 30 MeV (Refs. 3 and 4). Needless to say, a complete understanding of the relativistic corrections is not available.[5] Progress in this direction requiring among other things, a consistent treatment of the meson exchange currents.

Alternatively, it is possible to treat the gauge invariance constraint relativistically. In this case the use of electromagnetic current conservation in the low energy limit provides information on the behavior of the Lorentz-invariant amplitude. This results is known as Low's low photon energy theorem.[6] In the particular case of the deuteron photodisintegration the theorem has been rigorously proved by Sakita[7] and applied in an essentially parameter independent way in Refs. 8 and 9. (Hereafter we will refer to this as the soft photon approximation, SPA for short.)

The correct description of the total cross section is not enough to test the reliability of the SPA (Ref. 10). Further support to this approach comes from the relatively good description obtained in the forward direction above 30 MeV, and from the possible evidence of relativistic effects on the angular distribution. What we mean in the last point is the following. The experimental data show that the maximum of the differential cross section shifts towards smaller angles as the photon energy is increased. This is naturally understood in the SPA as a relativistic propagator effect. More precisely,[11] the SPA predicts[8] that the maximum of the differential cross section is located at $\theta=\arccos(v)$ ($v$ being the velocity of the outgoing proton in the center-of-mass frame) and that $r$, defined as:

$$r \equiv \frac{1}{\sigma_T}\left[\frac{d\sigma}{d\Omega}\right]_{\theta=\theta_{\max}} \tag{1}$$

is nearly constant. Unfortunately the available data is not enough to confirm or rule out such behavior.[12]

However we do not think that the value of the SPA is an alternative to provide a detailed description of all the observables associated with the deuteron photodisintegration. In fact we know[8,9,13] that this is only possible in a limited kinematical region. Instead, we believe that—since the SPA results are derived in a model-independent way from the relativistic treatment of the long wavelength limit for the photon—its main utility is as a guide to the dynamical NR calculations.

Among the limitations of this approach we can mention the impossibility of incorporating the structure of the deuteron and the failure of the amplitude to fulfill unitarity. These two deficiencies are reflected, as shown by the results of Refs. 8 and 9, in the range of applicability of the SPA. There are other problems which are not inherent to the SPA but to the way it is applied. Thus, for example, it is difficult to compare the results with those of the NR approach. In this respect it is worth mentioning the existing discrepancy regarding the forward direction. In the NR approach most of the contribution comes from $E1$ transition whereas Low's theorem leads to a cross section which in the NR limit reduces to a $M1$ transition. Moreover, as emphasized by Zieger, Grewer, and Ziegler,[14] the NR treatments predict a minimum around 15 MeV as compared to the monotonic fall obtained relativistically.[8,9,15]

In this paper we perform a multipolar analysis of the SPA to the deuteron photodisintegration. Thus, we derive model-independent expressions for the low energy behavior of the multipole amplitudes which follow from the relativistic treatment of the long wavelength limit for the photon. To our knowledge the only previous attempts to derive this kind of low energy theorem for the multipole amplitudes are those by Sakita and Goebel[16] and by Le Bellac.[17] In the first case the authors restrict to the electric dipole amplitudes. On the other hand, Le Bellac discusses both electric and magnetic dipole amplitudes, however his results are incorrect because he leaves out the $3F_2$ channel which is important for the low energy behavior and gauge invariance.[18] In a recent publication,[19] similar ideas are developed going beyond the SPA, unfortunately some of the approximations (concerning the off-mass shell NPD vertex) involved in the analysis are not consistent with gauge invariance and Low's theorem.[20]

As a further step we used the multipole amplitudes previously derived to implement unitarity. Besides their academic interest, this excercise could be of relevance when considering the differential cross section in the forward direction, since it is known[21] that the minimum occurring there, is related to the final state interaction.

Finally we also consider an extension derived from an approximated dispersion relation which has the proper analytical behavior. Although the agreement with the experimental data above 30 MeV is improved, both in the unitary, or in the dispersion relation approximation, our conclusion is that for the $^1S_0$ channel a more detailed analysis of the singularities is required.

This paper is organized as follows. In Sec. II we make some remarks on the relativistic treatment of gauge invariance. We briefly comment on Low's and Burnett-Kroll theorems. In Sec. III we present our results for the electric and magnetic dipole amplitudes. Those results were derived through the use of Low's low energy theorem and for that reason we will refer to them as the low energy theorem for the multipoles. In Sec. IV we discuss unitarity, the approximated dispersion relation and compare our results with the data. Finally in Sec. V we present our conclusions. We have relegated to two appendixes the kinematics and conventions we use as well as the general relation among the invariant and the multipole amplitudes, which to our knowledge has not been previously given.

## II. GAUGE INVARIANCE IN A RELATIVISTIC CONTEXT

This section deals with Low's and Burnett and Kroll low energy theorems.[6,22] Although this material is well known we have included this short section so as to stress some points we consider particularly important.

We can describe the deuteron photodisintegration by considering an infinite number of Feynman diagrams (in Fig. 1 we show some of them). If we could evaluate all those diagrams and be assured of including the off-mass shell effects and of properly developing the resulting amplitude around $\omega=0$, where $\omega$ is the photon energy, then we would find (schematically)

$$\mathcal{M}=\frac{T_1}{\omega}+T_2\omega^0+T_3\omega+\cdots \ . \tag{2}$$

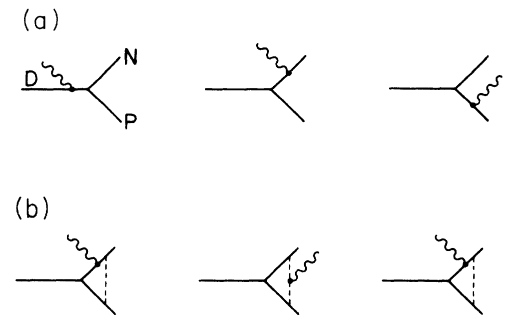


FIG. 1. Some of the Feynman diagrams contributing to the invariant amplitude of $^2\mathrm{H}(\gamma,n)^1\mathrm{H}$. The dotted line represents a $(\pi,\rho,\omega,\ldots,$ etc.) meson

Of course, we cannot accomplish such program and in fact it is not necessary since Low[6] has proved that gauge invariance exactly determines $T_1$ and $T_2$.

For the sake of completeness let us briefly outline the procedure to obtain $T_1$ and $T_2$ in the case under consideration.

We will use the following notations:

(i) $M$ and $m$ for the mass of the deuteron and the nucleons. (We neglect the $n$-$p$ mass difference.)

(ii) $k$, $d$, $p$, and $p'$ for the four-momenta of the photon, deuteron, proton, and neutron, respectively (we have thus $k+d=p+p'$).

(iii) $\varepsilon^\mu(k)$ and $\eta^\nu(d)$ for the polarization four-vectors of the photon and the deuteron.

(iv) $u_p(p)$ and $u_n(p')$ for the spinor fields of the proton and the neutron. Notice that $C\bar{u}_n^T(p')=v_n(p')$, where $C$ is the charge conjugation operator.

By covariance, the Feynman amplitude may be written as

$$\mathcal{M}=\varepsilon^\mu\eta^\nu\bar{u}_p(p)\mathcal{M}_{\mu\nu}C\bar{u}_n^T(p') \ , \tag{3}$$

According to Low's result, the first two terms, of order $(-1)$ and 0, in a power series in $k$ of the Feynman amplitude, can be known exactly; they depend only on global properties of the neutron, proton, and deuteron, and are given completely by the Born diagrams [Fig. 1(a)] and a contact term needed for gauge invariance to that order.

Taking then the expansion of the Born amplitude in a power series of $k$, requiring gauge invariance at each order (i.e., $k^\mu\eta^\nu\mathcal{M}_{\mu\nu}=0$), it is seen that the first two terms, of order $(-1)$ and 0 in $k$, can be exactly known.

We find then

$$\begin{aligned}\mathcal{M}_{\mu\nu}=e\bar{u}_p(p)\Bigg[&\left[A\gamma_\sigma+\frac{B}{2m}(p-p')_\sigma\right]\frac{g^{\sigma\rho}}{2d\cdot k}\left[2d_\rho g_{\mu\nu}+2M\frac{\mu_d}{3}(k_\rho g_{\mu\nu}-k_\nu g_{\mu\rho})\right]\\&+\left[\gamma_\mu+i\kappa_p\frac{\sigma_{\mu\rho}k^\rho}{2m}\right]\frac{\not p-\not k+m}{-2p\cdot k}\left[A\gamma_\nu+\frac{B}{2m}(p-p'-k)_\nu\right]\\&+\left[A\gamma_\nu+\frac{B}{2m}(p-p'+k)_\nu\right]\frac{\not p'-\not k+m}{-2k\cdot p'}i\kappa_n\frac{\sigma_{\mu\rho}k^\rho}{2m}-\frac{B}{2m}g_{\mu\nu}\Bigg]v_n(p') \ ,\end{aligned} \tag{4}$$

where $e$ is the proton charge, $\kappa_p$ and $\kappa_n$ the anomalous magnetic moments of the nucleons, $\mu_d$ the magnetic moment of the deuteron, and $A$ and $B$ the $d$-$np$ form factors with the three particles on shell.

Notice that besides the kinematical dependence on the wave functions (i.e., polarization vectors and spinors) and the four-momenta of the external particles, this amplitude depends only on the two form factors $A$ and $B$. In a nonrelativistic approximation it is possible to relate these to the $^3S_1$ and $^3D_1$ state amplitudes of the deuteron wave function. Since this has been worked in detail in Ref. 7 we only quote the result:

$$A_s=\frac{1}{2}\left[-2A-\frac{A}{6}\frac{|\mathbf{p}|^2}{m^2}-\frac{2B}{3}\frac{|\mathbf{p}|^2}{m^2}\right],$$
$$A_d=\frac{|\mathbf{p}|^2}{3m^2}(A+2B)\ . \tag{5}$$

Moreover, it is possible to relate $A$ and $B$ to more conventional parameters, namely, the scattering parameters of the neutron-proton system in a nonrelativistic approach. This is achieved by comparing the real part of the amplitude for elastic $n$-$p$ scattering at the deuteron pole as calculated relativistically and nonrelativistically using the effective range approximation. The results are

$$A=-\Gamma(1+\alpha),\quad B=3(m^2/\gamma^2)\alpha\Gamma\ , \tag{6}$$

where

$$\Gamma=-\left[\frac{8\pi\gamma/m}{(1+2\alpha^2)(1-p\gamma)}\right],$$

with $\alpha$ the asymptotic ratio of the $D$ and $S$ waves of the deuteron, $\rho$ is the triplet neutron-proton effective range and $\gamma$ is related to the deuteron binding energy.

$$\gamma^2=m^2-M^2/4\ .$$

For further details on this point we refer to the interested reader to the appendixes of Ref. 7.

Coming back to Low's theorem, from Eq. (2) it follows that for low photon energies we can expect that the SPA gives reliable results. But, on the other hand, near threshold ($\omega<4$ MeV) the rescattering of the final $N$-$P$ system introduces a strong momentum dependence in the amplitude, spoiling thus the applicability of the theorem.[24]

Using the amplitude obtained from Low's theorem, Burnett and Kroll derived a theorem for the square of the invariant amplitude. This is of interest to us because for $^2H(\gamma,N)^1H$ it requires the vanishing of the interference between $T_1$ and $T_2$. This fact may serve as a further test of gauge invariance once the square of the invariant amplitude has been taken (this requirement is not fulfilled by the multipole amplitudes derived by Le Bellac[17]).

## III. MULTIPOLE AMPLITUDES

The aim of this section is to establish a relation between the covariant amplitude Eqs. (3) and (4) and the more conventional multipole amplitudes. The steps involved in such a program are[33] the following.

(a) evaluate the helicity amplitudes.

(b) Write the helicity amplitudes in terms of the electric and magnetic multipole states.

(c) Using the results of (a) and (b) obtain the multipole amplitudes in terms of the helicity amplitudes projected on the rotation matrices (see below).

To carry out this program it is convenient to use a set of linearly independent invariant amplitudes. The ones we use are listed in Table I and will be denoted by $I_i$. Thus, any invariant amplitude describing the deuteron photodisintegration can be written in the form

$$\mathcal{M}=\sum_{i=1}^{12}H_iI_i\ . \tag{7}$$

Different models will lead to different $H$'s. In the SPA, when $\mathcal{M}$ is given by Eqs. (3) and (4), the $H$'s are those listed in Table II.

We introduce the helicity amplitudes $F_{\lambda_1\lambda_2\mu_1\mu_2}$, where $\lambda_1$, $\lambda_2$, $\mu_1$, and $\mu_2$ are the helicities of the photon, deuteron, proton, and neutron, respectively. The helicity amplitude is defined as the matrix element of the invariant amplitude with the helicities taking precisely the values

TABLE I. List of the twelve invariant amplitudes used in the text. Here $q=(p-p')/2$ and $Q=(p+p')/2$.

| $i$ | $I_i$ |
|---|---|
| 1 | $[(\epsilon\cdot q)(\eta\cdot k)-(\epsilon\cdot\eta)(q\cdot k)]/(2m^2)$ |
| 2 | $[(\epsilon\cdot\eta)(Q\cdot k)-(\epsilon\cdot Q)(\eta\cdot k)]/(2m^2)$ |
| 3 | $[(\epsilon\cdot q)(Q\cdot k)-(\epsilon\cdot Q)(q\cdot k)]\eta\cdot q/(m^4)$ |
| 4 | $-[(\epsilon\cdot Q)(q\cdot k)-(\epsilon\cdot q)(Q\cdot k)]\not{\eta}/(m^3)$ |
| 5 | $[(\epsilon\cdot\eta)\not{k}-(\eta\cdot k)\not{\epsilon}]/(2m)$ |
| 6 | $[(\epsilon\cdot Qk-Q\cdot k\epsilon)\eta\cdot k]/2m^3-[(\epsilon\cdot q)\not{k}-(q\cdot k)\not{\epsilon}]q\cdot\eta/(m^3)$ |
| 7 | $[(\epsilon\cdot qk-q\cdot k\epsilon)\eta\cdot k]/(2m^3)+[(\epsilon\cdot Q)\not{k}-(Q\cdot k)\not{\epsilon}]q\cdot\eta/(m^3)$ |
| 8 | $[\not{\epsilon},\not{k}]\eta\cdot k/(4m^2)$ |
| 9 | $[\not{\epsilon},\not{k}]\eta\cdot q/(2m^2)$ |
| 10 | $(q\cdot k[\not{\eta},\not{\epsilon}]-q\cdot\epsilon[\not{\eta},\not{k}]+2Q\cdot\epsilon\eta\cdot k-2\epsilon\cdot\eta Q\cdot k)/(2m^2)$ |
| 11 | $(Q\cdot k[\not{\eta},\not{\epsilon}]-Q\cdot\epsilon[\not{\eta},\not{k}]+2q\cdot\epsilon\eta\cdot k-2\epsilon\cdot\eta q\cdot k)/(2m^2)$ |
| 12 | $\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu k^\nu\eta^\rho\gamma^\sigma\gamma_5/(2m^2)$ |

TABLE II. Values of the twelve $H$'s as predicted by the SPA.

$$H_1 = eBm\left[\frac{1}{p\cdot k} - \frac{M}{m}\frac{\mu_d}{d\cdot k}\right]$$
$$H_2 = -emB/p\cdot k$$
$$H_3 = \frac{B}{A}H_4 = -2em^3B/(p\cdot k d\cdot k)$$
$$H_5 = -eAm\left[\frac{\mu_p}{p\cdot k} + \frac{\kappa_n}{k\cdot p'} - \frac{M}{m}\frac{\mu_d}{d\cdot k}\right]$$
$$H_6 = -\frac{B}{A}H_{11} = \frac{emB}{2}\left[\frac{\kappa_p}{p\cdot k} + \frac{\kappa_n}{k\cdot p'}\right]$$
$$H_7 = \frac{B}{A}H_{10} = -\frac{emB}{2}\left[\frac{\kappa_p}{p\cdot k} - \frac{\kappa_n}{k\cdot p'}\right]$$
$$H_8 = 0$$
$$H_9 = \frac{B}{2A}H_{12} = \frac{emB}{2}\left[\frac{\mu_p}{p\cdot k} - \frac{\kappa_n}{k\cdot p'}\right]$$

$\lambda_1$, $\lambda_2$, $\mu_1$, and $\mu_2$.

$$F_{\lambda_1\lambda_2\mu_1\mu_2} = \langle \lambda_1\lambda_2|\mathcal{M}|\mu_1,\mu_2\rangle \ . \qquad (8)$$

Since the proton has only two helicity states, whereas the deuteron and each of the nucleons have 3 and 2, respectively, then we will have a total of 24 helicity amplitudes. However due to parity conservation only 12 are independent. In order to shorten the notation we will work with the following 12 quantities:

$$F_1^{\pm} = \langle 1,1|\mathcal{M}|\pm\tfrac{1}{2},\pm\tfrac{1}{2}\rangle,\quad F_4^{\pm} = \langle 1,1|\mathcal{M}|\pm\tfrac{1}{2},\mp\tfrac{1}{2}\rangle \ ,$$
$$F_2^{\pm} = \langle 1,0|\mathcal{M}|\pm\tfrac{1}{2},\pm\tfrac{1}{2}\rangle,\quad F_5^{\pm} = \langle 1,0|\mathcal{M}|\pm\tfrac{1}{2},\mp\tfrac{1}{2}\rangle \ ,$$
$$F_3^{\pm} = \langle 1,-1|\mathcal{M}|\pm\tfrac{1}{2},\pm\tfrac{1}{2}\rangle,\quad F_6^{\pm} = \langle 1,-1|\mathcal{M}|\pm\tfrac{1}{2},\mp\tfrac{1}{2}\rangle \ ,$$

Notice that from each $F$ we can read the value of $\lambda_1$ $\lambda_2$, $\mu_1$, and $\mu_2$ and therefore $\lambda=\lambda_1-\lambda_2$ and $\mu=\mu_1-\mu_2$ are determined.

Let us now introduce:

$$a_i^{\pm}(J) = 2\sqrt{2}(2J+1)\int F_1^{\pm}(\theta,\varphi)D^J_{\lambda\mu}(\varphi,\theta,\varphi)d\Omega \ , \qquad (9)$$

where $J$ is the total angular momentum and $D^J_{\lambda\mu}$ are the rotation matrices.[33]

The multipole amplitudes are given in terms of the $a$'s, however the equations are so long that we have relegated them to Appendix B.[26]

Summarizing, in order to obtain a given multipole amplitude all we need is to know the $a$'s. These in turn depend only on the $H$'s. [This is seen by substituting Eq. (7) in Eq. (8).] Therefore, given a covariant model for the deuteron photodisintegration, or equivalently a set of Feynman diagrams, we must determine the $H$'s and from Eq. (9) derive the $a$'s.

The low energy theorem for the multipole amplitudes is obtained, of course, by using the $H$'s given by the SPA. In order to compare our results with the conventional SPA (Ref. 8) we will split the calculation as follows:

### A. Terms of order $1/\omega$

The terms of order $1/\omega$ are those proportional $H_3$ and $H_4$. Evaluating the helicity amplitudes associated to these terms we find (for notation see Appendix A or Ref. 8):

$$\begin{aligned}
&F_1^+ = F_1^- = -F_3^+ = -F_3^- = f_1\Lambda\sin\theta \ ,\\
&F_2^+ = F_2^- = \sqrt{2}f_1\Lambda\cos\theta \ ,\\
&F_4^+ = F_6^- = f_2\Lambda(1-\cos\theta) \ ,\\
&F_4^- = F_6^+ = f_2\Lambda(1+\cos\theta) \ ,\\
&F_5^+ = -F_5^- = \sqrt{2}f_2\Lambda\sin\theta \ ,
\end{aligned} \qquad (10)$$

with

$$\Lambda = \frac{v}{4\pi\sqrt{2}}\frac{\sin\theta}{1-v\cos\theta} \ ,$$
$$f_1 = \frac{4\pi}{\omega}\frac{Am^2+B\gamma^2}{\sqrt{2}m^2}, \quad f_2 = \frac{8\pi}{\omega}\frac{A(m^2-\gamma^2)}{\sqrt{2}mM} \ . \qquad (11)$$

In order to verify the correctness of these helicity amplitudes let us evaluate the differential cross section. Using Eq. (1) of Appendix A it is easily found that:

$$\left[\frac{d\sigma}{d\Omega}\right]_E = \frac{\alpha A^2m^2}{16\pi}\frac{v^2\sin^2\theta}{E\omega^3(1-v\cos\theta)^2} \qquad (12)$$

in agreement with the conventional SPA result.[8]

These helicity amplitudes lead to the following nonvanishing multipoles [see Eq. (B1)]:

$$\begin{aligned}
E_{J+1}[^3(J+1)_J] &= \frac{1}{2}\left[\frac{(2J+3)}{(2J+1)(J+2)}\right]^{1/2}\frac{1}{(2J+1)} \times[(1+J)f_1+Jf_2]I(J+1) \ ,\\
E_{J-1}[^3(J-1)_J] &= \frac{1}{2}\left[\frac{(2J-1)}{J(J-1)}\right]^{1/2} \times[Jf_1+(1+J)f_2]I(J-1) \ ,\\
E_{J-1}[^3(J+1)_J] &= \frac{1}{2}\left[\frac{(2J-1)(J+1)}{(J-1)}\right]^{1/2} \times\frac{f_1-f_2}{(2J+1)}I(J-1) \ ,\\
E_J(^3J_J) &= -\frac{1}{2}\left[\frac{2J+1}{J(J+1)}\right]^{1/2}f_2I(J) \ ,
\end{aligned} \qquad (13)$$

where

$$I(J) = \frac{4J(J+1)}{(2J+1)}\int_{-1}^{1}\frac{P_{J-1}(x)-P_{J+1}(x)}{(1/v)-x}dx \ , \qquad (14)$$

where $P_j(x)$ is the Legendre polynomial of order $j$.

Notice in particular that for arbitrary $J$ there are no magnetic multipoles. Thus the SPA is telling us that those are necessarily of order $\omega^0$ or higher. Even more important is that, to this order, there is no contribution in the forward direction. Therefore in the absence of final state interactions, the electric multipoles (since the magnetic ones are zero identically) have to combine to produce a vanishing $a_0, c_0$ (see Eqs. (A4) and (A5)]. In the dipole approximation this implies

$$-\sqrt{3}E1(^3P_2)+\sqrt{3}E1(^3P_0)+\frac{3}{\sqrt{2}}E1(^3F_2)=0\ ,$$
$$-\sqrt{3}E1(^3P_1)+\sqrt{3}E1(^3P_2)+\sqrt{2}E1(^3F_2)=0\ . \tag{15}$$

This constraint, for multipoles of arbitrary $J$, is naturally incorporated in Eq. (13).

### B. Terms of order $\omega^0$

These are obtained by collecting the order $\omega^0$ contribution from the twelve $H$'s. Although we have analytical expressions for the helicity and multipole amplitudes we will not present them since their length makes them useless. We only remark that using the helicity amplitudes so derived we have verified the Burnett-Kroll theorem (mentioned in Sec. I). Below we list the dipole amplitudes as a function of the c.m. velocity $v$ of the outgoing proton. This together with the dipole and quadrupole amplitudes, obtainable from Eq. (B1), will be dominant at low energies. In the following we will use, in agreement with the conventional SPA (Ref. 8), a vanishing $\alpha$ (the asymptotic ratio of the $D$ and $S$ waves of the deuteron).

$$\begin{aligned}
&M1(^1S_0)=11.679+8.905v^2\ ,\\
&E1(^3P_0)=-7.372v-1.021v^2\ ,\\
&E1(^1P_1)=1.58v-0.765v^2\ ,\\
&M1(^2S_1)=1.251v-0.612v^2\ ,\\
&M1(^3D_1)=18.139v-12.04v^2\ ,\\
&E1(^3P_1)=4.85v-1.64v^2\ ,\\
&M(^1D_2)=0.826v^2\ ,\\
&E1(^3P_2)=2.135v-0.70v^2\ ,\\
&E1(^3F_2)=-1.65v+0.75v^2\ ,\\
&M1(^3D_2)=0.49v^2\ .
\end{aligned} \tag{16}$$

Notice that to this order ($\omega^0$), we do get nonvanishing magnetic multipoles. Thus, in contrast with Siegert's theorem, the SPA, which is derived from the relativistic approach to the long wavelength limit for the photon, is able to give information on the behavior of the magnetic multipoles. In fact we would have expected this since, after all, magnetic moments are linked with the spin presence and quantum effects.

These multipoles give a nonvanishing contribution to the forward direction. We will not discuss this since, as it should be, the results are essentially the same as those of the SPA (Refs. 8 and 9).

## IV. UNITARITY

With the multipole amplitudes at hand it is now straightforward to impose unitarity. As it is well known,[25] in the case of a single wave, unitarity requires that the phase of the multipole equals the phase shift describing the elastic scattering of the $N$-$P$ system (in analogy with Watson's theorem in the $\pi^0$ photoproduction).

Thus, for example, for the $^1S_0$ channel we will write:

$$M1(^1S_0)=M1(^1S_0)_L e^{i\delta(^1S_0)} \tag{17}$$

Here $M1(^1S_0)_L$ stands for the amplitude obtained from the low energy theorem. An important point to remark is that this simple unitarization leads to an amplitude which disagrees in the zero energy limit with the SPA. The point is that, as we already mentioned, the low energy theorem determines all the terms of order $1/\omega$ and $\omega^0$ in the amplitude. Therefore in the zero energy limit the rescattering factor has to approach unity. Unfortunately Eq. (17) does not show such behavior. One way to render the unitarization procedure consistent with gauge invariance is to enforce at the same time the analytic properties of the amplitude. This can be done through the use of a dispersion relation,[16,25] which in the zero effective range approximation changes Eq. (17) to:

$$M1(^1S_0)=M1(^1S_0)_L e^{i\delta(^1S_0)}\left[\cos\delta+\frac{\gamma}{|\mathbf{p}|}\sin\delta\right]\ , \tag{18}$$

Where $|\mathbf{p}|$ is the magnitude of the momentum of the outgoing proton and $\gamma^2=m^2-M^2/4$.

It is easily seen that this multipole has the appropriated zero energy limit. For practical purposes however, at least in the case under consideration, both multipoles Eqs. (17) and (18) lead to the same numerical results.

In the case of coupled waves, using the Stapp convention, the unitarity relation is[25]

$$\begin{aligned}
\mathrm{Im}M_-=N\{&[\sin(\delta_++\delta_-)\\
&-\alpha\sin(\delta_+-\delta_-)]\mathrm{Re}M_- -\beta\,\mathrm{Re}M_+\}\ ,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\mathrm{Im}M_+=N\{&-\beta\,\mathrm{Re}M_+ +[\sin(\delta_++\delta_-)\\
&+\alpha\sin(\delta_+-\delta_-)]\mathrm{Re}M_-\}\ ,
\end{aligned}$$

where

$$N=\frac{\alpha\cos(\delta_+-\delta_-)-\cos(\delta_++\delta_-)}{\sin^2(\delta_++\delta_-)-\alpha^2\sin^2(\delta_+-\delta_-)-\beta^2}$$

and

$$\alpha=\cos 2\epsilon,\quad \beta=\mathrm{sins}2\epsilon\ ,$$

where $\epsilon$ is the mixing parameter and $\delta_-$ and $\delta_+$ are the phase shifts corresponding to the $j-1$ and $j+1$ waves.

The next step is to use the unitarized multipoles to evaluate the forward differential cross section. We will take the real part of the amplitudes as the one predicted by Low's theorem. Then using Eq. (19) we compute the corresponding imaginary parts. Using the phase shifts of

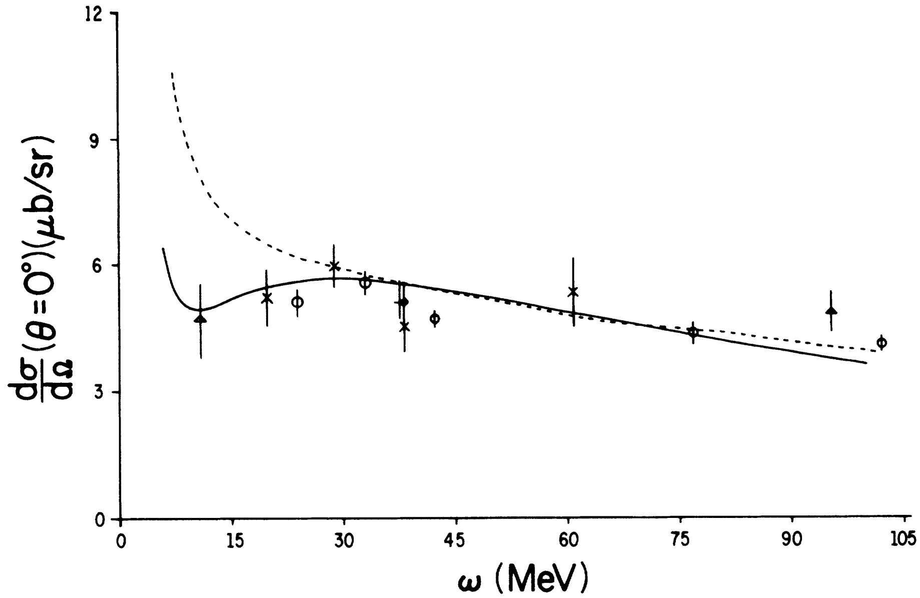


FIG. 2. Forward differential cross section, in microbarns, as a function of the laboratory photon energy (MeVs). Solid line (Ref. 4), dashed line this work. The data are ○ (Ref. 28), ▲(Ref. 29), − (Ref. 30), *x* (Ref. 12), ◊ (Ref. 31), △ (Ref. 14).

Arndt *et al.*,[27] we obtain the results shown in Fig. 2. (See Ref. 32). Although the agreement with the experimental data above 30 MeV is improved we see that the unitarization of the multipoles is not able to provide the necessary suppression of the $M1(^1S_0)$ required to obtain the minimum occurring at 15 MeV. This suggests that the very strong interaction that occurs in this channel spoils the validity of the SPA, implying thus that a more detailed analysis of the singularities contributing to the dispersion relation of this channel must be performed.

## V. CONCLUSIONS

Using a relativistic approach to the long wavelength limit for the photon we have derived expressions for the low energy behavior of the multipole amplitudes. We have shown that, in contrast to Siegert's theorem, the SPA is able to give information on the behavior of the magnetic multipoles. We have also pointed out that the simple unitarization of those lead to amplitudes which violate gauge invariance. This was remedied enforcing at the same time the analytic properties of the multipoles.

Comparison with the experimental data shows that the agreement is improved above 30 MeV. For energies below 30 MeV the dominant $M1(^1S_0)$ predicts too large a forward differential cross section. Thus a more detailed analysis of the singularities of this channel must be performed, since the quasibound state occurring there limits the applicability of the low energy theorem. On the other hand, the weaker coupling of the *N-P* system in the other partial waves indicates that the corrections will be correspondingly small. Discrepancies with the NR calculations regarding the forward differential cross section remain to be understood.


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## APPENDIX A

In terms of the invariant and helicity amplitudes the differential cross section is

$$\frac{d\sigma}{d\Omega}=\phi_s \sum_{\gamma,d,p,n} |\mathcal{M}|^2=2\phi_s {\sum}' |F_{\lambda_1\lambda_2\mu_1\mu_2}|^2 \tag{A1}$$

with

$$\phi_s=\frac{p}{6\omega}\left[\frac{8}{8\pi E}\right]^2 . \tag{A2}$$

Here $\omega$ is the photon energy, whereas $p$ and $E$ are the momentum and energy of the proton in the c.m. frame, $\theta$ will denote the c.m. photon-proton or deuteron-neutron scattering angle, $v$ is the c.m. proton velocity.

The sum on the left-hand side of Eq. (A1) is over the polarization of all the particles involved in the process, whereas the one on the right-hand side is over the deuteron, proton, and neutron helicities.

The coefficients of the conventional parametrization of the differential cross section are obtained in terms of the multipole amplitudes from the following equation:

$$\frac{d\sigma}{d\Omega}=a+b\sin^2\theta+c\cos\theta+d\sin^2\theta\cos\theta+e\sin^4\theta \tag{A3}$$

$$=\frac{\alpha\phi_s}{48\pi}(a_0+b_0\sin^2\theta+c_0\cos\theta+d_0\sin^2\theta\cos\theta+e_0\sin^4\theta)\ , \tag{A4}$$

where in the dipole approximation

$$\begin{aligned}
a_0&=\tfrac{4}{3}|l_1^+|^2+3|l_2|^2+18|E1(^1P_1)|^2\\
&\quad+|l_3^-|^2+|l_4^-|^2+|l_6^-|^2+|l_6^+|^2\ ,\\
b_0&=\tfrac{4}{3}|l_1^-|^2+|l_8|^2+|l_3^-|^2+|l_4^-|^2+|l_6^-|^2\\
&\quad-24|E1(^3P_1)|^2-\tfrac{16}{3}|E1(^3P_0)|^2\\
&\quad-6|E1(^1P_1)|^2-18|M1(^3D_1)|^2\\
&\quad-3|M1(^3S_1)|^2-4|M1(^1S_0)|^2\ ,\\
c_0&=\mathrm{Re}[-4\sqrt{2/3}\,l_1^{+*}l_4^- -2l_2^*l_3^- +2\sqrt{2}\,l_2^*l_5^-\\
&\quad-6\sqrt{2}E1^*(^1P_1)l_6^*]\ ,\\
d_0&=e_0=f_0=0
\end{aligned} \tag{A5}$$

with

$$\begin{aligned}
l_1^\pm&=-\sqrt{3}E1(^3P_2)\pm\sqrt{3}E1(^3P_0)+\frac{3}{\sqrt{2}}E1(^3F_2)\ ,\\
l_2&=-\sqrt{3}E1(^3P_1)+\sqrt{3}(^3P_2)+\sqrt{2}E1(^3F_2)\ ,\\
l_3^\pm&=3M1(^3D_1)\pm\tfrac{5}{2}M1(^3D_2)\ ,\\
l_4^\pm&=\sqrt{6}M1(^3D_1)\pm\sqrt{3}M1(^3S_1)\ ,\\
l_5^\pm&=\frac{5}{\sqrt{2}}M1(^3D_2)\pm3M1(^3S_1)\ ,\\
l_6^\pm&=2M1(^1S_0)\pm\sqrt{10}M1(^1D_2)\ ,\\
l_8&=3\sqrt{3}E1(^3P_1)+\sqrt{3}E1(^3P_2)+\sqrt{2}E1(^3F_2)\ .
\end{aligned}$$

## APPENDIX B

The relation among the multipole and helicity amplitudes are the following [See Eq. (9) for the definition of the $a$'s]:

$$M_{(J+1)}(^1J_J)=\frac{1}{4}\frac{1}{(2J+1)}\left[\frac{2J+3}{2J+1}\right]^{1/2}\left[-\sqrt{J+2}(a_1^+-a_1^-)-\left[\frac{2J(J+2)}{J+1}\right]^{1/2}(a_2^+-a_2^-)-\left[\frac{J(J-1)}{J+1}\right]^{1/2}(a_3^+-a_3^-)\right]\ ,$$

$$\begin{aligned}
E_{J+1}[^3(J-1)_J]=\frac{1}{4}\frac{\sqrt{2J+3}}{(2J+1)^2}\Bigg[&\sqrt{J(J+2)}(a_1^++a_1^-)-J\left[\frac{2(J+2)}{J+1}\right]^{1/2}(a_2^++a_2^-)\\
&+J\left[\frac{J-1}{J+1}\right]^{1/2}(a_3^++a_3^-)+\sqrt{(J+1)(J+2)}(a_4^++a_4^-)\\
&-\sqrt{(2J)(J+2)}(a_5^++a_5^-)+\sqrt{J(J-1)}(a_6^++a_6^-)\Bigg]\ ,
\end{aligned}$$

$$\begin{aligned}
E_{J+1}[^3(J+1)_J]=\frac{1}{4}\frac{\sqrt{(2J+3)}}{(2J+1)^2}\Bigg[&-\sqrt{(J+1)(J+2)}(a_1^++a_1^-)+\sqrt{2J(J+2)}(a_2^++a_2^-)\\
&-\sqrt{J(J-1)}(a_3^++a_3^-)+\sqrt{J(J+2)}(a_4^++a_4^-)\\
&-J\left[\frac{2(J+2)}{J+1}\right]^{1/2}(a_5^++a_5^-)+J\left[\frac{J-1}{J+1}\right]^{1/2}(a_6^++a_6^-)\Bigg]\ ,
\end{aligned}$$

$$E_J(^1J_J)=-\frac{1}{4\sqrt{2J+1}}\left[(a_1^+-a_1^-)+\left[\frac{2}{J(J+1)}\right]^{1/2}(a_2^+-a_2^-)-\left[\frac{(J-1)(J+2)}{J(J+1)}\right]^{1/2}(a_3^+-a_3^-)\right]\ ,$$

$$\begin{aligned}
M_J[^3(J-1)_J]=\frac{1}{4}\frac{1}{(2J+1)}\Bigg[&-\sqrt{J}(a_1^++a_1^-)-\left[\frac{2}{J+1}\right]^{1/2}(a_2^++a_2^-)\\
&+\left[\frac{(J-1)(J+2)}{J+1}\right]^{1/2}(a_3^++a_3^-)-\sqrt{J+1}(a_4^++a_4^-)\\
&-\sqrt{2/J}(a_5^++a_5^-)+\left[\frac{(J-1)(J+2)}{J}\right]^{1/2}(a_6^++a_6^-)\Bigg]\ ,
\end{aligned}$$

$$M_J[(^3J+1)_J]=\frac{1}{4}\frac{1}{(2J+1)}\left[\sqrt{J+1}(a_1^+ + a_1^-)+\sqrt{2/J}\,(a_2^+ + a_2^-)\right.$$
$$-\left[\frac{(J-1)(J+2)}{J}\right]^{1/2}(a_3^+ + a_3^-)-\sqrt{J}\,(a_4^+ + a_4^-)$$
$$\left.-\left[\frac{2}{J+1}\right]^{1/2}(a_5^+ + a_5^-)+\left[\frac{(J-1)(J+2)}{J+1}\right]^{1/2}(a_6^+ + a_6^-)\right], \tag{B1}$$

$$M_{J-1}(^1J_J)=\frac{1}{4}\frac{1}{2J+1}\left[\frac{2J-1}{2J+1}\right]^{1/2}\left[-\sqrt{J-1}(a_1^+ - a_1^-)-\left[\frac{2(J^2-1)}{J}\right]^{1/2}(a_2^+ - a_2^-)\right.$$
$$\left.-\left[\frac{(J+1)(J+2)}{J}\right]^{1/2}(a_3^+ - a_3^-)\right],$$

$$E_{J-1}[^3(J-1)_J]=\frac{1}{4}\frac{\sqrt{2J-1}}{(2J+1)^2}\left[\sqrt{J(J-1)}(a_1^+ + a_1^-)+\sqrt{2(J^2-1)}(a_2^+ + a_2^-)\right.$$
$$+\sqrt{(J+1)(J+2)}(a_3^+ + a_3^-)+\sqrt{J^2-1}(a_4^+ + a_4^-)+\sqrt{J+1}(a_5^+ + a_5^-)$$
$$\left.+(J+1)\left[\frac{J+2}{J}\right]^{1/2}(a_6^+ + a_6^-)\right],$$

$$E_{J-1}[^3(J+1)_J]=\frac{1}{4}\frac{\sqrt{2J-1}}{(2J+1)^2}\left[-\sqrt{J^2-1}(a_1^+ + a_1^-)-(J+1)\left[\frac{2(J-1)}{J}\right]^{1/2}(a_2^+ + a_2^-)\right.$$
$$\left.-(J+1)\left[\frac{J+2}{J}\right]^{1/2}(a_3^+ + a_3^-)+\sqrt{J(J-1)}(a_4^+ + a_4^-)+\sqrt{2(J^2-1)}(a_5^+ + a_5^-)\right],$$

$$M_{J+1}(^3J_1)=\frac{1}{4}\left[\frac{2J+3}{2J+1}\right]^{1/2}\frac{1}{2J+1}\left[\sqrt{J+2}(a_4^+ - a_4^-)-\left[\frac{2J(J+2)}{J+1}\right]^{1/2}(a_5^+ - a_5^-)+\left[\frac{J(J-1)}{J+1}\right]^{1/2}(a_6^+ - a_6^-)\right],$$

$$E_J(^3J_J)=\frac{1}{4}\frac{1}{\sqrt{2J+1}}\left[-(a_4^+ - a_4^-)-\left[\frac{2}{J(J+1)}\right]^{1/2}(a_5^+ - a_5^-)+\left[\frac{(J-1)(J+2)}{J(J+1)}\right]^{1/2}(a_6^+ - a_6^-)\right],$$

$$M_{J-1}(^3J_J)=\frac{1}{4}\frac{1}{2J+1}\left[\frac{2J-1}{2J+1}\right]^{1/2}\left[\sqrt{J-1}(a_4^+ - a_4^-)+\left[\frac{2(J^2-1)}{J}\right]^{1/2}(a_5^+ - a_5^-)\right.$$
$$\left.+\left[\frac{(J+1)(J+2)}{J}\right]^{1/2}(a_6^+ - a_6^-)\right],$$

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[1]See. G. Sachs, *Nuclear Theory* (Addison-Wesley, Reading, MA, 1953), p. 238, and references therein.

[2]J. L. Friar and S. Fallieros, Phys. Rev. C **29**, 1645 (1984).

[3]A. Cambi, B. Mosconi, and P. Ricci, J. Phys. G **10**, L11 (1984); Phys. Rev. Lett. **48**, 462 (1982); J. L. Friar, B. F. Gibson, and G. L. Payne, Phys. Rev. C **30**, 441 (1984); W. Jaus, and W. S. Woolcock, Nucl. Phys. **A431**, 669 (1984).

[4]K. M. Schmitt and H. Arenhövel, University of Mainz report, 1987 (unpublished). See also P. Wilhelm, H. Arenhövel, and W. Leideman, University of Mainz report, 1988 (unpublished).

[5]See Friar, Gibson, and Payne (Ref. 3).

[6]F. E. Low, Phys. Rev. **110**, 974 (1958); J. Pestieau, *ibid.* **160**, 1555 (1967).

[7]B. Sakita, Phys. Rev. **127**, 1800 (1962).

[8]J. Govaerts, J. L. Lucio M., A. Martinez, and J. Pestieau, Nucl. Phys. **A368**, 409 (1981); J. L. Lucio M., A. Martinez, and J. Pestieau, Nuovo Cimento **89A**, 292 (1985).

[9]H. W. Fearing, contribution to the *10th International Conference on Few Body Problems,* Karlsruhe, 1983, edited by B. Zeitnitz (Elsevier, Amsterdam, 1984), p. 99.

[10]H. Arenhövel, in *International School of Intermediate Nuclear Physics, Verona, Italy,* 1985, edited by Berge, Costa, and Shaerf (World Scientific, Singapore, 1986).

[11]This prediction, slightly modified at high $\omega$, is incorporated in the parametrization given in Lucio, Martinez, and Pestieau (Ref. 8). We will follow the notation of this reference.

[12]De Pascale *et al.*, Phys. Rev. C **32**, 1830 (1985).

[13]J. J. Alvarado, J. L. Lucio M., and J. Pestieau, Phys. Rev. C **40**, 531 (1989).

[14]A. Zieger, P. Grewer, and B. Ziegler, Few Body Systems **1**, 135 (1986).

[15]W. Y. Hwang, J. T. Londergane, and G. E. Walker, Ann. Phys. (N.Y.) **149**, 335 (1983).

[16]B. Sakita and C. Goebel, Phys. Rev. **127**, 1787 (1962).

[17]M. Le Bellac, Nuovo Cimento **L1x**, 110 (1969).

[18]It is worth mentioning that gauge invariance is a property not only of the operators but also of the states. Thus, by cutting the Hilbert space we can violate gauge invariance.

[19]S. I. Nogornyi, Yu. A. Kasatkin, E. V. Inopin, and I. K. Kirichenko, Yad. Fiz. **44**, 1171 (1986) [Sov. J. Nucl. Phys. **44**, 760 (1986)].

[20]J. L. Lucio M. and J. Pestieau (unpublished).

[21]H. Arenhövel and W. Leideman (private communication); Friar, Gibson, and Payne (Ref. 3).

[22]T. H. Burnett and N. M. Kroll, Phys. Rev. Lett. **20**, 86 (1968).

[23]H. W. Fearing, Phys. Rev. D **7**, 243 (1973).

[24]H. Chew, Phys. Rev. **123**, 377 (1961).

[25]M. Le Bellac, F. M. Renard, and J. Tran Thanh Van, Nuovo Cimento **33**, 594 (1964); *ibid.* **34**, 450 (1964). See also H. Huneke Diploma thesis, Bonn, 1980.

[26]For the helicity amplitudes we will follow the conventions of Ref. 25 where the first fermionic helicity refers to the proton. Notice that as far as the multipole moments is concerned the definition of $\theta$ may be taken either as in Ref. 8 or as in Ref. 23.

[27]Arndt *et al.*, Phys. Rev. D **28**, 97 (1983).

[28]R. J. Hughes, A. Zieger, H. Waffler, and B. Ziegler, Nucl. Phys. **A267**, 329 (1976).

[29]H. O. Meyer *et al.*, Phys. Rev. Lett. **52**, 11759 (1984).

[30]A. Ninane, *et al.*, Can. J. Phys. **62**, 1104 (1984).

[31]C. Dupont *et al.*, Nucl. Phys. **A445**, 13 (1985).

[32]It is important to realize that unitarity and rescattering are not equivalent. Indeed in the last case the real part of the amplitude is nontrivially affected whereas in the former it is necessary to appeal to the analytic properties of the amplitude to get such corrections.

[33]M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) **7**, 404 (1959).