# Self-consistent approximation to the solution of the Bethe-Salpeter equation

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A technique for the approximate solution of the Bethe-Salpeter equation is examined. The technique requires the solution of a pair of coupled equations for the relative-momentum and relative-energy dependence of the relativistic $T$ matrix. The solutions obey a self-consistency requirement as well as the usual elastic-unitarity constraint. It is also shown that the approximate $T$ matrix is stable under a single iteration in the exact four-dimensional equation at certain kinematic points, including the fully on-shell point. A model problem with an exactly solvable separable interaction is examined and exact, approximate, as well as three-dimensional reduction results are compared. The phase shifts calculated in this self-consistent approximation scheme are found to be in excellent agreement with the exact phase shifts.

## I. INTRODUCTION

We have been investigating the possibility of treating retardation effects in the Bethe-Salpeter (BS) equation[1] in a "self-consistent" manner. In this work we assume a factorized approximation (factorized in space and time, or equivalently in momentum and energy) to the solution of the BS equation. This leads to coupled equations which can be solved self-consistently. These equations treat the relative-time and relative-space degrees of freedom symmetrically and self-consistently, which is the source of our improved treatment of retardation effects. In an earlier paper[2] we reported calculations of a model problem (one scalar "meson" exchange between scalar "nucleons") for which exact solutions were available, and compared our results with "three-dimensional" reductions and with the exact results. That work, although very encouraging, relied on the use of a Wick rotation to obtain both the exact and our factorized approximation results. The Wick rotation[3] method, which is designed to deal with the singularities inherent in one-meson-exchange interactions, cannot be applied easily to more general types of interactions (such as two-meson-exchange crossed diagrams). In order to overcome the difficulties involved in the solution of the BS equation, we may find it convenient to separate those problems associated with the singularities of the interaction from those related to the treatment of retardation effects. With this in mind we have examined a model problem in which the interaction term in the BS equation is separable. For such interaction terms exact four-dimensional solutions of the BS equation are easily obtained. It is also straightforward for us to apply our approximation technique to the separable interaction problem without recourse to the Wick rotation. Hence we can study the efficacy of our approximation scheme for dealing with retardation effects and compare this approximation with various three-dimensional reduction schemes, as well as with the exact result. In this way we may help to eliminate our reservation that the smoothness of the Wick rotated solution might have been responsible for our earlier encouraging results.

Before we discuss the present approximation technique it may be appropriate for us to comment generally about the BS equation and the goal of approximation techniques for its solution. The BS equation is an integral equation for transition amplitudes which iterates a certain set of irreducible diagrams to all orders. Thus nonperturbative results can be obtained. In general there are an infinite number of irreducible diagrams which should be included in the "interaction" term. Whenever one deals with the BS equation in practice one begins with an approximate BS equation which contains some truncated set of irreducible diagrams. The solution of this approximate BS equation then contains all iterates of this truncated set of diagrams. It has been known for some time that the approximate BS equation generated by using only one-meson-exchange diagrams as an interaction term probably does not provide a good starting point for calculation of nucleon-nucleon scattering in a mesonic exchange model.[4] Indeed, the ladder BS equation does not satisfy a one-body limit criterion, as has been discussed by Gross.[4] Thus, if one wishes to use only one-meson-exchange diagrams as an interaction in a relativistic two-body equation, then there may be a better starting point than the BS equation. Some three-dimensional reductions of the BS equation, for example, do obey the proper one-body limit criterion when used with a one-meson-exchange interaction. However, the premise of our work is that one has a field-theoretic model of the nuclear force and that, in principle, one wishes to sum all possible diagrams which contribute to nucleon-nucleon

 

scattering. In such a systematic approach based upon a field-theoretic model (possibly including form factors), the BS equation seems to be the most feasible starting point. If that approach is pursued, then one should include, at the least, both ladder (one-meson-exchange) and crossed-box (irreducible two-meson-exchange) contributions to the interaction in order to obtain an equation which obeys the correct one-body limit. In fact, current state-of-the-art meson-exchange models of the nuclear force[5] do include both types of contributions to the interaction. Unfortunately, the BS equation is very difficult to solve when crossed-box diagrams are included. Thus one resorts to using approximations along the lines of the Blankenbecler-Sugar equation[6] or the Gross equation.[7] While such three-dimensional reductions might provide an adequate starting point when used with one-meson-exchange interactions, it is not at all clear whether or not they provide a reasonable approximation for higher-order interactions. Thus we present here further investigations of a new technique which seems to provide a better approximation for the solutions of equations of the BS type than do currently available practical alternatives. It is clearly intended as an approximation technique which eventually may be of use to those involved in state-of-the-art nuclear force calculations. Of course, the techniques developed here could also be applied to other problems requiring the solution of an equation of the BS type.

The rest of this paper is divided into three sections. In Sec. II we present the formalism for the factorized approximation for the transition matrix ($T$ matrix) and discuss the same approximation in terms of state vectors. In Sec. III the model problem with a separable interaction is discussed. Exact results as well as our approximate results and those from three-dimensional reductions are presented for a variety of parameter sets. The factorized approximation is seen to yield a good approximation to the exact results. The final section, Sec. IV, contains a brief summary and conclusions.

## II. FORMALISM

We take the BS equation for the $T$ matrix in operator notation as

$$T=U+Ug_0T \ , \tag{1}$$

where the two-particle interaction and propagator are $U$ and $g_0$, respectively. It is useful to define the "half-shell" $T$ as

$$|T\rangle \equiv T|\phi\rangle \ , \tag{2}$$

where $|\phi\rangle$ is the noninteracting, two-particle state vector (plane-wave solution). The noninteracting solution can be factored exactly into a relative-energy-dependent part, $|e_0)$, and a relative-momentum-dependent part, $|k_0)$, as

$$|\phi\rangle = |e_0)|k_0) \ . \tag{3}$$

The values $e_0$ and $k_0$ of relative energy and relative momentum are referred to as the on-shell values. In the center-of-momentum frame $e_0=0$ and $k_0$ is related to the total center-of-mass energy. We will use rounded bracket bras and kets, e.g., $|e_0)$, to denote state vectors which act only in the relative-energy or relative-momentum spaces. The BS equation, Eq. (1), for the half-shell $T$ matrix becomes

$$|T\rangle = U|e_0)|k_0) + Ug_0|T\rangle \ . \tag{4}$$

We then define an approximation to $|T\rangle$ via the relation

$$|\tilde{T}\rangle = U|e_0)|k_0) + Ug_0P_E|\tilde{T}\rangle \ , \tag{5}$$

where

$$P_E \equiv \frac{|E)(e_0|}{(e_0|E)} \ , \tag{6}$$

with $|E)$ an as yet undefined ket which acts only in the relative-energy space [thus $(e_0|E)$ is a scalar]. Note that $P_E$ is a projection operator, i.e., $P_E^2=P_E$, acting only in relative-energy space. From Eq. (5) we obtain the equation for $|\tilde{T}_e)\equiv(e_0|\tilde{T}\rangle$, a ket which acts only in relative-momentum space:

$$|\tilde{T}_e) = [(e_0|U|e_0)]|k_0) + \frac{(e_0|Ug_0|E)}{(e_0|E)}|\tilde{T}_e) \ . \tag{7}$$

Clearly $|\tilde{T}_e)$ is an approximation to $(e_0|T\rangle$, the exact half-shell $T$ matrix evaluated at the on-shell relative-energy point. As we will soon discuss, one way to interpret $|\tilde{T}_e)$ is as a relative-momentum-dependent "factor" in a factorized approximation of $|T\rangle$.

To develop the equations for the relative-energy-dependent factor, we define a projector $P_K$ as

$$P_K \equiv \frac{|K)(k_0|}{(k_0|K)} \ , \tag{8}$$

where $|K)$ is still undefined. This projector may be used to write another approximation to $|T\rangle$, viz.,

$$|\hat{T}\rangle = U|e_0)|k_0) + Ug_0P_K|\hat{T}\rangle \ , \tag{9}$$

from which we obtain

$$|\hat{T}_k) = [(k_0|U|k_0)]|e_0) + \frac{(k_0|Ug_0|K)}{(k_0|K)}|\hat{T}_k) \ . \tag{10}$$

The relative-energy-dependent ket, $|\hat{T}_k)$, is defined as $|\hat{T}_k)\equiv(k_0|\hat{T}\rangle$. This ket, $|\hat{T}_k)$, is an approximation to $(k_0|T\rangle$, the exact half-shell $T$ matrix evaluated at the on-shell relative-momentum point. Recall that the two projectors which we introduced are still incompletely defined, since the two kets $|E)$ and $|K)$ are still undefined. We now relate the kets, $|E)$ and $|K)$, in the projectors to the kets which solve Eqs. (7) and (10).

Notice that $|\tilde{T}\rangle$ and $|\hat{T}\rangle$ are independent approximations to $|T\rangle$. In order that these two approximations be compatible with each other, they must give the same results at kinematic points where they are both known. Because $|\tilde{T}\rangle$ is determined only at the on-shell relative-energy points $(e_0|\tilde{T}\rangle$, and $|\hat{T}\rangle$ is determined only at the on-shell relative-momentum points $(k_0|\hat{T}\rangle$, the only point for which they are both determined is the fully on-shell point. Thus we must require that

$$(k_0|(e_0|\tilde{T}\rangle=(e_0|(k_0|\hat{T}\rangle \tag{11}$$

or, equivalently

$$(k_0|\tilde{T}_e)=(e_0|\hat{T}_k) \ . \tag{12}$$

This is the *self-consistency condition.* Examining Eqs. (7) and (10) reveals that the self-consistency condition will be satisfied if

$$\frac{|E)|\tilde{T}_e)}{(e_0|E)}=\frac{|K)|\hat{T}_k)}{(k_0|K)} \ . \tag{13}$$

Because the relative-energy- and relative-momentum-dependent parts of the above equation must separately be equal, we find that

$$|\tilde{T}_e)=N\frac{|K)}{(k_0|K)} \tag{14}$$

and

$$|\hat{T}_k)=N\frac{|E)}{(e_0|E)} \ , \tag{15}$$

where $N$ is an arbitrary normalization. Thus we can rewrite Eqs. (7) and (10) as two coupled equations which incorporate the self-consistency condition, i.e.,

$$|\tilde{T}_e)=[(e_0|U|e_0)]|k_0)+\frac{(e_0|Ug_0|\hat{T}_k)}{(e_0|\hat{T}_k)}|\tilde{T}_e) \tag{16}$$

and

$$|\hat{T}_k)=[(k_0|U|k_0)]|e_0)+\frac{(k_0|Ug_0|\tilde{T}_e)}{(k_0|\tilde{T}_e)}|\hat{T}_k) \ . \tag{17}$$

One could, of course, have arrived at Eqs. (16) and (17) directly from Eq. (4) by assuming a factorized ansatz for $|T\rangle$, namely that

$$|T\rangle\approx\frac{|\tilde{T}_e)|\hat{T}_k)}{(k_0|\tilde{T}_e)}=\frac{|\tilde{T}_e)|\hat{T}_k)}{(e_0|\hat{T}_k)} \ . \tag{18}$$

This is the reason for interpreting $|\tilde{T}_e)$ and $|\hat{T}_k)$ as the relative-momentum- and relative-energy-dependent factors in a factorized approximation. The technique of deriving the factorized approximation from a projection operator starting point emphasizes the self-consistency between the two resulting equations.

An important property of the solution of the BS equation is two-body elastic unitarity.[8] In the scattering regime below particle production threshold this property corresponds to conservation of flux. All popular approximation schemes for the BS equation preserve this property and as a consequence have real phase shifts below particle, e.g., pion, production threshold.[9] The present approximation scheme preserves elastic unitarity as well, as we now show.

Although a general proof of elastic unitarity for the BS equation is difficult, the essential property of the equation which must be preserved is the discontinuity of the propagator at the on-shell point, i.e., $(e_0|(k_0|g_0-g_0^{\dagger}|k_0)|e_0)$. Since we can interpret our approximation technique as an approximation of the exact propagator, $g_0$, by $g_0P_E$, we only need to show that the discontinuity of $g_0P_E$ near the on-shell point is not different from $g_0$. Since we will consider here only $P_E$, which projects in relative-energy space, we can neglect relative-momentum dependences since they are unchanged. Although the exact propagator is diagonal in relative energy, it is convenient to rewrite it in terms of a general operator as

$$G_0(e_1,e_2)\equiv g_0(e_1)\delta(e_1-e_2) \ , \tag{19}$$

with the obvious notation $G_0(e_1,e_2)=(e_1|G_0|e_2)$. We note that our approximate propagator can be written

$$\tilde{G}_0(e_1,e_2)=g_0(e_1)\frac{(e_1|E)}{(e_0|E)}\delta(e_0-e_2) \tag{20}$$

(we assume delta function normalization of the relative-energy eigenstates). Since $(e_1|E)$ is a smooth function of $e_1$ near $e_1=e_0$, we conclude that

$$\tilde{G}_0(e_0,e_2)=G_0(e_0,e_2) \ . \tag{21}$$

This is a sufficient condition for the discontinuity to be correct and thus for elastic unitarity to hold for our factorized approximation. Indeed, we chose to have the bra $(e_0|$ in the definition of $P_E$ precisely in order to obtain an approximation scheme which preserves elastic unitarity. Since our self-consistency condition ensures that the values of $|\tilde{T}_e)$ and $|\hat{T}_k)$ are the same on shell, then elastic unitarity for $|\tilde{T}_e)$ implies elastic unitarity for $|\hat{T}_k)$ as well.

In nonrelativistic quantum mechanics one often has better intuition about state vectors than about half-shell $T$ matrices. Although the interpretation of state vectors in a relativistic theory is not identical to the nonrelativistic interpretation involving probability densities, it may be illuminating to examine a factorized approximation scheme for relativistic state vectors. We might define the relativistic two-body state vector, $|\Psi\rangle$, by the equation

$$|\Psi\rangle=|\phi\rangle+g_0U|\Psi\rangle \ . \tag{22}$$

This state vector will obey the usual relationship to the half-shell $T$ matrix, i.e.,

$$U|\Psi\rangle=T|\phi\rangle=|T\rangle \ . \tag{23}$$

At this point we could introduce projection operators and implement self-consistency at the on-shell point in exact analogy to the $T$-matrix formalism, however, we already know what the result will be. The two coupled equations for the state vector "factors" will be

$$|\tilde{\Psi}_e)=[(e_0|e_0)]|k_0)+\frac{(e_0|g_0U|\hat{\Psi}_k)}{(e_0|\hat{\Psi}_k)}|\tilde{\Psi}_e) \tag{24}$$

and

$$|\hat{\Psi}_k)=[(k_0|k_0)]|e_0)+\frac{(k_0|g_0U|\tilde{\Psi}_e)}{(k_0|\tilde{\Psi}_e)}|\hat{\Psi}_k) \ . \tag{25}$$

Although these equations are formally very similar to the factorized $T$-matrix equations, they are actually much more difficult to deal with. Since the plane-wave state,

$|\phi\rangle$, is a scattering state, it obeys delta function normalization. Therefore the inhomogeneous terms in Eqs. (24) and (25), $(e_0|e_0)|k_0)$ and $(k_0|k_0)|e_0)$, are singular, as are the solutions, $|\bar{\Psi}_e)$ and $|\hat{\Psi}_k)$. In addition, the unitarity proof which we used for the $T$ matrix relied upon manipulations of an approximation to the propagator. Clearly, when dealing with state vectors one does not have the projector in a position to act upon the propagator. Thus the elastic unitarity constraints are more difficult to implement. For these reasons we concentrate on the $T$ matrix where the solutions are normalizable functions.

If one wishes to implement the self-consistent approach in terms of state vectors, it is better to define an approximation to the state vector, $|\Psi\rangle \approx |\bar{\Psi}\rangle$, by means of the factorized $T$ matrix, i.e.,

$$|\bar{\Psi}\rangle \equiv |\phi\rangle + g_0 \frac{|\tilde{T}_e)|\hat{T}_k)}{(k_0|\tilde{T}_e)} \ . \tag{26}$$

We will not pursue further discussions of state vectors in this paper. The last formal aspect of the factorized approximation which we will discuss in this section is the stability of the on-shell value of the factorized approximation for the $T$ matrix under iteration.

Given any approximate solution for the half-shell $T$ matrix one may attempt to systematically improve the approximation by iterating it in the exact equation. Thus if $|T^{(0)}\rangle$ is a zeroth-order approximation to $|T\rangle$, then the first iterate, $|T^{(1)}\rangle$, is defined by

$$|T^{(1)}\rangle = U|e_0)|k_0) + Ug_0|T^{(0)}\rangle \ . \tag{27}$$

Clearly the only solution which is unchanged at all kinematic points under iteration is the exact solution. The factorized approximation has the property that its on-shell value is unchanged after a single iteration, i.e.,

$$(e_0|(k_0|T^{(1)}\rangle = (e_0|(k_0|T^{(0)}\rangle \ .$$

Thus we say that the on-shell value is stable under one iteration. This stability indicates that the on-shell value of the factorized $T$ matrix might provide a good approximation to the exact on-shell $T$ matrix.

The proof of stability is straightforward. We have that

$$|T^{(0)}\rangle \equiv \frac{|\tilde{T}_e)|\hat{T}_k)}{(k_0|\tilde{T}_e)} = \frac{|\tilde{T}_e)|\hat{T}_k)}{(e_0|\hat{T}_k)} \ ; \tag{28}$$

thus

$$|T^{(1)}\rangle = U|e_0)|k_0) + Ug_0 \frac{|\tilde{T}_e)|\hat{T}_k)}{(k_0|\tilde{T}_e)} \ . \tag{29}$$

Examining $|T^{(1)}\rangle$ at the on-shell point we find that

$$(e_0|(k_0|T^{(1)}\rangle = (e_0|(k_0|U|e_0)|k_0) + (e_0|(k_0|Ug_0 \frac{|\tilde{T}_e)|\hat{T}_k)}{(k_0|\tilde{T}_e)} \ . \tag{30}$$

Using either Eq. (16) or Eq. (17), we then have that

$$(e_0|(k_0|T^{(1)}\rangle = (k_0|\tilde{T}_e) = (e_0|\hat{T}_k) = (e_0|(k_0|T^{(0)}\rangle \ , \tag{31}$$

as claimed. In fact, it is easily seen that the factorized approximation is stable for one iteration at all values for which either the relative energy is on shell or the relative momentum is on shell. We note that three-dimensional approximations for the $T$ matrix do not have this stability property.

## III. SEPARABLE INTERACTION CASE

We now turn to the special case of a separable interaction and examine numerical results for our approximation technique for the $T$ matrix. As we will show, if the interaction in the BS equation is "separable," then we can analytically solve the equation for the exact $T$ matrix. In addition, both the factorized approximation equations and the three-dimensional reduction equations can be solved exactly as well. We do not claim here that the interaction in a realistic BS equation can be accurately approximated by a separable interaction of rank one, however it provides an easy test case for examining approximation schemes in a four-dimensional equation of the BS type. Other authors have also taken advantage of the convenience of separable potentials in relativistic formulations of two-particle interactions.[10] We note that Tjon and collaborators[11] have also looked at solutions of three-body BS equations with separable interaction terms. Further, there has been an interesting serious effort[12] made to generate realistic relativistic separable interactions suitable for insertion in the BS equation.

We choose our separable interaction, $U_S$, to be

$$U_S \equiv \frac{U|\eta\rangle\langle\eta|U}{\langle\eta|U|\eta\rangle} \ , \tag{32}$$

where both $U$ and $|\eta\rangle$ are arbitrary. Thus the BS equation with this interaction is given by

$$T = \frac{U|\eta\rangle\langle\eta|U}{\langle\eta|U|\eta\rangle} + \frac{U|\eta\rangle\langle\eta|U}{\langle\eta|U|\eta\rangle} g_0 T \ . \tag{33}$$

This equation for the fully off-shell $T$ matrix can be solved exactly; its solution is easily verified to be

$$T = \frac{U|\eta\rangle\langle\eta|U}{\langle\eta|U|\eta\rangle - \langle\eta|Ug_0U|\eta\rangle} \ . \tag{34}$$

Similarly, we can solve Eqs. (7) and (10) for the factorized approximation. We find that

$$|\tilde{T}_e) = \frac{(e_0|U|\eta\rangle\langle\eta|U|e_0)|k_0)}{\langle\eta|U|\eta\rangle - \langle\eta|Ug_0|E)(e_0|U|\eta\rangle/(e_0|E)} \ . \tag{35}$$

Note that $|\tilde{T}_e) = N_e(e_0|U|\eta\rangle$, where $N_e$ is a scalar. The solution to Eq. (10) for $|\hat{T}_k)$ is

$$|\hat{T}_k) = \frac{(k_0|U|\eta\rangle\langle\eta|U|e_0)|k_0)}{\langle\eta|U|\eta\rangle - \langle\eta|Ug_0|K)(k_0|U|\eta\rangle/(k_0|K)} \ , \tag{36}$$

and again $|\hat{T}_k) = N_k(k_0|U|\eta\rangle$. The self-consistency conditions, Eqs. (14) and (15), are used to eliminate $|E)$ and $|K)$. Since Eqs. (35) and (36) are independent of the normalizations of $|E)$ and $|K)$, we immediately see that

$$|\tilde{T}_e\rangle=\frac{(e_0|U|\eta\rangle\langle\eta|U|e_0)|k_0)}{\langle\eta|U|\eta\rangle-\langle\eta|Ug_0(k_0|U|\eta\rangle(e_0|U|\eta\rangle/(e_0|(k_0|U|\eta\rangle}\ . \tag{37}$$

This is to be compared to the exact result

$$(e_0|T\rangle=\frac{(e_0|U|\eta\rangle\langle\eta|U|e_0)|k_0)}{\langle\eta|U|\eta\rangle-\langle\eta|Ug_0U|\eta\rangle}\ . \tag{38}$$

The self-consistent solution for $|\hat{T}_k)$ is

$$|\hat{T}_k)=\frac{(k_0|U|\eta\rangle\langle\eta|U|e_0)|k_0)}{\langle\eta|U|\eta\rangle-\langle\eta|Ug_0(e_0|U|\eta\rangle(k_0|U|\eta\rangle/(k_0|(e_0|U|\eta\rangle}\ , \tag{39}$$

which is to be compared to the exact result

$$(k_0|T\rangle=\frac{(k_0|U|\eta\rangle\langle\eta|U|e_0)|k_0)}{\langle\eta|U|\eta\rangle-\langle\eta|Ug_0U|\eta\rangle}\ . \tag{40}$$

The factorized solution clearly obeys the self-consistency condition since the ket $(k_0|U|\eta\rangle$, which acts only in relative-energy space, commutes with the ket $(e_0|U|\eta\rangle$, which acts only in relative-momentum space. Note that with a separable interaction one can solve the self-consistent equations analytically. In most other situations one would be forced to solve the coupled equations using an iterative procedure. In a previously examined problem[2] it was found that the iterative procedure had good convergence properties. We have no need to discuss iterative methods here.

Let us examine the analytic solutions in Eqs. (37)–(40). We notice that both the exact results for $(e_0|T\rangle$ or $(k_0|T\rangle$ and the factorized results for $|\tilde{T}_e)$ or $|\hat{T}_k)$ can be written in the generic form:

$$|\bar{T}_x)=\frac{(x|U|\eta\rangle\langle\eta|U|e_0)|k_0)}{\langle\eta|U|\eta\rangle-\langle\eta|Ug_0\bar{U}|\eta\rangle}\ , \tag{41}$$

where $(x|$ is either $(e_0|$ or $(k_0|$ and

$$\bar{U}_{\text{exact}}|\eta\rangle=U|\eta\rangle \tag{42}$$

and

$$\bar{U}_{\text{factorized}}|\eta\rangle=\frac{(e_0|U|\eta\rangle(k_0|U|\eta\rangle}{(e_0|(k_0|U|\eta\rangle}\ . \tag{43}$$

Thus the difference between the exact and factorized result is the replacement of $\bar{U}_{\text{exact}}|\eta\rangle$ by $\bar{U}_{\text{factorized}}|\eta\rangle$ in Eq. (41). Note that this replacement leaves $\bar{U}|\eta\rangle$ unchanged if either the relative momentum is the on-shell value or the relative energy is the on-shell value. Otherwise the replacement provides a factorized interpolation for $\bar{U}|\eta\rangle$. However, one integrates $\bar{U}|\eta\rangle$ over all values of relative momentum and relative energy in Eq. (41) with the weighting function $\langle\eta|Ug_0$. Thus, the accuracy of the factorized approximation depends upon the difference in these weighted integrals of $\bar{U}|\eta\rangle$. Notice that the weighting function has a double pole at the on-shell point, corresponding to both "nucleons" being on shell. Therefore the on-shell point is strongly weighted in the integral. Let us emphasize again that $\bar{U}_{\text{exact}}|\eta\rangle$ and $\bar{U}_{\text{factorized}}|\eta\rangle$ are the same at the on-shell point.[13]

Let us now discuss a specific separable interaction and some numerical results for the $s$-wave phase shifts. We write the BS equation for a single partial wave of the half-shell $T$ matrix with a separable interaction as

$$T_{\omega p}=V_{\omega p}+\frac{i}{\pi^2}\int d\omega'\int q^2dq\frac{V_{\omega p}V_{\omega' q}}{V_{0\hat{p}}}G_{\omega' q}T_{\omega' q}\ , \tag{44}$$

with the propagator defined as

$$G_{\omega q}\equiv\frac{-1}{[(\sqrt{s}/2+\omega)^2-E_q^2+i\epsilon]\cdot[(\sqrt{s}/2-\omega)^2-E_q^2+i\epsilon]}\ , \tag{45}$$

where $E_q^2=q^2+m^2$. All equations here are in the zero total-momentum frame. The normalization of $T_{\omega p}$ is chosen so that the phase shifts, $\delta_l$, are given in terms of the on-shell $T$ matrix, $T_{0\hat{p}}$, by

$$T_{0\hat{p}}=\frac{i\sqrt{s}}{\hat{p}}(1-e^{2i\delta_l})\ . \tag{46}$$

The on-shell relative energy is $\omega=0$ and the on-shell relative momentum is $p=\hat{p}$, which is defined in terms of the total center-of-mass energy squared via $\sqrt{s}=2E_{\hat{p}}$. The range of integration for relative energy is $(-\infty,\infty)$ and for relative momentum is $(0,\infty)$. We, somewhat arbitrarily, choose the separable interaction to be defined according to

$$V_{\omega p}\equiv\frac{\lambda}{\omega^2-p^2-\mu^2+i\epsilon}\ . \tag{47}$$

Our choice of a singular interaction is motivated by the singular nature of realistic meson-exchange interactions. Of course, the simple pole nature of the singularities in $V_{\omega p}$ is different from the cut singularities typical of meson exchange.[14] However, we felt that this interaction might be more interesting than a nonsingular interaction such as a Gaussian. The coupling constant $\lambda$ and "meson" mass $\mu$ are both adjustable.

The expressions for the on-shell $T$ matrices are easily determined to be

$$T_{0\hat{p}}^{\text{exact}}=\frac{V_{0\hat{p}}^2}{V_{0\hat{p}}-(i/\pi^2)\int d\omega\int q^2dq\ V_{\omega q}^2G_{\omega q}} \tag{48}$$

and

$$T_{0\hat{p}}^{\text{factorized}} = \frac{V_{0\hat{p}}^2}{V_{0\hat{p}} - (i/\pi^2)\int d\omega \int q^2 dq\, V_{\omega q} G_{\omega q} (V_{\omega \hat{p}} V_{0q} / V_{0\hat{p}})} \,. \tag{49}$$

For comparison purposes we will also examine the phase shifts given by six three-dimensional reductions. It was pointed out in the Introduction that three-dimensional reductions are not necessarily designed to give accurate approximations for arbitrary BS equations. However, they are the only other approximation techniques against which comparisons can be made. Additionally, they are the only viable approximation techniques which can be used with higher-order BS equations. The three-dimensional approximations can be viewed as a direct replacement of the propagator $G_{\omega p}$ by an approximate propagator $G_{\omega p}^{\text{3D}}$. Let us write these in the general form

$$G_{\omega p}^{\text{3D},\alpha\beta} \equiv -\frac{i\pi\delta(\omega-\gamma_p^\alpha)}{E_p - \sqrt{s}/2 - i\epsilon} \Lambda_{E_p}^\beta \,. \tag{50}$$

We consider two different functions $\gamma_p^\alpha$, given by $\gamma_p^{(1)} \equiv 0$ and $\gamma_p^{(2)} \equiv E_p - \sqrt{s}/2$. We will consider three functions $\Lambda_{E_p}^\beta$, defined by

$$\Lambda_{E_p}^{(1)} \equiv \frac{1}{2E_p\sqrt{s}} \,, \tag{51}$$

$$\Lambda_{E_p}^{(2)} \equiv \frac{1}{2E_p(E_p+\sqrt{s}/2)} \,, \tag{52}$$

and

$$\Lambda_{E_p}^{(3)} \equiv \frac{1}{4E_p^2} \,. \tag{53}$$

The six combinations formed from $\alpha=1,2$ and $\beta=1,2,3$ are the same six three-dimensional approximations which were compared by Woloshyn and Jackson.[15] In the figures, we will use the same labeling convention as was used by Woloshyn and Jackson, namely: $A$ for $\alpha=2$, $\beta=1$; $B$ for $\alpha=2$, $\beta=2$; $C$ for $\alpha=2$, $\beta=3$; $D$ for $\alpha=1$, $\beta=1$; $E$ for $\alpha=1$, $\beta=2$; and $F$ for $\alpha=1$, $\beta=3$. Names commonly associated with some of these approximations are as follows: $A$—the Gross equation,[7] $C$—the Kadyshevsky equation,[16] and $E$—the Blankenbecler-Sugar equation.[6] The on-shell $T$ matrix can be written as

$$T_{0\hat{p}}^{\text{3d}} = \frac{V_{0\hat{p}}^2}{V_{0\hat{p}} - (i/\pi^2)\int d\omega \int q^2 dq\, V_{\omega q}^2 G_{\omega q}^{\text{3D}}} \tag{54}$$

for the three-dimensional reductions. Of course, the relative-energy integration can be done immediately using the delta function in $G_{\omega q}^{\text{3D}}$ which reduces the integral to a one-dimensional integral.

Because of the simple form of the interaction in Eq. (47), the integrals which appear in the expressions for the on-shell $T$ matrices can be easily evaluated. In all cases we evaluated the relative-energy integrals analytically and the relative-momentum integrals numerically. Let us discuss the results for four different parameter sets whose results seemed typical of those we examined. The scale for the parameters is set by the mass $m$ which is chosen to be $m=4.9$ fm$^{-1}$ for every case. The two extreme cases for the mass $\mu$ are chosen to be a light "meson" mass, $\mu=0.7$ fm$^{-1}$, and a heavy "meson" mass, $\mu=4.9$ fm$^{-1}$. For each of these "meson" masses an attractive and repulsive interaction is shown. The results are shown in Figs. 1–4.

In Fig. 1 the results for a heavy "meson" mass ($\mu=4.9$ fm$^{-1}$) and a repulsive interaction ($\lambda=338$ fm$^{-3}$) are shown. The differences between the exact and approximate results are shown in two separate ways. First, in the upper half of Fig. 1 the $s$-wave phase shifts, $\delta_0$, are shown in radians versus the on-shell relative-momentum, $\hat{p}$, in fm$^{-1}$. The energy range for the figures is from scattering threshold to one-meson production threshold. The exact phase shift is represented with the unlabeled solid line. The factorized approximation phase shifts are shown with the crosses which, more or less, follow the exact results. The other six solid lines, which all show phase shifts which are too large, represent the six three-

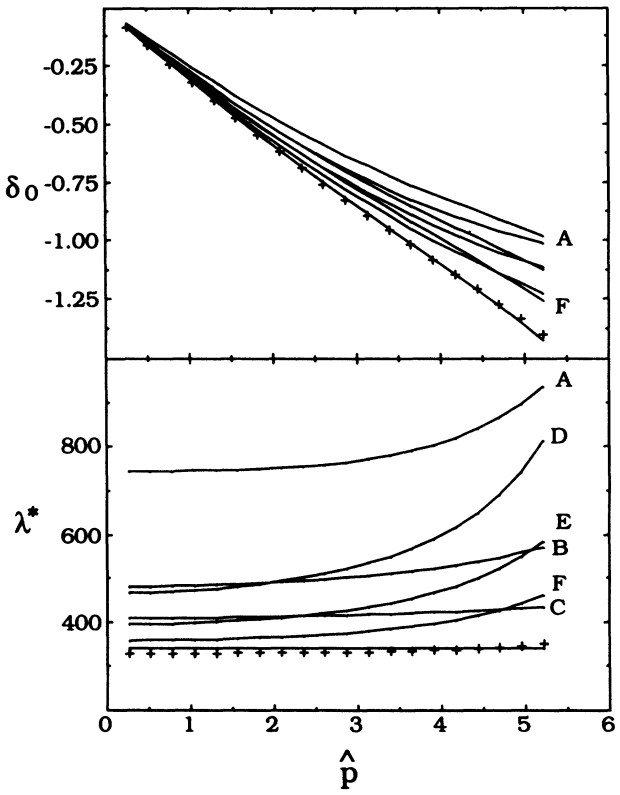


FIG. 1. Plots of $s$-wave phase shifts, $\delta_0$, in radians and effective coupling constants, $\lambda^*$, in fm$^{-3}$ vs on-shell relative-momentum, $\hat{p}$, in fm$^{-1}$. Unlabeled line indicates exact results, crosses indicate factorized approximation results, lines labeled $A$–$F$ indicate three-dimensional reduction results. Parameters are $m=4.9$ fm$^{-1}$, $\mu=4.9$ fm$^{-1}$, and $\lambda=338$ fm$^{-3}$.

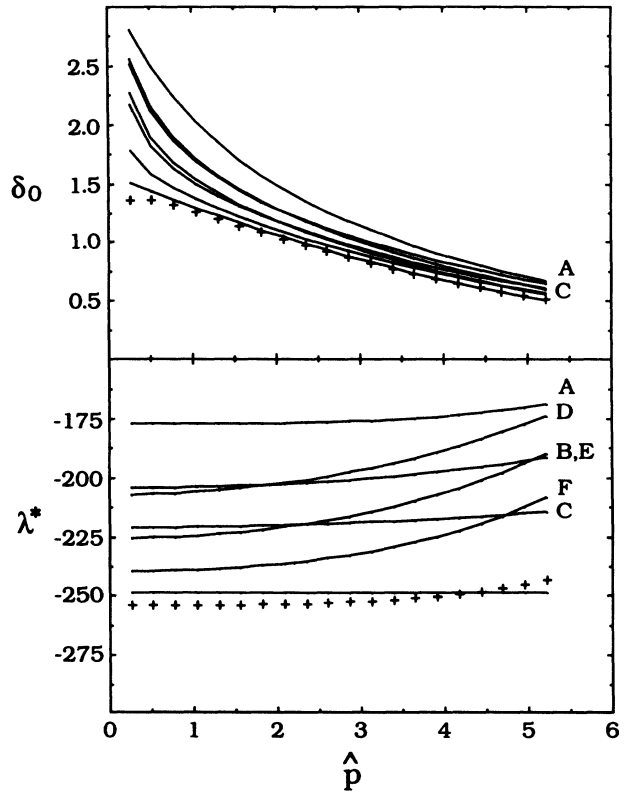


FIG. 2. Same as Fig. 1 except the parameters are $m=4.9$ fm$^{-1}$, $\mu=4.9$ fm$^{-1}$, and $\lambda=-249.5$ fm$^{-3}$.

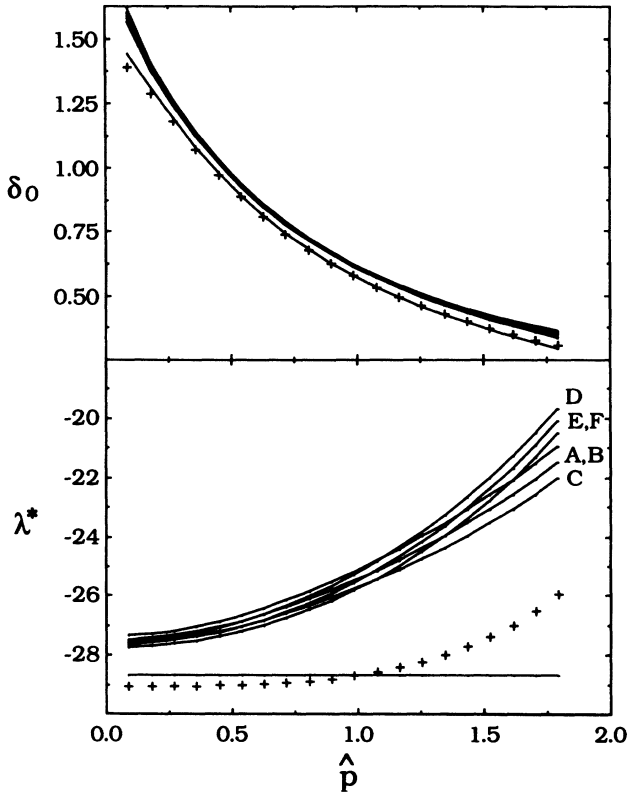


FIG. 4. Same as Fig. 1 except the parameters are $m=4.9$ fm$^{-1}$, $\mu=0.7$ fm$^{-1}$, and $\lambda=-28.7$ fm$^{-3}$.

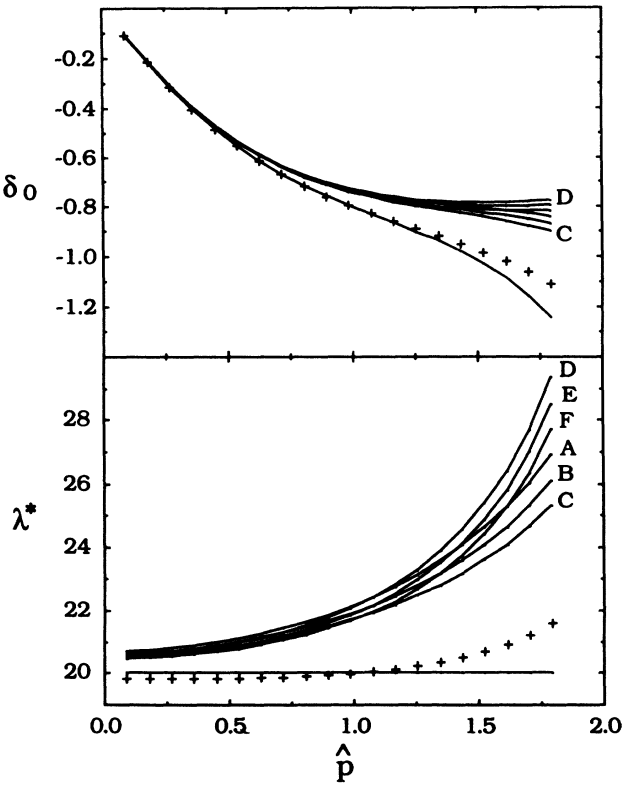


FIG. 3. Same as Fig. 1 except the parameters are $m=4.9$ fm$^{-1}$, $\mu=0.7$ fm$^{-1}$, and $\lambda=20$ fm$^{-3}$.

dimensional reductions, $A-F$, discussed above. The ordering of the three-dimensional reductions is more easily seen in the lower half of Fig. 1, which shows effective coupling constants.

The effective coupling constant $\lambda^*$ for a given approximation scheme is defined as the coupling constant which if used in that approximation would give the exact $s$-wave phase shift. These obviously are functions of energy. The lower half of Fig. 1 shows the effective coupling constants for each approximation scheme vs $\hat{p}$. In the plot of $\lambda^*$ vs $\hat{p}$ the exact coupling constant is shown by the unlabeled solid line which is fixed at the constant value of $\lambda=338$ fm$^{-3}$. As in the phase-shift plot, the effective coupling constant for the factorized approximation is represented by crosses. The factorized approximation's $\lambda^*$ is fairly close to the exact $\lambda$ for the energies shown. However, differences at low energies are more easily seen in the $\lambda^*$ vs $\hat{p}$ plots than in the phase-shift plots. The six three-dimensional reductions are easily distinguishable and are represented by the labeled solid lines. Notice that the effective coupling constant can differ substantially from the exact value even when the phase shifts show only minor differences, for example, in Fig. 1 all the phase shifts go to zero as $\hat{p}$ goes to zero. However, the $\lambda^*$'s do not all converge to the exact $\lambda$. One might also notice that the $\lambda^*$'s corresponding to different three-dimensional reductions show differing amounts of energy dependence. However, it is the purpose of this article to compare the factorized approximation to the exact and

three-dimensional reduction results, not to compare the three-dimensional reductions among themselves. Thus, the major point which should be stressed is that the amount of variability possible within the standard three-dimensional reductions is much larger than the typical difference between the exact and factorized approximation results. Similar conclusions can be drawn from Figs. 2–4 as well.

In Fig. 2 the results for a heavy "meson" mass ($\mu=4.9$ fm$^{-1}$) and an attractive interaction ($\lambda\approx-250$ fm$^{-3}$) are shown. The coupling constant is chosen so that the exact result has a zero-energy bound state. Thus the exact phase shift goes to $\pi/2$ as $\hat{p}$ goes to zero. The factorized approximation underbinds slightly at low energies and has a resonance just above scattering threshold. The three-dimensional reductions all overbind and, as such, all have phase shifts which go to $\pi$ as $\hat{p}$ goes to zero. Because of the choice of a coupling constant which gives a zero-energy bound state, the phase-shift differences are accentuated near $\hat{p}$ equals zero. However, the effective coupling constants show a behavior very similar to that from the repulsive interaction in Fig. 1.

Figures 3 and 4 show results for a light "meson" mass ($\mu=0.7$ fm$^{-1}$). The results for a repulsive interaction with $\lambda=20$ fm$^{-3}$ are shown in Fig. 3 while the results for an attractive interaction corresponding to a zero-energy bound state ($\lambda\approx-29$ fm$^{-3}$) are shown in Fig. 4. The effective coupling constants tend to show a larger energy dependence when a lighter "meson" is considered. However, the generally good agreement between the factorized approximation and exact results remains. As was the case for the heavy "meson" mass, one sees that the factorized approximation tends to underbind slightly at low energies while the three-dimensional reductions all overbind. Our conclusions from these studies is that the encouraging results which were obtained for the Wick-rotated factorized approximation[2] have been reproduced in a model problem which does not use a Wick rotation. Thus, it does not seem unreasonable to expect that the factorized approximation would provide a useful approximation scheme for realistic higher-order BS equations.

## IV. SUMMARY AND CONCLUSIONS

Our goal in this and in our previous work[2] has been to develop an approximation technique which can accurately approximate the solutions of the fully relativistic BS equation. As such, we have chosen to investigate two problems for which exact solutions of the BS equation are obtainable, namely, a one-meson-exchange interaction[2] and a separable interaction. In both cases and for all parameter sets examined, our approximation technique has provided a remarkably good approximation to the exact BS phase shifts. It should be emphasized that our goals have not been identical with the goals of previous work on approximation techniques for the BS equation. In the late 1960s and early 1970s a great deal of work centered on finding simple relativistic equations to approximate the BS equation. However, the goal of that earlier work was to find an equation which when used with a one-boson-exchange interaction would provide an approximation to the one-plus-two-boson-exchange BS equation. That work was motivated by the fact that it was impractical at that time to include two-boson-exchange interactions even in a reduced (three-dimensional) relativistic equation. That restriction is no longer true. Indeed, all state-of-the-art meson-exchange models of the nuclear force contain two-boson-exchange (crossed-box) interaction terms. However, it is still not practical to solve the full BS equation exactly with these interactions. Since the amount of variability among the available three-dimensional reductions is large and no easy systematic corrections are available, it is clear that an accurate approximation technique for the fourth-order BS equation is desired. That has been the motivation for our investigations of this new approximation technique.

The BS equation is a four-dimensional equation (three spatial and one temporal relative coordinates). It is the fourth dimension, relative time, which differentiates the relativistic and nonrelativistic two-body problems. It is the relative-time variable which accounts for retardation effects arising from the finite speed of propagation of the interaction, e.g., mesons. In the usual three-dimensional reductions the relative-time degree of freedom (actually, the conjugate coordinate, relative energy) is fixed by a particular ansatz in order to obtain an equation which is similar in form to the nonrelativistic Lippman-Schwinger equation. Our approach differs in that we treat the relative energy and relative momentum in a symmetric fashion, thereby obtaining reduced equations in both relative momentum and relative energy. These two separate approximations are then required to be consistent with each other for kinematic points where the solution is fixed simultaneously by both equations. Each of these two approximations is also required to satisfy an elastic-unitarity constraint which ensures the reality of the phase shifts below the inelastic threshold. If we write our approximation scheme with the aid of a pair of projection operators of the generic form $P_E=|E)(e|/(e|E)$ and $P_K=|K)(k|/(k|K)$ where $|e)$ and $|k)$ are eigenstates of relative energy and relative momentum, respectively, then elastic unitarity constrains $e$ and $k$ to be the on-shell values and self-consistency fixes $|E)$ and $|K)$ in terms of the solution kets, $|\tilde{T}_e)$ and $|\hat{T}_k)$. This was discussed in detail in Sec. II. Thus our rather general form of approximation is completely fixed by our two constraints, self-consistency and elastic unitarity. The two equations which must be solved are both similar in form to the nonrelativistic equation, except that one of them is for a function of relative energy. The self-consistency requirement couples the two equations. Because relative energy is treated as a dynamical variable, we expected that our approximation would do a better job of including the correct retardation effects. Indeed, it was found that the exact model problem phase shifts were approximated much better by our approximation than by the three-dimensional reductions for the two problems which we examined. The results for the ladder Bethe-Salpeter equation were shown in our previous work[2] and the results for a separable interaction were shown in Sec. III of this work. Thus, we anticipate that our technique could provide a better means of approximating the solution to

realistic BS equations than other currently available techniques. Such calculations will, by no means, be simple. The inclusion of crossed-box diagrams in the somewhat simpler three-dimensional reduction schemes is already near the present limits of computational ability. However, our approximation technique is still very much simpler to implement than an exact solution technique.

Finally, a stability property of our approximate solution was discussed. In particular, our approximation schemes can be viewed as the determination of approximations to the $T$ matrix along fixed lines in relative-energy and relative-momentum space. For our approximation there is one line in each of relative energy and relative momentum and they intersect at the fully on-shell point. The self-consistency condition requires that the approximations agree at the point where the grid lines intersect. The stability property states that if the approximate solution is iterated in the exact four-dimensional equation, then the first iteration will not change the results on any of the grid lines. This property indicates that the approximate solution should, in general, provide a good approximation on the fixed grid lines (which includes the fully on-shell point). Additionally we have begun investigating correction schemes for the self-consistent technique. That work will be presented in a separate publication.

In conclusion, the model problems investigated in this and in a previous work[2] indicate that the factorized approximation seems to be a viable candidate for the accurate approximation of phase shifts for realistic Bethe-Salpeter equations. Such work can check the validity of relativistic field-theoretic models of the nuclear force and could provide a starting point for future work on the relativistic three-body problem.

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[13]As an aside, we note that if one examines the exact $T$ matrix for the following choice of separable interaction

$$U_S' \equiv \frac{(e_0|U|\eta\rangle(k_0|U|\eta\rangle\langle\eta|U}{(e_0|(k_0|U|\eta\rangle\langle\eta|U|\eta\rangle} ,$$

then one finds that the exact on-shell $T$ matrix for $U_S'$ is identical to the on-shell factorized $T$ matrix for $U_S$. This is not too surprising since the each half-shell $T$ matrix for $U_S'$ is exactly factorizable in relative energy and relative momentum.

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