Validity of the adiabatic rotational model in the case of the hexadecupole operator

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(Received 23 March 1990)

The validity of the rotational model expressions relating the $E4$ transition probabilities and the intrinsic hexadecupole moments is examined in the light of the recent data on the inelastic electron-scattering form factors for the $0^+ \rightarrow 4^+$ transitions in some medium-mass nuclei.

The rotational spectra in nuclei are usually discussed in the framework of the unified model of Bohr and Mottelson (BM).[1] The model is characterized by the decoupling of single-particle and rotational motions; the wave function of a yrast state in quasirotational nuclei is given, for example, by the product of the symmetric top wave function and an intrinsic state describing, respectively, the collective rotation and the single-particle motion. The rotational model relates the reduced electric 2^L-pole transition probabilities to the intrinsic 2^L-pole moments—expectation values of the operator Q_0^L with respect to the intrinsic state— in a simple manner:

$$B(EL;J_i^+ \rightarrow J_f^+) = \begin{bmatrix} J_i & L & J_f \\ 0 & 0 & 0 \end{bmatrix}^2 \langle \Phi | Q_0^L | \Phi \rangle^2 . \quad (1)$$

Relation (1) has been used extensively in the past to obtain Nilsson parameters for quadrupole deformations in various mass regions from the $E2$ matrix elements resulting from Coulomb excitation cross sections or the lifetime measurements.[2] The BM prescription has also often been invoked to obtain semiquantitative estimates[3] of the reduced transition probabilities for electric quadrupole transitions, $B(E2;J_i^+ \rightarrow J_f^+)$, in terms of the intrinsic quadrupole moments, $\langle Q_0^2 \rangle$, resulting form either phenomenological models, such as the Nilsson model, or microscopic Hartree-Fock (HF)/Hartree-Fock-Bogolubov (HFB) descriptions.

In the context of a general discussion of the shape-collective aspects of nuclear dynamics, a study of the electric hexadecupole matrix elements is a logical extension of the studies involving the electric quadrupole operator. The difficulties associated with the extraction of the $B(E4)$ values have, however, hindered progress on this topic in light- and medium-mass nuclei; the $E4$ decay branches are unmeasurably small compared to the competing $E2$ branches. An attempt some time ago at verifying relation (1) by Zumbro *et al.*[4] was not successful because of the smallness of the muonic hyperfine splitting caused by the $E4$ interaction.

Recently a large number of medium-mass nuclei have been subjected to inelastic electron-scattering experiments.[5–7] These experiments have provided significant data involving electroexcitation form factors covering a broad momentum-transfer range (with $q = 0.5-3.0$ fm^{-1}) for the $0^+ \rightarrow 2^+$ as well as $0^+ \rightarrow 4^+$ transitions. It has become possible to obtained fairly reliable estimates of the heretofore unmeasured $E4$ transition probabilities, $B(E4;0^+ \rightarrow 4^+)$, via an extension of the form factor data in to the momentum transfer of the photon point, $q = [E(4^+) - E(0^+)]$.

In Fig. 1 we present the observed form factors for the electroexcitation of the yrast 2^+ and 4^+ levels in the nuclei ^{46,48,50}Ti, ^{50,52,54}Cr, and ^{54,56}Fe. The occurrence of the $1f_{7/2}$ subshell closure in the $N = 28$ isotones is signalled by a number of observed features such as the enhanced $[E(2^+) - E(0^+)]$ separations in the observed spectra. This empirical feature has also been substantiated by a large number of microscopic calculations[8] in the $2p$-$1f$ shell which have predicted, as an obvious implication of the $1f_{7/2}$ subshell closure, a *simultaneous* reduction in the various multipole moments of the intrinsic states associated with the isotones with $N = 28$. The use of Eq. (1) then immediately suggests *dips* in the values of the electric quadrupole *as well as* hexadecupole transition probabilities in the $N = 28$ isotones compared to the values of these quantities in the neighboring nuclei with $N = 26, 30$. Since the electroexcitation form factors for small momentum transfers can be considered proportional to the corresponding transition probabilities for the sake of qualitative discussion, one expects the first maxima ($q \simeq 0.8$ fm^{-1}) of $|F(q)|^2(0^+ \rightarrow 2^+)$ and $|F(q)|^2(0^+ \rightarrow 4^+)$ in the isotones with $N = 28$ to be *smaller* than the corresponding first maxima in the isotones with $N = 26, 30$. The results presented in Fig. 1 show that the observed $|F(q)|^2(0^+ \rightarrow 2^+)$ *do* display this trend in the Ti, Cr as well as Fe isotopes. In sharp contrast to this, the observed hexadecupole form factors, $|F(q)|^2(0^+ \rightarrow 4^+)$, *fail* to provide any indication of the subshell closure at $N = 28$. In fact the $0^+ \rightarrow 4^+$ form factor in $^{50}_{22}$Ti$_{28}$ at its first maximum is seen to be *larger* than the form factor at its first peak in $^{48}_{22}$Ti$_{26}$. The first maximum of the $(0^+ \rightarrow 4^+)$ form factor in $^{52}_{24}$Cr$_{28}$ is also *larger* than the first peaks of the form factors for the same transition in the nuclei $^{50,54}_{24}$Cr$_{26,30}$.

In view of this observation one is prompted to question the efficacy of expression (1) for relating the reduced $E4$ transition probabilities and the hexadecupole moments of the intrinsic states. The purpose of this paper is to examine the validity of the usual rotor-model prescription for the hexadecupole operator. We first present a quantitative assessment of relation (1) for the quadrupole as well

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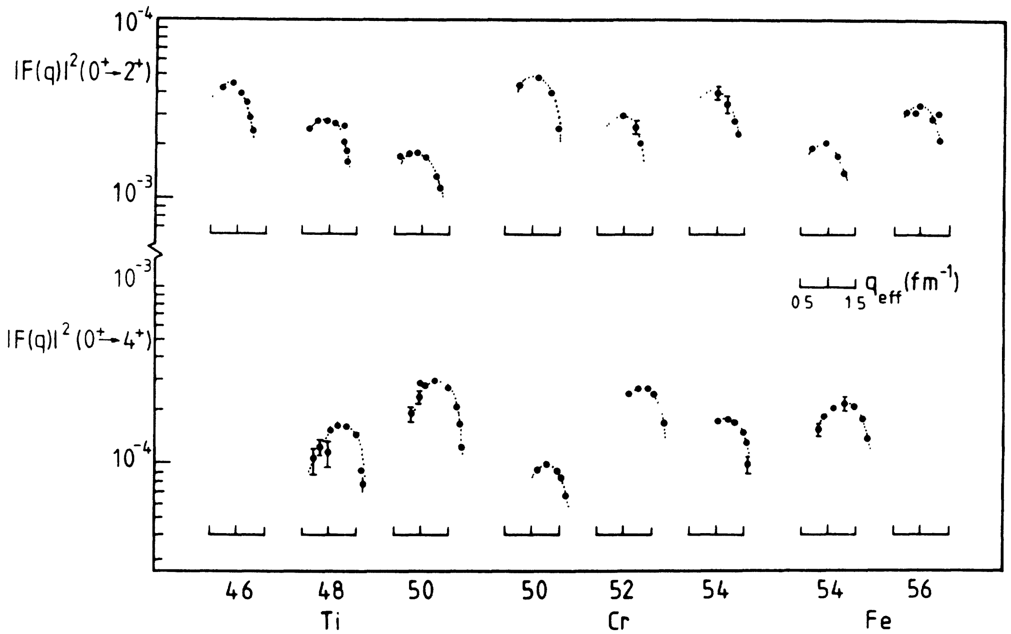

FIG. 1. The observed electroexcitation form factors for $0^+\to2^+$ as well as $0^+\to4^+$ transitions in some $2p$-$1f$ shell nuclei.

as hexadecupole operators by carrying out explicit angular momentum projection on the HFB variational intrinsic states (PHFB) for a number of $2p$-$1f$ shell nuclei. The results reveal dramatic differences between the quadrupole and the hexadecupole operators vis-à-vis the semiquantitative reliability of the rotor-model predictions. An examination of the conditions under which the Peierls-Yoccoz projection formalism[9] permits the recovery of the BM prescription offers significant clues concerning the inefficacy of the model in the case of the hexadecupole operator. It is seen that the huge mismatch between the microscopic values of the $E4$ matrix elements and their rotor-model estimates arises mostly due to the large variances or fluctuations of the hexadecupole operator with respect to the variational intrinsic states.

We have considered here, as illustrative examples, the HFB intrinsic states in the nuclei ^{48,50,52}Ti, ^{50,52,54}Cr, and ^{52,54}Fe resulting from the realistic effective interactions given by Kuo and Brown[10,11] for the $(2p_{1/2},2p_{3/2},1f_{5/2},1f_{7/2})$ space. The one-body part of the Hamiltonian for the valence nucleons is characterized by the observed ^{41}Ca spectrum. We have given in Table I the multipole moments of the axially symmetric self-consistent states. We have also presented here the $E2$ and $E4$ transition probabilities for the $0^+\to2^+$ and $0^+\to4^+$ transitions, respectively, resulting from explicit angular momentum projection along with their estimates based on the rotor-model prescription.

TABLE I. The intrinsic multipole moments as well as the reduced electric transition probabilities $B(E2;0^+\to2^+)$ and $B(E4;0^+\to4^+)$ involving the quadrupole and the hexadecupole operators, respectively, for some doubly even $2p$-$1f$ shell nuclei. The reduced transition probabilities resulting from explicit angular momentum projection on the HFB intrinsic states (PHFB) have been compared with the rotor-model predictions (RM). The intrinsic 2^L-pole moments have been given in units of b^L, where b is the oscillator parameter. The reduced transition probabilities for the $(0^+\to2^+)$ and $(0^+\to4^+)$ transitions have been given in units of $e^2\times10^{-50}$ cm^4 and $e^2\times10^{-102}$ cm^8, respectively.

Nucleus	$\langle Q_0^2\rangle$	$[B(E2)]_{PHFB}$	$[B(E2)]_{RM}$	$\langle Q_0^4\rangle$	$[B(E4)]_{PHFB}$	$[B(E4)]_{RM}$
^{48}Ti	16.4	4.7	3.7	21.3	9.6	156.0
^{50}Ti	−9.4	2.5	1.2	18.9	25.6	79.2
^{52}Ti	15.7	4.9	3.6	20.0	14.9	150.4
^{50}Cr	26.3	10.7	9.7	23.2	5.2	192.2
^{52}Cr	11.4	3.7	1.9	15.7	13.7	92.6
^{54}Cr	28.7	12.5	10.6	33.7	18.8	449.2
^{52}Fe	20.8	8.0	6.2	−19.3	13.1	140.5
^{54}Fe	9.0	3.0	1.2	−7.5	14.1	22.3

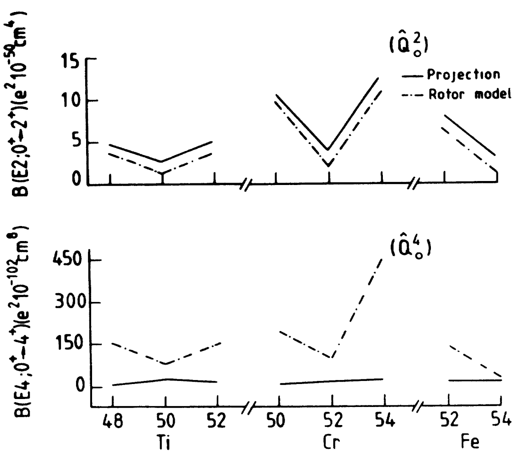

FIG. 2. The $B(E2;0^+\rightarrow 2^+)$ as well as $B(E4;0^+\rightarrow 4^+)$ values in some $2p$-$1f$ shell nuclei calculated in the framework of the projected HFB method. The dash-dotted line joins the rotor-model estimates based on the Eq. (1).

A graphical presentation of the results (see Fig. 2) shows that the rotor-model predictions are in qualitative agreement with the PHFB results for the $B(E2;0^+\rightarrow 2^+)$ values. In fact, for the nuclei with $N\neq 28$, even a quantitative agreement between the rotor model and the PHFB estimates is obtained; the maximum discrepancy is only about 27% of the latter.

In sharp contrast to the situation for the $B(E2;0^+\rightarrow 2^+)$ values, the results presented in Table I and Fig. 2 indicate a dramatic failure—both in the quantitative *as well as* qualitative sense—of the rotor-model prescription as applied to the hexadecupole operator. In fact the rotor-model estimates are larger than the projected HFB ones by *an order of magnitude* in most of the cases considered here.

In what follows we examine qualitatively the reasons for the inadequacy of the rotor-model prescriptions as applied to the hexadecupole operator. In the framework of this model, the state with angular momentum J and projection M (in the laboratory frame) is given by

$$|\Psi^J_{M(K)}\rangle=\left[\frac{(2J+1)}{8\pi^2}\right]D^J_{MK}(\Omega)|\Phi_K\rangle\ . \qquad (2)$$

Using these wave functions the matrix element of the Q_0^L operator between the states J_i^π and J_f^π (belonging to the $K=0$ band) can be written

$$\langle\Psi^{J_f}_{M(0)}|Q_0^L|\Psi^{J_i}_{M(0)}\rangle=\left[\frac{(2J_i+1)}{(2J_f+1)}\right]^{1/2}\times\begin{bmatrix}J_i & L & J_f\\ M & 0 & M\end{bmatrix}\begin{bmatrix}J_i & L & J_f\\ 0 & 0 & 0\end{bmatrix}\times\langle\Phi_0|Q_0^L|\Phi_0\rangle\ . \qquad (3)$$

The expression for $B(EL,J_i\rightarrow J_f)$ given by Eq. (1) is then obtained by carrying out the sum

$$(2J_i+1)^{-1}\sum_{M_i,M_f}|\langle\Psi^{J_f}_{M_F(0)}|Q_0^L|\Psi^{J_i}_{M_i(0)}\rangle|^2\ .$$

We next consider the matrix element of the Q_0^L operator between the states of good angular momentum projected from the microscopic (and usually variational) intrinsic states. The latter can be written as

$$|\Psi^J_{M(K)}\rangle=\left[\frac{(2J+1)}{8\pi^2}\right]\int D^J_{MK}(\Omega)R(\Omega)|\Phi_K\rangle d\Omega\ , \qquad (4)$$

where $R(\Omega)$ is the rotation operator. Employing the projected wave functions the matrix element $\langle\Psi^{J_f}_{M(0)}|Q_0^L|\Psi^{J_i}_{M(0)}\rangle$ can be expressed as

$$\left[\frac{(2J_i+1)}{2\sqrt{(N_{J_i}N_{J_f})}}\right]\begin{bmatrix}J_i & L & J_f\\ M & 0 & M\end{bmatrix}\sum_\mu\begin{bmatrix}J_i & L & J_f\\ 0 & \mu & \mu\end{bmatrix}\times\int d^{J_f}_{0\mu}(\beta)\langle\Phi_0|e^{-i\beta J_y}Q^L_\mu|\Phi_0\rangle d(\cos\beta)\ , \qquad (5)$$

where

$$N_j=[(2J+1)/2]\int d^J_{00}(\beta)\langle\phi_0|e^{-i\beta J_y}|\Phi_0\rangle(\cos\beta)\ . \qquad (6)$$

An insertion of the complete set of states between the operators $\exp(-i\beta J_y)$ and Q^L_μ in Eq. (5) yields

$$\langle\Psi^{J_f}_{M(0)}|Q_0^L|\Psi^{J_i}_{M(0)}\rangle=\left[\frac{(2J_i+1)}{(2J_f+1)}\right]\sqrt{N_{J_f}/N_{J_i}}\begin{bmatrix}J_i & L & J_f\\ M & 0 & M\end{bmatrix}\begin{bmatrix}J_i & L & J_f\\ 0 & 0 & 0\end{bmatrix}\langle\Phi_0|Q_0^L|\Phi_0\rangle+\left[\frac{(2J_i+1)}{2\sqrt{(N_{J_i}N_{J_f})}}\right]\begin{bmatrix}J_i & L & J_f\\ M & 0 & M\end{bmatrix}\times\sum_{n=(1p\text{-}1h)}\sum_\mu\langle\Phi_n|Q^L_\mu|\Phi_0\rangle\begin{bmatrix}J_i & L & J_f\\ 0 & \mu & \mu\end{bmatrix}\int d^{J_f}_{0\mu}(\beta)\langle\Phi_0|e^{-i\beta J_y}|\Phi_n\rangle d(\cos\beta)\ . \qquad (7)$$

This *exact* expression facilitates an identification of the steps involved in recovering the prescription embodied in Eq. (3). The desired steps follow.

(i) In medium-mass nuclei $\langle J^2 \rangle$ varies between 40 and 50. In heavier deformed nuclei the mean value of J^2 is expected to exceed 100. It is, therefore, quite reasonable to employ the relation[12]

$$N_J \simeq [(2J+1)/\langle J^2 \rangle] \exp[-J(J+1)/\langle J^2 \rangle] \langle J^2 \rangle \gg 1$$

to obtain

$$(N_{J_f}/N_{J_i}) \cong [(2J_f+1)/(2J_i+1)] \; .$$

With this approximation the first term in (7) becomes *exactly* equal to the rhs of Eq. (3).

(ii) As emphasized by Ripka,[13] the overlap functions $\langle \Phi_0 | \exp) - i\beta J_y) | \Phi_n \rangle$ have significant nonvanishing contributions only at the values $\beta \sim 0$ and $\beta \sim \pi$ provided $\langle J^2 \rangle \gg 1$. This automatically restricts the μ summations to just $\mu = 0$ since

$$\lim_{\beta \to 0, \pi} d^L_{0\mu}(\beta) = \delta_{0\mu} \; .$$

The overall magnitude of the second term in (7) is therefore mainly governed by the quantities $\langle \Phi_n | Q^L_0 | \Phi_0 \rangle$.

(iii) One next needs an *additional* assumption—and this has not been appreciated heretofore— that the matrix elements $\langle \Phi_0 | Q^L_0 | \Phi_{1p\text{-}1h} \rangle$ are sufficiently small.

A simple measure of the quantitative importance of the matrix elements $\langle \Phi_0 | Q^L_0 | \Phi_n \rangle$ is provided by the fluctuations

$$[\langle (Q^L_0)^2 \rangle - \langle Q^L_0 \rangle^2]^{1/2} \equiv \left[\sum_{n=1p\text{-}1h} |\langle \Phi_0 | Q^L_0 | \Phi_n \rangle|^2 \right]^{1/2} .$$

In Table II we have given the calculated values of the fluctuations $[\langle (Q^2_0)^2 \rangle - \langle Q^2_0 \rangle^2]^{1/2}$ and $[\langle (Q^4_0)^2 \rangle - \langle Q^4_0 \rangle^2]^{1/2}$ associated with the quadrupole and the hexadecupole operators, respectively, for the variational intrinsic states in some $2p$-$1f$ shell nuclei. We find that the fluctuations associated with the quadrupole operator are always less than 23% of the magnitude of the $\langle Q^2_0 \rangle$ values for the isotones with $N \neq 28$, and are about 40% of the magnitude of the $\langle Q^2_0 \rangle$ values in the cases with $N = 28$. On the other hand, in the case of the hexadecupole operator, the fluctuations are comparable to—and in some cases even *larger* than—the magnitudes of $\langle Q^4_0 \rangle$ values themselves.

These results explain the anomalous behavior of the hexadecupole operator vis-à-vis the rotational model; whereas the reasonably small values (lying in the range 0.01–0.42) of the ratio $[\langle (Q^2_0)^2 \rangle - \langle Q^2_0 \rangle^2]^{1/2} / |\langle Q^2_0 \rangle|$ still permit the recovery of the rotor-model limit to a significant extent, a similar situation is *not* realized in the case of the hexadecupole operator since the large fluctuations of this operator (indicating significantly non-negligible magnitudes of the individual matrix elements $\langle \Phi_n | Q^4_0 | \Phi_0 \rangle$) *invalidate* the assumption involved in step (iii) above. The results given here also rationalize the noticeably reduced quantitative efficacy of the model for obtaining the $E2$ transition probabilities in the nuclei with $N = 28$ that was noted earlier in terms of an appreciable increase (by nearly a factor of 2) in the fluctuation of the quadrupole operator in these nuclei. It may be mentioned here that the relative smallness of the fluctuations of the Q^2_0 operator for intrinsic states obtained with realistic effective interactions is not entirely unanticipated in view of the (well-known) quadrupole-quadrupole (qq) dominance[14] of the latter.

It turns out that a replacement of the realistic effective interactions by schematic ones such as the qq interaction (often used in the structural studies in heavy nuclei) does not lead to significantly reduced fluctuations of the Q^4_0 operator, and therefore does not support in general the application of the rotor model for *this* operator. We have given in Table II the values of the Q^4_0 fluctuations for the SU(3) eigenstates for the nuclei considered here. The ratio $[\langle (Q^4_0)^2 \rangle - \langle Q^4_0 \rangle^2]^{1/2} / |\langle Q^4_0 \rangle|$ is still quite large (lying in the range 0.4–2.1) although the quadrupole fluctuations are rigorously zero in this case.

In summary, we have sought to examine here the reasons underlying the observed inapplicability of the usual rotor-model prescriptions in the context of the hexadecupole collectivity in medium-mass nuclei. It turns out that an important factor that characterizes the differences between the quadrupole and the hexadecupole operator is the significantly larger magnitude of the fluctuation of the latter with respect to the (variational)

TABLE II. The fluctuations associated with the quadrupole and the hexadecupole operators for the variational intrinsic states in some $2p$-$1f$ shell nuclei resulting from the Kuo-Brown (KB) effective interaction (Ref. 10). The last column gives the fluctuations of the hexadecupole operator associated with the SU(3) eigenstates for various nuclei.

Nucleus	$[\langle (Q^2_0)^2 \rangle - \langle Q^2_0 \rangle^2]^{1/2}_{\mathrm{KB}}$	$[\langle (Q^4_0)^2 \rangle - \langle Q^4_0 \rangle^2]^{1/2}_{\mathrm{KB}}$	$[\langle (Q^4_0)^2 \rangle - \langle Q^4_0 \rangle^2]^{1/2}_{\mathrm{SU(3)}}$
^{48}Ti	3.8	20.2	13.4
^{50}Ti	3.9	16.9	08.5
^{52}Ti	3.5	20.8	11.2
^{50}Cr	0.3	23.6	14.7
^{52}Cr	4.6	22.7	14.8
^{54}Cr	3.5	21.2	12.7
^{52}Fe	4.5	22.8	15.9
^{54}Fe	3.9	23.5	15.8

intrinsic states resulting from the realistic effective interactions—a feature that greatly hinders the recovery of the rotor-model limit in a microscopic perspective.

The authors wish to thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics. One of us (S.K.S.) would like to acknowledge discussions with Professors R.K. Bhaduri, Professor J. de Boer, Professor M.G. Huber, Professor V. Paar, and Professor P. Ring.

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