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Equivalence between deep energy-dependent and shallow angular-momentum-dependent potentials

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Marchenko inversion followed by Baye's supersymmetric transformation scheme determines shallow, l-dependent but energy-independent singular local potentials that are strictly phase equivalent to given nonlocal resonating group model interactions. The resulting potential has none of the Pauli forbidden states present in the corresponding resonating group model interaction, but ensures that a generalized Levinson theorem is still valid. The intermediate step of Marchenko inversion is introduced to transform the energy-dependent deep local potentials of Horiuchi, which are phase and spectrum equivalent to the resonating group model interaction, into an energy-independent potential, suitable for the application of Baye's supersymmetric scheme for the removal of the Pauli forbidden states.

I. INTRODUCTION

There is a long standing controversy as to whether nucleus-nucleus potentials should be shallow or deep. This was discussed in detail by Aoki and Horiuchi[1] in the context of the resonating group model (RGM hereafter) approach to two-cluster scattering. The nonlocal RGM interactions support not only physical bound states, but also have redundant solutions and consequently satisfy a modified Levinson theorem.[2] Aoki and Horiuchi[1,3] used the Wentzel-Kramers-Brillouin (WKB) approximation in their studies to determine deep equivalent local potentials (ELP) that have a substantial energy dependence and, at most a weak l dependence. This approach is generally called the "RGM+WKB" approach. These ELP are deep because they have to support the so-called Pauli forbidden states (PFS) corresponding to the redundant solutions of the RGM interaction.

But equally good fits to the same phase-shift "data" were obtained by using shallow phenomenological potentials that are strongly l dependent. The phenomenological potentials for $\alpha+\alpha$ scattering determined by Ali and Bodmer[4] are of this type. However, with no PFS these shallow interactions do not reproduce the high-energy behavior of the RGM phase shifts. But, recently it has been shown[5] that by including a $1/r^2$ singularity, microscopic phase shifts can be reproduced in agreement with the extension of the Levinson theorem as proposed by Swan.[6] Subsequently Baye[7] was able to show that the supersymmetry applied to the Schrödinger equation establishes an exact relation between deep and shallow, but singular, local potentials, both of which are energy independent. The potentials related in this way are exactly phase equivalent, while their wave functions are related, by a differential operator. However, they produce a different number of bound states while preserving the Levinson relation, since, due to the exact phase equivalence, the difference $\delta_l(0)-\delta_l(\infty)$ is the same in each case. The shallow potential has the required r^{-2} singularity to make this possible.

Baye successfully applied this procedure to the case of $\alpha+\alpha$ scattering. Using the energy-dependent local potential of Buck *et al.*,[8] which approximates the ELP of the RGM interaction in the low-energy region, he applied the supersymmetric relations to eliminate the two unphysical PFS for $l=0$ and the one PFS for $l=2$ and obtained shallow but strongly l-dependent potentials. These potentials compared well with the well-known shallow potentials of Ali and Bodmer.[4] Therefore, they offer a more rigorous explanation of the phenomenological shallow potentials[5] by relating them to microscopic RGM interactions.

However, the exact ELP to a RGM interaction is intrinsically energy dependent, since the nonlocality of the RGM interaction is represented by the energy dependence of the equivalent local potential. Aoki and Horiuchi[1,9] explicitly derived the ELP for the $\alpha+\alpha$ nonlocal RGM interaction and found a substantial energy dependence over a more extended energy region. This energy dependence becomes even more pronounced for the scattering of heavier clusters than $\alpha+\alpha$, e.g., $\alpha+{}^{16}O$.[10]

Unfortunately the supersymmetric relations cannot be used to generate energy-independent shallow potentials from energy-dependent deep potentials. This also follows from their being intrinsically related to inverse scattering theory as discussed by Sukumar.[11] Therefore, the application of Baye's procedure depends crucially on the approximation of the energy-dependent ELP of Horiuchi, by an energy-independent local potential (at low energies in his example). Such an approximation, however, neglects the energy dependence, which simulates the nonlocality of the RGM interaction. It is this nonlocality of the RGM which is responsible for the Pauli forbidden states in the first place.

Fortunately there is a way out of this dilemma as we will demonstrate in detail in the next sections. From the phase shifts and bound states (including the PFS) of the energy-dependent ELP, we construct by means of the exact inverse scattering formalism of Marchenko,[12] an energy-independent but l-dependent equivalent local potential. This potential is deep and nonsingular and has the same PFS as the ELP. Consequently it obeys the generalized Levinson theorem

$$\delta_l(0)-\delta_l(\infty)=n_P\pi+n_{\rm PFS}\pi \;, \tag{1}$$

where n_P equals the number of physical bound states and $n_{\rm PFS}$ equals the number of Pauli forbidden states.

In a second step we eliminate the PFS by means of supersymmetry according to Baye's prescription. The potential obtained in this way is shallow and even more strongly l dependent. It is also singular and satisfies the same Levinson relation and therefore represents a rigorous explanation of the phenomenological shallow potentials based on extensions of the Levinson theorem in the context of the RGM theory and Horiuchi's "RGM+WKB" approach.

II. SCATTERING OF TWO DINEUTRONS

To illustrate our method, we consider a particularly simple model example of the scattering of two dineutrons in the RGM, which was originally studied by Kukulin et al.[13] The internal spatial function of the dineutron is chosen as simple as possible, i.e., as a translationally invariant shell-model function of a $(0s)^2$ configuration in its ground state in a harmonic-oscillator well of width parameter α:

$$\varphi(r)=(\alpha/\pi)^{3/4}\exp(-\tfrac{1}{2}\alpha r^2)\;,\qquad \alpha=m\omega/2\hbar\;. \tag{2}$$

The (2+2) system corresponds to the shell-model configuration s^2p^2. In this system one S state of the relative motion is forbidden by the Pauli principle and has the wave function

$$\phi_0(\mathbf{R})=(2\alpha/\pi)^{3/4}\exp(-\alpha R^2)\;. \tag{3}$$

The effective interaction is chosen to be of the form

$$V(r)=-V_0(w+mP_x)\exp(-\gamma r^2)\;, \tag{4}$$

where w and m are the usual Wigner and Majorana admixtures, respectively, and P_x is the Majorana operator. The direct potential is then given by

$$V_D(R)=-V_0\left[\frac{2\alpha}{2\alpha+\gamma}\right]^{3/2}(4w-2m)\times\exp\left[-\frac{2\alpha\gamma}{2\alpha+\gamma}R^2\right]\;. \tag{5}$$

The exchange kernel of the RGM is more complex consisting of several terms given in Ref. 13. The solution $\phi(R)$ of the integrodifferential equation of the RGM is not necessarily orthogonal to the function $\phi_0(R)$, but a physically meaningful solution $\tilde\phi(R)$ must be.

We therefore have $\tilde\phi(R)\equiv P_0\phi(R)$ with $P_0\equiv 1-|\phi_0\rangle\langle\phi_0|$. Kukulin et al.[13] have shown that the RGM equation can be simplified as follows:

$$(T_R+V_D+\hat V_N-E)\phi(R)=0\;, \tag{6}$$

where

$$\hat V_N\phi(R)\equiv\int d\mathbf{R}'U_N\left[\frac{R+R'}{2}\right]\times\exp[-(\tfrac{1}{2}\alpha+\gamma)(\mathbf{R}-\mathbf{R}')^2]\phi(\mathbf{R}') \tag{7}$$

and

$$U_N(R)=-V_0(4m-2w)\left[\frac{4\alpha}{\alpha+2\gamma}\right]^{3/2}\exp(-2\alpha R^2) \tag{8}$$

with the additional orthogonality condition

$$\langle\phi|\phi_0\rangle=0\;. \tag{9}$$

The values of the parameters of the NN force are given by[13]

$$V_0=73\text{ MeV}\;,\quad \gamma=0.46\text{ fm}^{-2}\;,\quad w=m=0.5$$

while

$$\alpha=0.22\text{ fm}^{-2}\;.$$

The ELP of the nonlocal interaction of Eq. (7) is obtained by using Horiuchi's "WKB+RGM" method, which in this case reduces to the old Perey-Buck procedure. The ELP, V_D+V_L, phase equivalent to the potential V_D+V_N, is determined from

$$V_L(E,R)\approx U_N(R)\exp\left\{\frac{m}{\hbar^2(\alpha+2\gamma)}[V_D(R)+V_L(E,R)-E]\right\}\;. \tag{10}$$

Note that this potential is independent of l.

This energy-dependent potential has a forbidden $0s$ state with a binding energy of 8.3 MeV, which, according to Kukulin et al.,[13] has a wavefunction very closely approximating $\phi_0(R)$. Therefore, the orthogonality condition Eq. (9) is automatically satisfied in this case. They also found that in the low-energy region for $E\approx 0$, $V_L(R)$ can be approximately represented by

$$V_L(R)\approx -V_{0L}\exp(-\kappa R^2)\;, \tag{11}$$

where $V_{0L}=20.5$ MeV and $\kappa=0.30\text{ fm}^{-2}$.

The ELP, however, has a substantial energy dependence which at low energies can be represented by

$$V_{0L}(E) \approx V_{0L}(E=0)\exp(-E/E_0) ,$$

where $E_0=84$ MeV. For $E \le E_0$ the well depth of

$$V(r)=V_D(R)+V_L(E,R)$$

has the approximate form $V_0(E) \approx V_0 - \frac{1}{4}E$ which clearly indicates that it is not negligible at all. This also shows up in the fact that $V_L(R)$ given by Eq. (11) has a binding energy of only 6.74 MeV instead of 8.3 MeV for $V_L(E,R)$ given by Eq. (10).

III. ENERGY-INDEPENDENT DEEP POTENTIALS BY MARCHENKO'S INVERSION METHOD

To be able to apply the supersymmetric scheme of Baye we first have to generate E-independent but l-dependent potentials which have the same spectra as the ELP of Eq. (10), including the PFS, by means of the Marchenko theory.

The Marchenko spectral inversion equation for each angular momentum l is given by

$$K_l(r,s)+F_l(r,s)+\int_r^\infty K_l(r,t)F_l(t,s)dt=0 , \qquad (12)$$

where the input kernel $F_l(r,s)$ is obtained from the S matrix via

$$F_l(r,s)=(2\pi)^{-1}\int_{-\infty}^{\infty} h_l^{(+)}(kr)[1-S_l(k)]h_l^{(+)}(ks)dk + \sum_{i=1}^{n_P+n_{\rm PFS}} M_i h_l^{(+)}(k_i,r)h_l^{(+)}(k_i,s) , \qquad (13)$$

and where the $h_l^{(+)}$ are Riccati-Hankel functions for outgoing wave conditions, while the M_i are the bound-state normalization constants (where both the physical and forbidden bound states are included). The local potential producing $S_l(k)$ is obtained from the diagonal part of the output kernel by

$$V_l(r)=-2\frac{d}{dr}K_l(r,r) . \qquad (14)$$

We represent $S_l(k)$ in rational form by

$$S_l(k)=\prod_{n=1}^{N}\frac{k+a_n}{k-a_n} , \qquad (15)$$

which allows us to perform the integral in Eq. (13) analytically and solve Eq. (12) for $K_l(r,s)$ (Refs. 14–16) assuming arbitrary values for the normalization constants M_i. It can be shown[14,15] that for a unique set of values of the M_i, only a short-range potential is obtained. Otherwise the potentials $V_l^M(r)$ determined from Eq. (14) have long-range tails. In our simple example only one Pauli forbidden state occurs for $l=0$ at -8.3 MeV. The corresponding bound-state normalization parameter M_1 is determined according to the short-range criterium as described in Refs. 14 and 15 and is given by $M_1=9.91$. In Fig. 1 we show $V_D(r)$, $V_D(r)+V_L(0,r)$, $V_D(r)+V_L(E_b,r)$ (where $E_b=-8.3$ MeV), and $V_0^M(r)$, the potential obtained by the Marchenko inversion of the whole spectrum of $V_D(r)+V_L(E,r)$. It is seen that the Marchenko potential is quite deep and supports the corresponding Pauli forbidden state at -8.3 MeV.

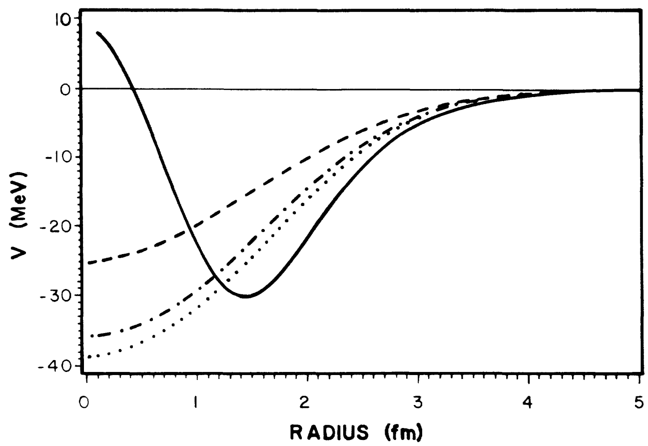

FIG. 1. The direct potential $V_D(r)$— — —, the ELP at zero energy $V_D(r)+V_L(0,r)$ —·—·—·, and at $E_b=-8.3$ MeV, i.e., $V_D(r)+V_L(E_b,r)$ · · · ·, and the potential $V_0^M(r)$ ——, obtained by the Marchenko inversion of the whole spectrum of $V_D(r)+V_L(E,r)$ for $l=0$, are plotted.

IV. REMOVAL OF PAULI FORBIDDEN STATE

We now proceed to remove the PFS at $E_b=-8.3$ MeV from the spectrum of $V_0^M(r)$ using Baye's supersymmetric scheme.[7]

In this method the Hamiltonian, in units where $\hbar^2/2m=1$, is given by

$$H_0=\left[-\frac{d^2}{dr^2}+V_0\right] ,$$

where V_0 includes the centrifugal, Coulomb, and nuclear potentials. The Hamiltonian H_0 and its supersymmetric partner H_1 are factorized as

$$H_0=A_0^+A_0^-+\epsilon_0,\quad H_1=A_0^-A_0^++\epsilon_0 , \qquad (16)$$

where

$$A_0^-=(A_0^+)^\dagger=-\frac{d}{dr}+\frac{d}{dr}\ln\Psi_0(\epsilon_0) .$$

The nodeless wave function $\Psi_0(\epsilon_0)$ is the solution of the Schrödinger equation at the factorization energy ϵ_0. We must therefore have $\epsilon_0 \le E_0^{(0)}$, where $E_0^{(0)}$ is the ground-state energy of H_0, while the eigenfunctions $\Psi_0(E)$ and $\Psi_1(E)$ of H_0 and H_1 are related by

$$\Psi_1(E)=(E-\epsilon_0)^{-1/2}A_0^-\Psi_0(E) . \qquad (17)$$

The choice of $\epsilon_0=E_0^{(0)}$ makes the spectrum of H_1 identical to the spectrum of H_0, except for the absence of the ground state of H_0. In this case the potential corresponding to H_1 is given by

$$V_1=V_0-2\frac{d^2}{dr^2}\ln\Psi_0(E_0^{(0)}) . \qquad (18)$$

For small values of r we have

$$V_1 \sim V_0 + 2(l+1)r^{-2} \sim (l+1)(l+2)r^{-2} ,$$

which behaves as a nonsingular potential with angular momentum l. However, V_1 and V_0 are not phase equivalent. To achieve this, the supersymmetric partner H_2 of H_1 with $\epsilon_1 = E_0^{(0)}$ as factorization energy is required. Since $\epsilon_1 < E_1^{(0)}$ the ground-state energy of H_1, the function $\Psi_1(\epsilon_1)$ is not square integrable and behaves asymptotically as $\exp(+\gamma_0 r)$, where $\gamma_0 = (-E_0^{(0)})^{1/2}$, but can be chosen to be regular at $r=0$. The bound-state spectra of H_1 and H_2 are identical. To any physical state of H_1 with energy E corresponds a state of H_2 with the wave function

$$\Psi_2(E) = (E_0^{(0)} - E)^{-1} A_1^- A_0^- \Psi_0(E) , \tag{19}$$

where $\Psi_2(E)$ and $\Psi_0(E)$ have the same phase shift $\delta_2(E) = \delta_0(E)$ and where $\Psi_2(E)$ is normalizable for the bound states. The corresponding potential is given by

$$V_2 = V_0 - 2\frac{d^2}{dr^2} \ln[\Psi_0(E_0^{(0)})\Psi_1(E_0^{(0)})] , \tag{20}$$

which behaves for small r like

$$V_2 \sim V_0 + 2(2l+3)r^{-2} \sim (l+2)(l+3)r^{-2} . \tag{21}$$

The singular potential V_2 is shallow and has one bound state less than V_0. From the equality of the phase shifts it follows that $\delta_l(0) - \delta_l(\infty)$ is the same for V_0 and V_2, in spite of V_2 having one bound state less than V_0 as proved by Swan.[6] This two-step process can be iterated to remove all the nonphysical bound states.

V. RESULTS

Applying the procedure described in the previous section to eliminate the PFS at -8.3 MeV from the deep potential $V_0^M(r)$, obtained by the Marchenko inversion for $l=0$ as shown in Fig. 1, we find the corresponding phase equivalent shallow potential $V_2(r)$ having no bound state. It is shown in Fig. 2, together with the unphysical potential $V_1(r)$ obtained in the intermediate step which corresponds to Eq. (18). It is seen that the potential $V_2(r)$ is purely repulsive in spite of having a phase shift with the property $\delta_0(0) - \delta_0(\infty) = \pi$, like the original $V_0(E,r)$ and $V_0^M(r)$. This is possible since, as discussed in the introduction, it has an r^{-2} singularity.

Finally, similar to Baye for $\alpha + \alpha$ scattering, we take the potential $V_D(r) + V_L(0,r)$, which approximately represents the ELP at low energies and is energy independent, as the starting point of the supersymmetric construction of the shallow potential $\tilde{V}_2$. This potential has a PFS at $E_b = -6.74$ MeV. In this case the Marchenko inversion as an intermediate step can be omitted due to the absence of an E dependence. This shallow potential is compared in Fig. 3, to the one derived previously for $l=0$, using $V_D(r) + V_L(E,r)$ by employing the Marchenko inversion. The corresponding difference in their phase shifts is shown in Fig. 4. It is seen that neglecting the energy dependence of $V_L(E,r)$ of Eq. (10) and replacing it by $V_L(0,r)$, does have a considerable effect on the phase

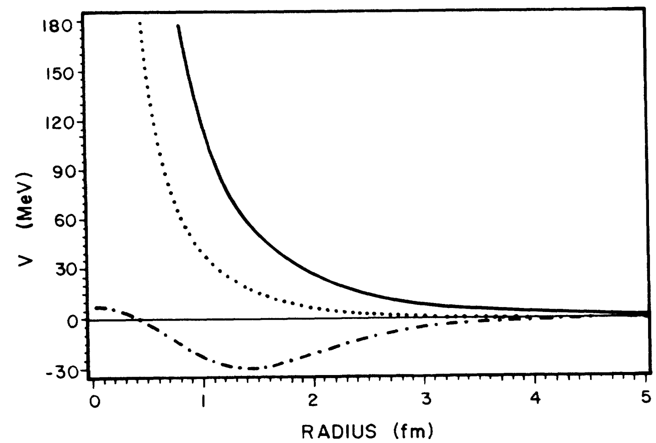

FIG. 2. The potentials $V_0^M(r)$ — · — · — · for S waves obtained by the Marchenko inversion of the corresponding spectra produced by the full equivalent local potential $V_D(r) + V_L(E,r)$ is shown. We also plot the potential $V_1(r)$ · · · · obtained by the first step described by Eq. (18) in supersymmetric procedure to remove the PFS and the final potential $V_2(r)$ ——, which is phase equivalent to $V_0^M(r)$ but has no bound state. The potential $V_2(r)$ has a r^{-2} singularity described by Eq. (21).

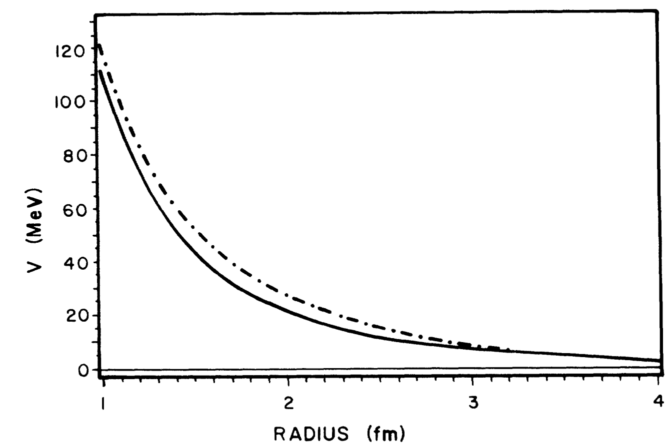

FIG. 3. The potential $V_2(r)$ — · — · — · is compared to the potential $\tilde{V}_2(r)$ —— obtained by applying Baye's method to remove the PFS at -6.74 MeV of the energy-independent potential $V_D(r) + V_L(0,r)$.

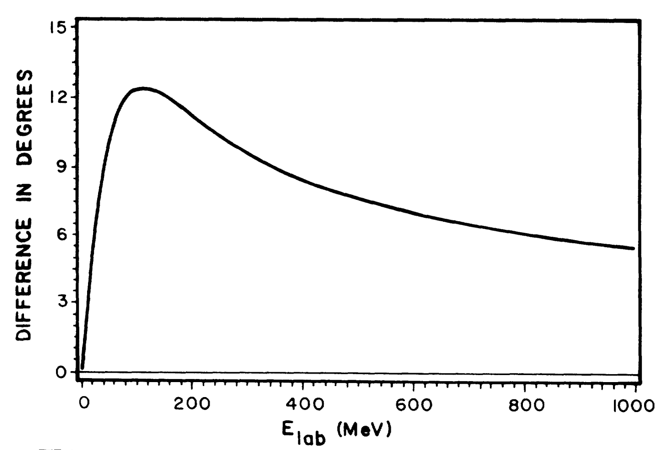

FIG. 4. The difference of the phase shifts of the potentials $\tilde{V}_2(r)$ and $V_2(r)$. For the definition of the potentials see previous figure captions.

shift even at low energies. The potentials themselves also differ substantially in the region from 1 to 3 fm.

VI. CONCLUSIONS

We have presented a mathematically rigorous scheme to obtain shallow singular potentials which are strictly phase equivalent to a given RGM nonlocal interaction excluding the Pauli forbidden states while still preserving the $\delta_l(0)-\delta_l(\infty)$ given by the generalized Levinson theorem for the RGM phase shifts.

The scheme does not depend on the neglect of the energy dependence of the deep local potential of the Horiuchi equivalent to the RGM. Its aim is achieved by converting the energy dependence of the ELP into an l dependence by means of the Marchenko inversion method in a first step. In a second step these deep l-dependent potentials are transformed into shallow l-dependent singular potentials by using Baye's supersymmetric scheme to remove the Pauli forbidden states. This two-step procedure, in general, increases the l dependence, as compared to the one obtained when an approximate energy-independent ELP, valid only for low energies, is employed (as in Baye's example for $\alpha+\alpha$ scattering).

In the first step (Marchenko inversion) the phase shift and bound-state spectrum of the RGM interaction can be used directly as input. The construction of the deep ELP is not even necessary for this purpose. In this way exact phase equivalence between the RGM interaction and the shallow potentials can be achieved. This could be of some importance if the energy-dependent ELP of Horiuchi is not quite phase equivalent to the RGM interaction, within the accuracy required. Even in this case, however, the construction of the deep local and energy-dependent ELP to the RGM interaction could still be very useful for comparison purposes.

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