Constituent quark model of the nucleon-nucleon interaction

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We use the semirelativistic constituent quark model in conjunction with the resonating group method to obtain a nonlocal Schrödinger equation for the two-nucleon system. As the quark-quark color exchange potential we use the lattice gauge theory result: the sum of the Breit potential and a linearly rising confining potential. Both the Breit potential and the quark kinetic energies are treated consistently to order $(v/c)^2$ in the quark speeds; however, we ignore relativistic corrections to the confining potential and to the kinetic energy of relative motion of the nucleons. We distinguish carefully between the exact formulation and the approximations we use to make the calculations tractable. We present the resulting nonlocal nucleon-nucleon Schrödinger equation explicitly and analytically.

I. INTRODUCTION

The purpose of this paper is to present a nucleon-nucleon interaction derived from the quark substructure of hadrons. Our approach is based on the results of lattice gauge theory. Of particular importance in the calculation is the (i) consistent and systematic inclusion of relativistic corrections up to order $(v/c)^2$ and (ii) complete and correct incorporation of the Pauli principle. As will be demonstrated, the direct nucleon-nucleon potential vanishes due to the color algebra and only the exchange interaction survives. Since this is proportional to the overlap of the two nucleons, the resulting di-nucleon potential is of short range. Furthermore, the various tensorial terms present in the quark-quark potential (scalar, spin-spin, spin-orbit, tensor, etc.) will be shown to give rise to corresponding terms in the nucleon-nucleon potential.

The nucleon-nucleon interaction results from the exchange of quarks in the overlap region and we shall present guidance[1] to our full mathematical derivation. There have been several investigations[2-8] and we shall elucidate the relationship of our work to at least some of these later in the paper. Our calculations result in a nonlocal Schrödinger equation for the two-nucleon problem:

$$\int d^3r' \left[\frac{P^2}{M}\delta^3(\mathbf{r}-\mathbf{r}') + U_0(\mathbf{r},\mathbf{r}') + U_c(\mathbf{r},\mathbf{r}') + U(\mathbf{r},\mathbf{r}') - EK(\mathbf{r},\mathbf{r}') \right] \Psi(\mathbf{r}') = 0 \quad (1)$$

where $\mathbf{r}$ is the internucleon separation. $U_0(\mathbf{r},\mathbf{r}')$ arises from the quark kinetic energy, $U_c(\mathbf{r},\mathbf{r}')$ comes from the quark confining potential contribution, and $U(\mathbf{r},\mathbf{r}')$ is derived from the short-range Breit potential[9] contribution. This equation is solved numerically in a following paper[10] for the positive parity isospin singlet and triplet cases. Our purpose here is to present the derivation of the equation and the computation of the integrands.

It has been obvious since the discovery of quarks that nucleon constituents will play a key role in the nucleon-nucleon interaction. Liberman[11] and Barry[12] advocated a nonrelativistic quark model treatment. However, since constituent quarks are expected to have a mass of about one-third the nucleon mass, their confinement to a sphere having the proton's radius would lead to $(v/c)^2$ of the order of 0.3 to 0.4. Hence relativistic corrections, at least to first order, are required both to the kinetic and potential energy terms (the Breit potential). Indeed, we find that keeping only the quark QCD $1/r$ term, leads to a poor estimate of the *nuclear* scalar interaction and obviously fails to produce the important *nuclear* spin-spin, spin-orbit and tensor interactions.

Since the late 1970's, the real question has been on how to correctly implement the Pauli principle. This has been attempted by using the resonating group method or a method essentially equivalent. Unfortunately several of the references previously mentioned make a subtle error in the implementation of the Pauli principle. Basically all have assumed either *ab initio* or as an approximation that the effective two-body nuclear interaction is given by

$$V_{NN} = \left\langle \varphi_A\varphi_B \left| \sum_{i=1}^{3}\sum_{j=4}^{6} V_{ij}\mathcal{P} \right| \varphi_A\varphi_B \right\rangle \tag{2}$$

in which V_{ij} is the potential between quarks i and j. φ_A and φ_B are wave functions describing nucleons A,B where A is comprised of quarks 1-3 and B is comprised of quarks 4-6. $\mathcal{P}$ is the operator which antisymmetrizes $\varphi_A\varphi_B$ with respect to all six quarks. Since $\mathcal{P}$ does not commute with the summed potential terms, the V_{NN} produced this way is *not Hermitian*.

If φ_A, φ_B are not exact eigenstates of the separate three quark Hamiltonians, V_{NN} will remain *nonzero* in the limit of infinite internucleon separation. This will lead to a nucleon-nucleon interaction which will not bind the deuteron.

This spurious behavior occurs because the corresponding kinetic energy is written as $T = T_A + T_B + T_R$ where T_A, T_B are the kinetic energies of the quarks belonging to one cluster relative to the center of mass of the cluster and T_R is the kinetic energy of the relative motion of the clusters. The relative kinetic energy in the two-nucleon problem then is

$$T_{NN} = \langle \varphi_A\varphi_B | T_R\mathcal{P} | \varphi_A\varphi_B \rangle. \tag{3}$$

This also has a non-Hermitian term since again $\mathcal{P}$ does not commute with the T_R operator. If φ_A, φ_B were exact eigenstates of the clusters A and B, then the non-Hermitian parts of T_{NN} and V_{NN} would cancel. Because of the complexity[13] of the exact φ, one would not choose it; indeed, that would make an already difficult computational problem even more formidable. We correct for this situation in an average way, however, by defining

$$V_{NN} = \left\langle \varphi_A\varphi_B \left| \left(\sum_{i=1}^{3}\sum_{j=4}^{6} V_{ij} - \langle V_A + V_B \rangle \right) \mathcal{P} \right| \varphi_A\varphi_B \right\rangle \tag{4}$$

and

$$T_{NN} = \langle \varphi_A\varphi_B | (T - \langle T_A + T_B \rangle)\mathcal{P} | \varphi_A\varphi_B \rangle. \tag{5}$$

This makes both V_{NN}, T_{NN} separately Hermitian regardless of whether the φ_A, φ_B are exact or not; if they are, it presents no correction. Furthermore this approach eliminates the spurious long-range behavior which arises from allowing errors in estimating the energy of the three-quark single nucleon system to propagate into the di-nucleon Schrödinger equation. The use of Eqs. (4) and (5) makes our work quite different from those referenced.

There are still other differences. First of all we apply Eqs. (4) and (5) to the first-order relativistic corrections, for the reasons given earlier. Secondly, we keep all the terms of the Breit potential. Thirdly, while we do not explicitly employ the "hidden color" method used by Harvey,[5] all the hidden color phenomena are fully and correctly incorporated.

In Sec. II we discuss the general N-quark Hamiltonian while in Sec. III we focus on the three-quark problem briefly to introduce some ideas that will be used in Sec. IV. That section deals with the particular implementation of the resonating group method particular to the six-quark problem. In Sec. V we indicate the simplifying approximations necessary to keep the problem tractable.

II. THE HAMILTONIAN

The nuclear force is defined to be the interaction between hadrons. The underlying field theory is the quark-gluon $SU(3)_C$ non-Abelian gauge theory.[14] All hadrons, including nuclei, must be eigenfunctions of the many-body n-quark Hamiltonian. From lattice gauge calculations[15] it is now known that for nonrelativistic velocities, the gluon and antiquark degrees of freedom freeze out and low energy quantum chromodynamics collapses to a potential theory with massive quarks: the nonrelativistic quark model. The proceeding statement has been amply demonstrated for the two quark interaction.

The n-quark Hamiltonian which includes only the two-quark potential is

$$H(n) = nm + \sum_{i=1}^{n} T_i + \frac{1}{2}\sum_{i\neq j=1}^{n}\sum_{\alpha=1}^{8} \lambda_i^\alpha \lambda_j^\alpha V_{ij}, \tag{6}$$

where

$$T_i = \frac{p_i^2}{2m} - \frac{p_i^4}{8m^3c^2}. \tag{7}$$

The λ_i^α are the $SU(3)_C$ color generators of the quarks, and

$$V_{ij} = V_{ij}^{\mathrm{OGE}} + V_{ij}^{\mathrm{CONF}}. \tag{8}$$

The quarks are assumed to have the same effective mass m (isospin symmetry) and units are chosen so that $c = 1$. The two-quark potential is split into two pieces: a long-range confining piece and a short range one-gluon exchange piece. From lattice gauge theory one expects[16]

$$V_{ij}^{\mathrm{CONF}} = -\frac{k}{\mu}(1 - e^{-\mu r}) \tag{9}$$

where $r = |\mathbf{r}_i - \mathbf{r}_j|$. In Eq. (9), μ^{-1} is a few fermi; Eq. (9) comes from explicit lattice gauge theory numerical results.[16] For $\mu r << 1$, V_{ij}^{CONF} is approximately $-kr$ while for $\mu r >> 1$, V_{ij}^{CONF} is approximately $-k/\mu$. The breakdown of the linear confining potential occurs due to quark-antiquark pair creation which shields the original quarks: when the pair becomes sufficiently far apart such that the energy in the string exceeds $2m$, the string breaks, pair creation occurs and nonrelativistic quantum mechanics no longer applies. The Hamiltonian becomes a field operator with explicit gluon and antiquark degrees of freedom. Thus the long-range contribution in Eq. (9) is not a low energy potential and hence we have excluded it from Eq. (8). Our general formalism is, however, independent of the particular choice of V^{CONF}. Ignoring this term completely, however, has led to some needless concern about spurious van der Waals forces between hadrons.[17] The one-gluon short-range potential (abbreviated V^{OGE}) is

$$V_{ij}^{\mathrm{OGE}} = \alpha_s\{\ r^{-1} - \tfrac{1}{2}m^{-2}r^{-3}[r^2\mathbf{p}_i\cdot\mathbf{p}_j + \mathbf{r}\cdot(\mathbf{r}\cdot\mathbf{p}_i)\mathbf{p}_j] - m^{-2}\pi\delta^3(\mathbf{r})[1+\tfrac{8}{3}\mathbf{s}_i\cdot\mathbf{s}_j]$$
$$-\tfrac{1}{2}m^{-2}r^{-3}[\mathbf{r}\times\mathbf{p}_i\cdot(\mathbf{s}_i+2\mathbf{s}_j) - \mathbf{r}\times\mathbf{p}_j\cdot(\mathbf{s}_j+2\mathbf{s}_i)] + m^{-2}r^{-5}[r^2\mathbf{s}_i\cdot\mathbf{s}_j - 3(\mathbf{s}_i\cdot\mathbf{r})(\mathbf{s}_j\cdot\mathbf{r})]\} \tag{10}$$

where $\mathbf{s}_i$ is the spin of the ith quark.

There will be an n-quark wave function $\Psi = \Psi(\mathbf{r}_1,\mathbf{s}_1,\mathbf{t}_1,\ldots,\mathbf{r}_n,\mathbf{s}_n,\mathbf{t}_n)$ with $\mathbf{r}_i,\mathbf{s}_i,\mathbf{t}_i$, respectively, the position, spin, and isospin of the ith quark. This wave function will satisfy the fundamental equation

$$H\Psi = E\Psi. \tag{11}$$

Our goal is to use Eq. (11) for $n = 6$ (the two nucleon system) to derive a Schrödinger equation involving only the nucleon variables. Since we use $\langle V_A\rangle$ etc., we need to look briefly at the $n = 3$ problem.

III. THE n=3 HAMILTONIAN

The Hamiltonian of Eq. (6) for $n = 3$, $H(3)$, can be expressed in Jacobi coordinates defined by

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{r}_1+\mathbf{r}_2-2\mathbf{r}_3) \tag{12}$$

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{r}_1-\mathbf{r}_2) \tag{13}$$

$$\mathbf{R} = \frac{1}{3}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3), \tag{14}$$

We denote the momenta canonically conjugate to $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, and $\mathbf{R}$, by $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{P}$, respectively. The Hamiltonian is independent of $\mathbf{R}$, so that the state can be a momentum eigenstate of the center of mass motion. The Hamiltonian can be expanded in increasing powers of $\mathbf{P}$ as

$$H(3) = H_0(\boldsymbol{\lambda},\boldsymbol{\rho},\mathbf{q},\mathbf{p}) + \mathbf{P}\cdot\mathbf{H}(\lambda,\rho,\mathbf{q},\mathbf{p}) + \frac{1}{6m}P^2$$
$$+P^2H_1(\boldsymbol{\lambda},\boldsymbol{\rho},\mathbf{q},\mathbf{p})$$
$$+\sum_{i,j}P_iP_jQ_{ij}(\boldsymbol{\lambda},\boldsymbol{\rho},\mathbf{q},\mathbf{p}) - \frac{P^4}{216m^3}\ , \tag{15}$$

where the subscripts i,j are now spatial indices and Q has the rotational properties of a quadrupole operator. Each of H_0, $\mathbf{H}$, H_1 and Q can be written as a sum of contributions arising from Eqs. (7) and (8), as follows:

$$H_0 = T_0 + V_0,\ \mathbf{H} = \mathbf{T}+\mathbf{V},$$
$$H_1 = T_1 + V_1,\ \ Q_{ij} = T_{ij}^Q + V_{ij}^Q. \tag{16}$$

The explicit forms of T_0, V_0, etc., are easily developed, but will not be needed for our discussion here.

Several important insights are readily discernable. First, we note that in a nonrelativistic theory only the terms of H_0 quadratic in momenta or independent of momenta, and $P^2/6m$, would occur in Eq. (15). The remaining terms arise from the relativistic corrections to both the kinetic energy and the potential energy. They are therefore of order $(v/c)^2$, where v is the quark speed. Now suppose that $\varphi(\boldsymbol{\lambda},\boldsymbol{\rho},\mathbf{P})\exp(i\mathbf{P}\cdot\mathbf{R})$ is an eigenfunction of $H(3)$ corresponding to an eigenvalue $E(\mathbf{P})$ which is the lowest eigenvalue compatible with the fixed value $\mathbf{P}$ for the momentum of the nucleon. Clearly $\varphi(\boldsymbol{\lambda},\boldsymbol{\rho},\mathbf{0}) \equiv \varphi_0(\boldsymbol{\lambda},\boldsymbol{\rho})$ is the ground state internal wave function for the baryon in its rest frame, and $E(\mathbf{0}) = M$ the rest energy of the baryon. The momentum dependence in $\varphi(\boldsymbol{\lambda},\boldsymbol{\rho},\mathbf{P})$ arises because the internal coordinates $\boldsymbol{\lambda},\boldsymbol{\rho}$ are not invariant when the baryon is given a Lorentz boost from rest to momentum $\mathbf{P}$, in contrast to the case when the boost is generated by a Galilean transformation. Such a momentum dependence is expected in order to produce the Lorentz contraction of the moving nucleon. However, $\varphi(\boldsymbol{\lambda},\boldsymbol{\rho},\mathbf{P})$ will differ from $\varphi_0(\boldsymbol{\lambda},\boldsymbol{\rho})$ only by contributions of order $(v/c)^2$. If Eq. (15) were exactly correct to order $(v/c)^2$, then to the same accuracy $E(\mathbf{P})$ would be given by $E(\mathbf{P}) = M+(P^2/2M)-(P^4/8M^3)$, where $M = E(\mathbf{0})$ is the rest energy of the baryon. It is also legitimate to use lowest order perturbation theory to obtain, with the same accuracy,

$$\int d^3\lambda\, d^3\rho\ \varphi_0(\boldsymbol{\lambda},\boldsymbol{\rho})H(3)\varphi_0(\boldsymbol{\lambda},\boldsymbol{\rho}) = M+\frac{P^2}{2M}-\frac{P^4}{8M^3}, \tag{17}$$

where

$$H_0(\boldsymbol{\lambda},\boldsymbol{\rho},\mathbf{q},\mathbf{p})\varphi_0 = M\varphi_0. \tag{18}$$

This discussion is of crucial importance in sorting out the relative motion of the nucleons from their internal structure. However, we shall discuss the procedure for doing so only in the context of the simplified theory described in Sec. V. Before we develop $H(6)$ we would note the physical significance of $H(3)$.

Kiefer[13] has solved Eq. (11) for $n = 3$. Using $m = 320$ MeV as the constituent quark mass, $\alpha_s/\pi = 0.678$ and $k = 6.8\times10^4$ MeV^2, Kiefer was able to predict all the known $N,\Delta\ J^\pi$ states and all the known photon decay amplitudes for transitions to the nucleon ground state with a χ^2 averaged error of 6.4 percent. For all the 38 $N,\Delta\ J^\pi$ states, he found $(v/c)^2 < 0.43$, which justifies the semirelativistic approximation to about the 10 percent level. Kiefer employed all the terms in $H(3)$ that would seem to resolve the problem of the supposed[18] inadequacy of $H(3)$ (with the two-quark potential truncated to include only the central and spin-orbit terms) to describe the baryon spectrum.

IV. THE n=6 VARIATIONAL SOLUTION

A six-quark system in an overall color singlet must have a color state belonging to the five dimensional [2,2,2] rep-

resentation of the symmetric group S_6, i.e., the group of permutations of the quarks, while, in order to satisfy the Pauli principle, the space-spin-isospin state must belong to the complementary [3,3] irreducible representation. If we use a mixed notation of wave functions for coordinate dependence and ket vectors for the state dependence on the discrete variables, i.e., spin s, isospin t, and color c, then the antisymmetry of the six-quark state is expressed symbolically by

$$|\Psi\rangle = \{(\Phi|st))^{[3,3]}|c\rangle^{[2,2,2]}\}^{[1,1,1,1,1,1]}. \tag{19}$$

Now let us consider the six quarks to be partitioned into cluster A containing quarks numbered 1, 2, and 3, with cluster B containing quarks 4, 5, and 6. The color state of cluster A can be either a singlet (associated with the [1,1,1] representation of S_3) or one of two octets (associated with the two components of the [2,1] representation of S_3), and similarly for cluster B. The five color states of the [2,2,2] representation of S_6, may be labeled by the color multiplets of the two clusters, and Eq. (19) can be rewritten as

$$|\Psi\rangle = (\Phi|st))_{11}|1_A 1_B\rangle + (\Phi|st))_{88}|8_A 8_B\rangle + (\Phi|st))_{88'}|8_A 8'_B\rangle + (\Phi|st))_{8'8}|8'_A 8_B\rangle + (\Phi|st))_{8'8'}|8'_A 8'_B\rangle, \tag{20}$$

where, for example, $|8_A 8_B\rangle$ is the color state in which cluster A is in the color octet state 8_A, cluster B is likewise in color 8_B, and the two color octet states are coupled to form an overall color singlet.

Equation (20) displays explicitly the "hidden color" that has played a major role in some quark model calculations.[5] This expansion, however, obscures the true physical situation by giving a preferred role to a *particular* partitioning of the six quarks into two clusters of three quarks each. We note that $|\Psi\rangle$ in Eq. (20) can be reconstructed by antisymmetrization applied to just one of the terms on the right side. (This is because the direct product representation [2,2,2]⊗[3,3] of S_6 contains the antisymmetric representation [1,1,1,1,1,1] *precisely* once.) Hence, *exactly the same physical state* as Eq. (20) may be written (N = normalization constant)

$$|\Psi\rangle = \frac{N}{6!}\sum_{\pi\in S_6}(-1)^{\pi}\,\pi[(\Phi|st))_{11}\,|1_A 1_B\rangle], \tag{21}$$

where the permutation signature, $(-1)^{\pi}$, is $\pm$ according to whether the permutation π is even or odd. Now $|1_A 1_B\rangle$ is antisymmetric under any odd permutation of the three quarks 1,2,3 or of the three quarks 4,5,6, and is also antisymmetric under the simultaneous interchange of 1 with 4, 2 with 5, and 3 with 6 (i.e., $A \longleftrightarrow B$). The space-spin-isospin function $(\Phi|st))_{11}$ possesses the complementary symmetries. When this symmetry is exploited, we find that the completely antisymmetric 6-quark state $|\Psi\rangle$ is given by

$$|\Psi\rangle = N\mathcal{P}[(\Phi|st))_{11}|1_A 1_B\rangle]\,, \tag{22}$$

where

$$\mathcal{P} = \tfrac{1}{10}[\,1 - (14) - (15) - (16) - (24) - (25) - (26) - (34) - (35) - (36)], \tag{23}$$

We note that Ψ of Eq. (20) or its equivalent form of Eq. (22) contains equal mixtures of all possible 3-quark color states. Once the quark coordinates are explicitly integrated out to reveal the nucleon-nucleon interaction, this color composition is no longer explicit. Nontheless, this information carries into the nucleon-nucleon interaction and hence into any two nucleon dynamics. This is consistent with the analysis of Brodsby and Ji[6] who found this equal distribution of color clusters in the deuteron wave function.

Let us further examine the space-spin-isospin state of the two color singlet clusters, with quarks 1,2,3 occupying cluster A and quarks 4,5,6 making up cluster B. We shall adopt coordinates defined by Eqs. (12)–(14) with a subscript A referring to cluster A and a B (with 1,2,3 replaced by 4,5,6) for cluster B. Let $\{|\Phi_\alpha(\boldsymbol{\lambda},\boldsymbol{\rho})\sigma_\alpha,\tau_\alpha)\}$ be a complete set of eigenstates of $H_0(\boldsymbol{\lambda},\boldsymbol{\rho},\mathbf{q},\mathbf{p})$ of Eq. (15), i.e., a complete set of states for a (color singlet) baryon at rest. Here σ_α is the total angular momentum or "spin" of the baryon, and τ_α is its isospin. The dependence on the coordinates $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$ is placed inside the ket symbol in recognition of the need for the coupling of orbital and quark spin angular momentum to yield the physical baryon spin σ_α, whose coupling is to be understood. Then $(\Phi|st))_{11}$ of Eqs. (20) and (22) can be expanded as

$$(\Phi|st))_{11} = \sum_{\alpha,\beta,S,T}\psi^{ST}_{\alpha\beta}(\mathbf{r})|\phi_\alpha(\boldsymbol{\lambda}_A,\boldsymbol{\rho}_A),\,\phi_\beta(\boldsymbol{\lambda}_B,\boldsymbol{\rho}_B),\,(\sigma_\alpha\sigma_\beta)S,\,(\tau_\alpha\tau_\beta)T\rangle\,, \tag{24}$$

where the center-of-mass motion of the pair of baryons is omitted since it separates from the balance of the problem. On the right $\psi^{ST}_{\alpha\beta}(\mathbf{r})$ is related to the probability amplitude that cluster A of quarks 1,2,3 will be in state α, while cluster B is in state β, with the clusters coupled to have total spin S and isospin T. From the antisymmetry of $(\Phi|st))_{11}$ under the interchange of all three quarks between the clusters, it follows that

$$\psi^{ST}_{\alpha\beta} = (-1)^{1+\sigma_\alpha+\sigma_\beta-S+\tau_\alpha+\tau_\beta-T}\,\psi^{ST}_{\alpha\beta}(-\mathbf{r}). \quad (25)$$

We use a variational procedure to establish a set of coupled equations for the functions $\psi^{ST}_{\alpha\beta}$ in Eq. (24). The variational quantity is

$$\begin{aligned} A &\equiv \langle\Psi|H|\Psi\rangle + \eta(1-\langle\Psi|\Psi\rangle) \\ &= N^2 \sum_{\alpha\beta ST}\sum_{\alpha'\beta'S'T'}\int dV\,\psi^{ST*}_{\alpha\beta}(\mathbf{r}) \\ &\quad\times\langle\phi^*_\alpha\phi^*_\beta\,STC|(H-\eta)\mathcal{P}\psi^{S'T'}_{\alpha'\beta'}(\mathbf{r})|\phi_{\alpha'}\phi_{\beta'}\,S'T'C\rangle + \eta, \end{aligned} \quad (26)$$

where $dV = d^3\lambda_A\,d^3\rho_A\,d^3\lambda_B\,d^3\rho_B\,d^3r$ and is invariant under arbitrary permutations of the six quarks, and we have written

$$|STC\rangle \equiv |(\sigma_\alpha\sigma_\beta)S,(\tau_\alpha\tau_\beta)T\rangle|1_A1_B\rangle. \quad (27)$$

The parameter η is the Lagrange multiplier for the normalization constraint $\langle\Psi|\Psi\rangle = 1$. The variational equations are

$$\frac{\partial A}{\partial\eta} = 0\,, \quad (28)$$

which ensures normalization, and

$$\frac{\partial A}{\partial\psi^{S'T'*}_{\alpha'\beta'}} = 0, \quad (29)$$

which establishes the coupled equations for the $\psi^{ST}_{\alpha\beta}$ functions. In Eq. (28), the variations must be consistent with Eq. (25). In deriving Eq. (26), we have used the symmetry of H under quark permutations, and the effective idempotency of $\mathcal{P}$ acting on any state in the expansion of Eq. (24). The same symmetry of H, together with the permutation symmetry built into the states $|\phi_\alpha\phi_\beta STC\rangle$, allows us to replace $\mathcal{P}$ in Eq. (23) by the simpler expression $\frac{1}{10}[1-9(36)]$, and similar manipulations permit the expectation value of H in Eq. (26) to be expressed in comparatively few terms. This is discussed more fully in Sec. VI. Equation (28) leads to the normalization condition

$$\sum_{\alpha\beta ST}\sum_{\alpha'\beta'S'T'}\int d^3r\,d^3r'\,\psi^{ST*}_{\alpha\beta}(\mathbf{r})K^{ST,S'T'}_{\alpha\beta,\alpha'\beta'}(\mathbf{r},\mathbf{r}')\psi^{S'T'}_{\alpha'\beta'}(\mathbf{r}') = 1\,, \quad (30)$$

where the kernel K is given by

$$\begin{aligned} K^{ST,S'T'}_{\alpha\beta,\alpha'\beta'} = \frac{N^2}{20}\{&\delta_{S'S}\delta_{T'T}[\delta_{\alpha'\alpha}\delta_{\beta'\beta}\delta^3(\mathbf{r}'-\mathbf{r}) + \epsilon'\delta_{\alpha\beta'}\delta_{\alpha'\beta}\delta^3(\mathbf{r}+\mathbf{r}')] \\ &- 9[\langle\alpha\beta STC|\delta^3(\mathbf{r}'-(36)\mathbf{r})(36)|\alpha'\beta'S'T'C\rangle \\ &+ \epsilon'\langle\alpha\beta STC|\delta^3(\mathbf{r}'+(36)\mathbf{r})(36)|\beta'\alpha'S'T'C\rangle]\} \end{aligned} \quad (31)$$

in which

$$(36)\mathbf{r} = \frac{1}{3}\mathbf{r} + \frac{2\sqrt{6}}{9}(\boldsymbol{\lambda}_A - \boldsymbol{\lambda}_B)$$

$$\epsilon' = (-1)^{1+\sigma_{\alpha'}+\sigma_{\beta'}+\tau_{\alpha'}+\tau_{\beta'}-S'-T'} \quad (32)$$

and for any $\mathcal{O}$,

$$\langle\alpha\beta STC|\mathcal{O}|\alpha'\beta'S'T'C\rangle = \int d^3\lambda_A\,d^3\lambda_B\,d^3\rho_A\,d^3\rho_B\langle\phi^*_\alpha\phi^*_\beta\,STC|\mathcal{O}|\phi_{\alpha'}\phi_{\beta'}\,S'T'C\rangle. \quad (33)$$

Equation (31) incorporates the symmetry of the $\psi^{ST}_{\alpha\beta}(\mathbf{r})$ expressed in Eq. (25), and in deriving Eq. (31), use has been made of the result

$$\epsilon'\epsilon\langle\beta\alpha STC|(36)|\beta'\alpha'S'T'C\rangle = \langle\alpha\beta STC|(36)|\alpha'\beta'S'T'C\rangle\,, \quad (34)$$

where ϵ is defined analogously to ϵ', but using the unprimed quantum numbers.

The 6-quark Hamiltonian, Eq. (6), can be written as

$$H(6) = H_A(3) + H_B(3) + V_{AB}\,, \quad (35)$$

where $H_A(3), H_B(3)$ are the 3-quark Hamiltonians, Eq. (15), for quarks 1,2,3 and quarks 4,5,6, respectively, and $V_{AB} = H(6) - H_A(3) - H_B(3)$. The overall center-of-mass motion factors from the problem and is therefore ignored in all that follows, so that the state $|\Psi\rangle$ refers only to the relative motion of the six quarks. Since the kinetic energy contributions to $H(6)$ are single particle operators, they do not contribute to V_{AB}. The matrix element of V_{AB} *vanishes* between states in which the clusters A and B are color singlets (easily proven by the Wigner-Eckhart theorem), and therefore V_{AB} contributes only an *exchange* term to the expectation value of $H(6)$ in the state $|\Psi\rangle$. The *absence* of a direct term, due to the color exchange nature of the quark-quark interaction, is in contradistinction to the otherwise analogous situation in molecular physics. It is now easy to show that

$$\langle\Psi|H(6)|\Psi\rangle = \sum_{\alpha\beta ST}\sum_{\alpha'\beta'S'T'}\int d^3r\, d^3r'\, \psi^{ST*}_{\alpha\beta}(\mathbf{r})H^{ST,S'T'}_{\alpha\beta,\alpha'\beta'}(\mathbf{r},\mathbf{r}')\psi^{S'T'}_{\alpha'\beta'}(\mathbf{r}') \ , \tag{36}$$

where

$$\begin{aligned} H^{ST,S'T'}_{\alpha\beta,\alpha'\beta'}(\mathbf{r},\mathbf{r}') = \frac{N^2}{20}\{\ & \delta_{S'S}\delta_{T'T}[\langle\alpha\beta STC|\delta^3(\mathbf{r}-\mathbf{r}')(H_A+H_B)|\alpha'\beta' STC\rangle \\ & + \epsilon'\langle\alpha\beta STC|\delta^3(\mathbf{r}+\mathbf{r}')(H_A+H_B)|\beta'\alpha' STC\rangle] \\ & - 9[\langle\alpha\beta STC|\delta^3(\mathbf{r}'-(36)\mathbf{r})(H_A+H_B+V_{AB})(36)|\alpha'\beta' S'T'C\rangle \\ & + \epsilon'\langle\alpha\beta STC|\delta^3(\mathbf{r}'+(36)\mathbf{r})(H_A+H_B+V_{AB})(36)|\beta'\alpha' S'T'C\rangle]\}. \end{aligned} \tag{37}$$

We note that both $K^{ST,S'T'}_{\alpha\beta,\alpha'\beta'}$ and $H^{ST,S'T'}_{\alpha\beta,\alpha'\beta'}$ satisfy the following symmetry rules, which we write out explicitly only for K.

$$\begin{aligned} K^{ST,S'T'}_{\alpha\beta,\alpha'\beta'}(\mathbf{r},\mathbf{r}') &= \epsilon' K^{ST,S'T'}_{\alpha\beta,\beta'\alpha'}(\mathbf{r},-\mathbf{r}') \\ &= \epsilon K^{ST,S'T'}_{\beta\alpha,\alpha'\beta'}(-\mathbf{r},\mathbf{r}') \\ &= \epsilon\epsilon' K^{ST,S'T'}_{\beta\alpha,\beta'\alpha'}(-\mathbf{r},-\mathbf{r}') \ . \end{aligned} \tag{38}$$

The variational condition Eq. (29) is now easily seen to yield the coupled-channel nonlocal eigenvalue equations

$$\sum_{\alpha'\beta'S'T'}\int d^3r'\,[H^{ST,S'T'}_{\alpha\beta,\alpha'\beta'}(\mathbf{r},\mathbf{r}') - \eta K^{ST,S'T'}_{\alpha\beta,\alpha'\beta'}(\mathbf{r},\mathbf{r}')]\psi^{S'T'}_{\alpha'\beta'}(\mathbf{r}') = 0 \ . \tag{39}$$

In principle, these must be solved subject to the normalization condition Eq. (30). In practice, it is obvious that some simplifying approximations are necessary. Before discussing these, however, it is appropriate to comment on the general formalism developed so far.

Equations (30) and (39) give an *exact formulation* of the problem of six quarks in an overall color singlet, *according to the Hamiltonian in Eq. (6)*. No new approximations are introduced beyond those already implicit in the Hamiltonian.

The formulation of Eqs. (30) and (39) describes the *relative motion of two color singlet clusters*, without regard to which quarks belong to which cluster.

Although this formalism avoids the explicit use of "hidden color," *the phenomena normally associated with "hidden color" are fully and correctly incorporated under the guise of exchange contributions to K and H.*

The interpretation of $\psi^{ST}_{\alpha\beta}(\mathbf{r})$, at least when the relative motion of the clusters is nonrelativistic, is as the probability amplitude for finding one cluster in internal state α, the other in state β, the intrinsic angular momentum of the two clusters coupled to S, with total isobaric spin T, and the centers of mass of the clusters separated by $\mathbf{r}$. However, because of the nondiagonal exchange contribution to $K^{ST,S'T'}_{\alpha\beta,\alpha'\beta'}(\mathbf{r},\mathbf{r}')$, this interpretation is exact only in the limit of large $\mathbf{r}$.

The interaction between the two clusters is supplied only by the exchange contribution to $H^{ST,S'T'}_{\alpha\beta,\alpha'\beta'}(\mathbf{r},\mathbf{r}')$. Thus the color algebra gives a null direct force between colorless clusters and only the exchange term survives. As mentioned before, this is in contradistinction to the intermolecular potential in chemistry where both the direct and exchange interactions are present. Consequently, the range of the interaction is controlled by the overlap between the states $|\phi_\alpha(\boldsymbol{\lambda}_A,\boldsymbol{\rho}_A),\phi_\beta(\boldsymbol{\lambda}_B,\boldsymbol{\rho}_B)STC\rangle$ and $(36)|\phi_{\alpha'}(\boldsymbol{\lambda}_A,\boldsymbol{\rho}_A),\phi_{\beta'}(\boldsymbol{\lambda}_B,\boldsymbol{\rho}_B)S'T'C\rangle$. Since this decreases exponentially or Gaussian-like with large internucleon separation, the nuclear force is short ranged. The truncation and other simplifications necessary to make the calculation practical are given in the next section.

V. SIMPLIFYING APPROXIMATIONS

The first simplification is to restrict the set of internal states to the nucleon ground state. In view of the nucleon-delta mass difference, ignoring the coupling of the delta and nucleon channels is reasonable for low energy nucleon-nucleon interactions. Our success in the deuteron calculation[10] adds support to this view.

In principle, the exact internal ground state [i.e. the $H(3)$ solution] for the nucleon should be used in Eq. (24). Perusal of Kiefer's solution[13] reveals the nucleon wave function to be extremely complex. In order to reduce the computational burden, we felt free to replace the exact internal nucleon state by a simple approximation in which the spin and orbital angular momentum are decoupled. We choose

$$\phi_\alpha(\boldsymbol{\lambda},\boldsymbol{\rho})|\sigma,\tau\rangle = \frac{\beta^3}{\pi^{3/2}}\exp[-\tfrac{1}{2}\beta^2(\lambda^2+\rho^2)]|\sigma,\tau\rangle. \tag{40}$$

The single parameter β may be chosen either by fitting the proton charge radius or by minimizing the expectation value $\langle\alpha,\sigma,\tau|H_0|\alpha,\sigma,\tau\rangle$ of the energy; we have chosen the former. When Eq. (39) is restricted to a single state, it greatly simplifies. After some algebra in which we exploit the symmetry $\epsilon\psi^{ST}(-\mathbf{r}) = \psi^{ST}(\mathbf{r}), \epsilon = (-1)^{1-S-T}$ and drop irrelevant constant multiplying factors, we find the simplified problem

$$\int d^3r'\,[H(\mathbf{r},\mathbf{r}') - \eta K(\mathbf{r},\mathbf{r}')]\,\psi^{ST}(\mathbf{r}') = 0, \tag{41}$$

$$\int d^3r\, d^3r'\, \psi^{ST*}(\mathbf{r})K(\mathbf{r},\mathbf{r}')\psi^{ST}(\mathbf{r}') = 1, \tag{42}$$

where

$$K(\mathbf{r},\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}') - 9\langle\delta^3(\mathbf{r}'-(36)\mathbf{r})(36)\rangle \tag{43}$$

and

$$H(\mathbf{r},\mathbf{r}') = \langle\delta^3(\mathbf{r}'-\mathbf{r})(H_A+H_B)\rangle - 9\langle\delta^3(\mathbf{r}'-(36)\mathbf{r})(H_A+H_B+V_{AB})(36)\rangle \tag{44}$$

In these equations we have used the abbreviated notation

$$\langle\mathcal{O}\rangle = \int d^3\rho_A\, d^3\rho_B\, d^3\lambda_A\, d^3\lambda_B\, ,\phi^*(\boldsymbol{\lambda}_A,\boldsymbol{\rho}_A)\phi^*(\boldsymbol{\lambda}_B,\boldsymbol{\rho}_B) \times\langle STC|\mathcal{O}|STC\rangle\phi(\boldsymbol{\lambda}_A,\boldsymbol{\rho}_A)\phi(\boldsymbol{\lambda}_B,\boldsymbol{\rho}_B) \tag{45}$$

in which ϕ is the Gaussian given in Eq. (40) and the exchange operator (36), if it occurs in $\mathcal{O}$ is understood to operate on everything to its right. We have included the δ function inside the matrix element to indicate that it will be used to eliminate one of the internal variables in those terms involving (36).

If the Gaussian described the exact nucleon ground state, it would make no change (to order v^2/c^2 in the quark speeds) if we replaced H_A+H_B in Eq. (44) by

$$H_A+H_B \to H_A+H_B-\langle H_A+H_B\rangle+2M+\frac{P^2}{M}-\frac{P^4}{4M^3}. \tag{46}$$

[See Eq. (17) and the discussion preceding it.] We make this substitution as a consistent way of compensating for the poor estimate of the baryon ground state energy provided by our approximate wave function ϕ. We also write

$$\eta = 2M+E\ , \tag{47}$$

where E is the relative energy of the nucleon pair. As a final approximation, we drop the relativistic correction to the nucleon kinetic energy while retaining the corrections for the motion of the quarks inside each nucleon.

If we fail to make the substitution of Eqs. (46) and (47) when we use the approximate nucleon ground state Eq. (40), then the computation would be inconsistent: the Schrödinger equation that would have followed would have been distorted due to errors in the description of the internal nucleon wave function. The distortion would lead to several unphysical effects, including a nucleon-nucleon potential which would have been nonzero at large distances (corresponding to errors in estimating the nucleon rest energy) and a dependence in the kinetic energy for the relative motion of the nucleons on the parameters of the quark potential (corresponding to errors in estimating the P^2 terms). The use of Eqs. (46) and (47), which provides the best estimate of the P^2 terms and of η consistent with the approximate internal wave function, eliminates these errors.

In the next section we shall evaluate Eq. (41) in detail to extract the nucleon-nucleon interaction, but to clearly display how we use Eqs. (46) and (47) in Eq. (41), it is convenient to use an even more symbolic notation. We write Eq. (41) as

$$\langle(H_A+H_B+V_{AB})\mathcal{P}\rangle - \eta\langle\mathcal{P}\rangle = 0$$

where $\mathcal{P}$ is the operator $1-9(36)$ and $\langle\rangle$ means internal coordinate integration; the effect of the (36) operator is to require the δ function, $\delta^3(\mathbf{r}'-(36)\mathbf{r})$. We can add a term $\langle V_{AB}\rangle$ with impunity since the color algebra matrix elements cause this to vanish. Now we make the substitution of Eqs. (46) and (47) to obtain

$$\left\langle\left[(H_A+H_B+V_{AB})-\langle H_A+H_B\rangle+2M+\frac{P^2}{M}\right]\mathcal{P}\right\rangle - 2M\langle\mathcal{P}\rangle - E\langle\mathcal{P}\rangle = 0,$$

or equivalently

$$\frac{P^2}{M} - 9\left\langle\frac{P^2}{M}(36)\right\rangle + 2M\langle\mathcal{P}\rangle - 9\langle[H_A+H_B-\langle H_A+H_B\rangle](36)\rangle - 9\langle V_{AB}(36)\rangle - 2M\langle\mathcal{P}\rangle - E\langle\mathcal{P}\rangle = 0\ .$$

Now we write

$$\langle H_A+H_B\rangle = \langle H_A+H_B\rangle_0 + \langle H_A+H_B\rangle_P\ ,$$

where $\langle H_A+H_B\rangle_0$ means $P=0$ and $\langle H_A+H_B\rangle_P$ involves only these terms with momentum dependence P^2. The approximation of Eq. (46) is

$$\langle H_A+H_B\rangle_P = \frac{P^2}{M}\ ,$$

which we use to yield finally

$$\frac{P^2}{M} - 9\langle[H_A+H_B-\langle H_A+H_B\rangle_0+V_{AB}](36)\rangle - E\langle\mathcal{P}\rangle = 0. \tag{48}$$

This is the nonlocal nucleon-nucleon Schrödinger equation, Eq. (1), based on the starting Hamiltonian Eq. (6) and incorporating the approximation of keeping only the nucleon ground state as the internal scattering state. In the last sections we process Eq. (48) to Eq. (1) and see

how the tensorial properties of the quark-quark potential give rise to the tensorial properties of the nuclear interaction. In the accompanying paper[10] we demonstrate that its solutions give good results for the di-nucleon state. Furthermore its inherent *nonlocality* is the explanation for the experimental result that fitting low energy nucleon-nucleon interactions by sums of local phenomenological nuclear potentials requires a menagerie of functions.

VI. POTENTIAL ENERGY GENERIC TERMS

For the potential energy contributions, we must calculate

$$U(\mathbf{r},\mathbf{r}') = -9\langle[V_A + V_B - \langle V_A + V_B\rangle_0 + V_{AB}](36)\rangle. \tag{49}$$

Before we display the details of the integrations, we show that comparatively few generic terms occur. Consider first of all

$$\langle[V_A+V_B](36)\rangle = \langle[V_{12}+V_{13}+V_{23}+V_{45}+V_{46}+V_{56}](36)\rangle. \tag{50}$$

The wave functions implicit in Eq. (50) are invariant under $[S_3(A) \times S_3(B)] \overset{\sigma}{\times} S_2$ in which A involves 1,2,3, B involves 4,5,6 and $S_2 = [e,(14)(25)(36)]$ exchanges A and B. (The $\overset{\sigma}{\times}$ means semi-direct product). Now, consider V_{13} which we may write as $(12)V_{23}(12)$. But (12) commutes with (36), and hence V_{13} is equivalent to V_{23}. Similarly, V_{46} is equivalent to V_{56}. Therefore

$$\langle[V_A + V_B](36)\rangle = \langle[V_{12} + 2V_{23} + V_{45} + 2V_{56}](36)\rangle. \tag{51}$$

But $V_{45} = (14)(25)(36)V_{12}(14)(25)(36)$, and $V_{56} = (14)(25)(36)V_{23}(14)(25)(36)$, and since (14)(25)(36) commutes with (36) we have

$$\langle[V_A + V_B](36)\rangle = \langle[2V_{12} + 4V_{23}](36)\rangle. \tag{52}$$

Next, consider $\langle V_{AB}(36)\rangle$. We may write

$$V_{AB} = V_{14} + V_{15} + V_{16} + V_{24} + V_{25} + V_{26} + V_{34} + V_{35} + V_{36}. \tag{53}$$

By the arguments used earlier, the terms with subscripts 14, 15, 24, and 25 are all equivalent, as are also 16 and 26. Furthermore, because of the S_2 invariance, 34 and 35 are equivalent to 26. Thus

$$\langle V_{AB}(36)\rangle = 4\langle V_{14}(36)\rangle + 4\langle V_{26}(36)\rangle + \langle V_{36}(36)\rangle. \tag{54}$$

When we use Eqs. (52) and (54), Eq. (49) becomes

$$U(\mathbf{r},\mathbf{r}') = -9[2\langle V_{12}(36)\rangle + \langle V_{36}(36)\rangle + 4\langle V_{14}(36)\rangle + 4\langle V_{23}(36)\rangle + 4\langle V_{26}(36)\rangle] + 9\langle\langle V_A + V_B\rangle_0(36)\rangle. \tag{55}$$

We note that $U(\mathbf{r},\mathbf{r}')$ is manifestly Hermitian since all the terms save $\langle V_{23}(36)\rangle$ and $\langle V_{26}(36)\rangle$ are self-Hermitian and those two are Hermitian conjugates. These generic terms are illustrated graphically in Fig. 1.

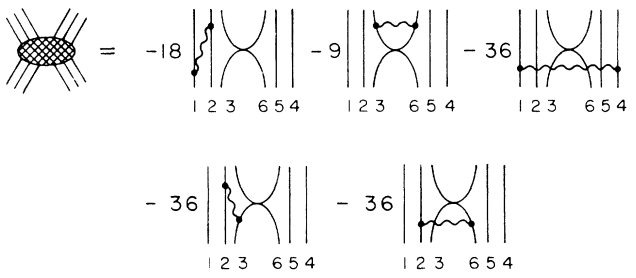

FIG. 1. Graphical illustration of the five generic terms. The solid lines represent quark trajectories and the wavy line represents the V_{ij} quark-quark interaction. Each of the five generic terms has its own weight.

With these findings in hand, we are ready to present our results. We display the kernel separately in the next section and then display the nonlocal potential term in Sec. VIII.

VII. THE KERNEL $K(\mathbf{r},\mathbf{r}')$

The kernel, $K(\mathbf{r},\mathbf{r}')$, is given by Eq. (43). We want to display its explicit form using the wave function Eq. (40). If we define $M(\mathbf{r},\mathbf{r}')$ by

$$K(\mathbf{r},\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}') + M(\mathbf{r},\mathbf{r}')\langle STC|(36)|STC\rangle, \tag{56}$$

then the integration implicit in Eq. (43) becomes explicitly

$$M(\mathbf{r},\mathbf{r}') = -9\int d^3\rho_A\, d^3\rho_B\, d^3\lambda_A\, d^3\lambda_B\, \phi^*(\boldsymbol{\lambda}_A,\boldsymbol{\rho}_A)\phi^*(\boldsymbol{\lambda}_B,\boldsymbol{\rho}_B)\delta^3(\mathbf{r}' - (36)\mathbf{r})(36)\phi(\boldsymbol{\lambda}_A,\boldsymbol{\rho}_A)\phi(\boldsymbol{\lambda}_B,\boldsymbol{\rho}_B). \tag{57}$$

The spin-isospin, color matrix element of the (36) exchange operator may be easily evaluated using standard angular momentum recoupling for the spin-isospin part and the standard S_6 symmetric group matrices for the color part. The resulting expression looks complicated when expressed in terms of the recoupling 9-j symbols, but reduces, after considerable algebra, to the following simple expression:

$$\langle STC|(36)|STC\rangle \equiv B^{ST} = \tfrac{100}{243}ST - \tfrac{22}{81}(S+T) + \tfrac{7}{27}. \tag{58}$$

The integration over the internal coordinates is reasonably straightforward and can be made if a change of variables is done from $\boldsymbol{\lambda}_A, \boldsymbol{\lambda}_B$ to $\boldsymbol{\sigma}, \boldsymbol{\tau}$ where

$$\boldsymbol{\sigma} = \tfrac{1}{2}(\boldsymbol{\lambda}_A + \boldsymbol{\lambda}_B),\ \boldsymbol{\tau} = \tfrac{1}{2}(\boldsymbol{\lambda}_A - \boldsymbol{\lambda}_B). \tag{59}$$

Then $(36)\boldsymbol{\sigma} = \boldsymbol{\sigma}$ and $(36)\boldsymbol{\tau} = \frac{2}{\sqrt{6}}\mathbf{r} - \frac{1}{3}\boldsymbol{\tau}$ while $(36)\mathbf{r} = \frac{1}{3}\mathbf{r} + \frac{4\sqrt{6}}{9}\boldsymbol{\tau}$. The result is

$$M(\mathbf{r},\mathbf{r}') = -\frac{3^6\sqrt{3}}{2^6}\frac{\beta^3}{\pi^{3/2}}\exp[-\beta^2(\tfrac{15}{16}r^2 + \tfrac{15}{16}r'^2 - \tfrac{9}{8}\mathbf{r}\cdot\mathbf{r}')]\,. \tag{60}$$

VIII. THE NONLOCAL POTENTIAL TERMS

The various potential terms of Eq. (1) arise from the first expectation value of Eq. (48). We begin with $H_A + H_B$. It is convenient to decompose $H_A + H_B$ into kinetic and potential energy terms. The kinetic energy terms in H_A and H_B are those given in Eq. (15) with the P^4 term ignored. In $\langle H_A + H_B\rangle_0$ only T_0 occurs. The constant $6m$ cancels and these kinetic energy contributions to Eq. (48) yield

$$U_0(\mathbf{r},\mathbf{r}') \equiv -9\langle\delta^3(\mathbf{r}' - (36)\mathbf{r})(T'(A) + T'(B) - \langle T'(A) + T'(B)\rangle_0)(36)\rangle$$

where $T' = T - 3m$. In more detail this is

$$U_0(\mathbf{r},\mathbf{r}') = -9\int d^3\rho_A\, d^3\rho_B\, d^3\lambda_A\, d^3\lambda_B\, \phi^*(\boldsymbol{\lambda}_A,\boldsymbol{\rho}_A)\phi^*(\boldsymbol{\lambda}_B,\boldsymbol{\rho}_B)\delta^3(\mathbf{r}' - (36)\mathbf{r})$$
$$\times\langle STC|[T'(\boldsymbol{\lambda}_A,\boldsymbol{\rho}_A) + T'(\boldsymbol{\lambda}_B,\boldsymbol{\rho}_B) - \langle T'_0(A) + T'_0(B)\rangle_0](36)|STC\rangle\phi(\boldsymbol{\lambda}'_A,\boldsymbol{\rho}_A{}')\phi(\boldsymbol{\lambda}'_B,\boldsymbol{\rho}_B{}'), \tag{61}$$

in which $\boldsymbol{\rho}_A{}' = (36)\boldsymbol{\rho}_A$, etc. As with the kernel, the integrations are easily done after a change of variables from $\boldsymbol{\lambda}_A, \boldsymbol{\lambda}_B$ to $\boldsymbol{\sigma}, \boldsymbol{\tau}$ given by Eq. (59). In all the following nonlocal potential terms it is most convenient to use alternative variables $\mathbf{x}, \mathbf{y}, \mathbf{u}$, and $\mathbf{v}$ defined as follows

$$\mathbf{x} = \tfrac{3}{4}\beta(\mathbf{r} + \mathbf{r}'),\quad \mathbf{y} = \beta(\mathbf{r} - \mathbf{r}')\,,$$
$$\mathbf{u} = \frac{3\sqrt{5}}{20}\beta(3\mathbf{r}' - \mathbf{r}),\quad \mathbf{v} = \frac{3\sqrt{5}}{20}\beta(3\mathbf{r} - \mathbf{r}'). \tag{62}$$

These combinations of $\mathbf{r}$ and $\mathbf{r}'$ occur naturally in the generic terms. In terms of these variables then

$$U_0^{ST} = \frac{\beta^2}{64m}B^{ST}M(\mathbf{r},\mathbf{r}')\left(168 - 8(2x^2 + 9y^2) - \frac{\beta^2}{m^2}[190 - \tfrac{5}{3}(68x^2 + \tfrac{657}{16}y^2) - 5x^4 + \tfrac{135}{16}x^2y^2 + \tfrac{1863}{256}y^4]\right), \tag{63}$$

where we have explicitly attached the spin, isospin labels S and T to U_0. Similarly the kernel $K(\mathbf{r},\mathbf{r}')$ has S and T labels and we may rewrite Eq. (56) as

$$K^{ST} = \delta^3(\mathbf{r} - \mathbf{r}') + B^{ST}M(\mathbf{r},\mathbf{r}')\,, \tag{64}$$

and it is sometimes useful to have $M(\mathbf{r},\mathbf{r}')$ in terms of $\mathbf{x}$ and $\mathbf{y}$ as

$$M(\mathbf{r},\mathbf{r}') = -\frac{729\sqrt{3}}{64}\frac{\beta^3}{\pi^{3/2}}\exp\left[-\beta^2\left(\frac{x^2}{3} + \frac{3y^2}{4}\right)\right]. \tag{65}$$

In the other potential energy terms, the integrals often result in various confluent hypergeometric functions. We use

$$G_{i,j}(z) = {}_1F_1(i, j - 1/2; z)e^{-z}\,. \tag{66}$$

The linear confinement has the form

$$V_{ij} = -\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j\, k|\mathbf{r}_i - \mathbf{r}_j|\,,$$

and as indicated earlier we ignore the damping for large $|\mathbf{r}_i - \mathbf{r}_j|$ over to the exponential term. The resulting nonlocal potential term is

$$U_c^{ST}(\mathbf{r},\mathbf{r}') = \frac{2k}{3\beta\sqrt{\pi}}B^{ST}M(\mathbf{r},\mathbf{r}')[-8\sqrt{2} - 3\sqrt{\pi}\,y - 4G_{2,2}(x^2) + 4\sqrt{5}G_{2,2}(u^2) + 4\sqrt{5}G_{2,2}(v^2)]. \tag{67}$$

The $U(\mathbf{r},\mathbf{r}')$ of Eq. (1) can be decomposed into various tensorial terms which follow directly from the corresponding

terms in Eq. (10). Thus we have

$$U^{ST} = U^{ST}_{cen} + U^{ST}_{so} + U^{ST}_{ss} + U^{ST}_{ten} + U^{ST}_{mom}. \tag{68}$$

Each of these tensorial pieces is made of the weighted sum of generic terms as displayed in Eq. (55). After the change of variables is performed and the resulting integration done, these terms are complicated hypergeometric functions. We display a little of the details in the appendix. Enormous simplification can result by judicious use of the recursion relationships among the hypergeometric functions.[19] The central term is

$$U^{ST}_{cen}(\mathbf{r},\mathbf{r}') = -\tfrac{1}{3}(\alpha_s/\sqrt{\pi})\beta B^{ST} M(\mathbf{r},\mathbf{r}')\Bigg[\sqrt{2}\left(4-\frac{\beta^2}{m^2}\right) - 4\left(2G_{1,2}(x^2) - \frac{\beta^2}{m^2}\exp(-x^2)\right) + \frac{32\sqrt{5}}{5}\left(G_{1,2}(u^2) + G_{1,2}(v^2) - \frac{2\beta^2}{5m^2}[\exp(-u^2)+\exp(-v^2)]\right) - \frac{8\sqrt{\pi}}{3}\left(\frac{1}{y} - \frac{4\pi\beta^2}{9m^2}\delta^3(\mathbf{y})\right)\Bigg]. \tag{69}$$

The spin-orbit term is

$$U^{ST}_{so}(\mathbf{r},\mathbf{r}') = (\alpha_s/\sqrt{\pi})\frac{\beta^2}{m^2}\beta^3 M(\mathbf{r},\mathbf{r}')[i(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{S}] \times\left(12\,F^T G_{1,3}(x^2) + \frac{8\sqrt{\pi}}{3}G^T y^{-3} - \frac{16\sqrt{5}}{25}[F^T - G^T][G_{1,3}(u^2)+G_{1,3}(v^2)]\right). \tag{70}$$

The spin-spin term yields

$$U^{ST}_{ss}(\mathbf{r},\mathbf{r}') = \tfrac{32}{27}(\alpha_s/\sqrt{\pi})(\beta^2/m^2)\beta M(\mathbf{r},\mathbf{r}')\left(\frac{\sqrt{2}}{4}A^{ST}_{12} - A^{ST}_{14}\exp(-x^2) - \tfrac{8}{27}\pi^{3/2}A^{ST}_{36}\delta^3(\mathbf{y}) + \frac{16\sqrt{5}}{25}A^{ST}_{23}[\exp(-u^2)+\exp(-v^2)]\right). \tag{71}$$

The tensor term is

$$U^{ST}_{\rm ten}(\mathbf{r},\mathbf{r}') = \tfrac{8}{405}(\alpha_s/\sqrt{\pi})\frac{\beta^2}{m^2}\beta M(\mathbf{r},\mathbf{r}') \times\left(-\frac{\sqrt{2}}{27}\,k^T_{14}\,G_{1,4}(x^2)S_{AB}(\mathbf{x}) - \tfrac{5}{3}\sqrt{\pi}\,k^T_{36}\,y^{-5}S_{AB}(\mathbf{y}) + \frac{192\sqrt{5}}{25}\,k^T_{23}[G_{1,4}(u^2)S_{AB}(\mathbf{u}) + G_{1,4}(v^2)S_{AB}(\mathbf{v})]\right) \tag{72}$$

in which $S_{AB}(\mathbf{x})$ is the tensor operator given as usual by

$$S_{AB}(\mathbf{x}) = 3(\boldsymbol{\sigma}_A\cdot\mathbf{x})(\boldsymbol{\sigma}_B\cdot\mathbf{x}) - x^2(\boldsymbol{\sigma}_A\cdot\boldsymbol{\sigma}_B). \tag{73}$$

All these terms are obtainable without considerable effort, but the momentum term requires a good deal more work. The answer is

$$U^{ST}_{mom}(\mathbf{r},\mathbf{r}') = (\alpha_s/\sqrt{\pi})\frac{\beta^2}{m^2}\beta M(\mathbf{r},\mathbf{r}')B^{ST}\left(\frac{\sqrt{2}}{18}(8-9y^2) + \tfrac{10}{27}y^{-1} - \tfrac{1}{9}y^{-3}|\mathbf{x}\times\mathbf{y}|^2 + \frac{16\pi}{81}\delta^3(\mathbf{y}) + \tfrac{16}{9}G_{12}(x^2) - y^2[\tfrac{3}{2}G_{23}(x^2) - \tfrac{1}{2}G_{13}(x^2)] + \tfrac{1}{5}(\mathbf{x}\cdot\mathbf{y})^2 G_{24}(x^2) + W(\mathbf{u},\mathbf{v}) + W(\mathbf{v},\mathbf{u})\right) \tag{74}$$

where

$$W(\mathbf{u},\mathbf{v}) = \frac{32\sqrt{5}}{75}\left(\tfrac{23}{288}e^{-v^2} + \tfrac{41}{576}G_{11}(v^2) - \tfrac{567}{576}G_{12}(v^2) - \frac{u^2}{24}G_{12}(v^2) + \mathbf{u}\cdot\mathbf{v}\frac{11v^2}{36}G_{24}(v^2) + u^2v^2\tfrac{5}{72}G_{24}(v^2) - (\mathbf{u}\cdot\mathbf{v})^2\tfrac{5}{12}G_{24}(v^2)\right). \tag{75}$$

In the spin-orbit term, there occur the functions F^T and G^T. These are given as

$$F^T = -\tfrac{1}{3}\langle ST||\sigma_1(36)||ST\rangle/\sqrt{S(S+1)(S+2)} \quad (76)$$

and

$$G^T = -\tfrac{1}{3}\langle ST||\sigma_3(36)||ST\rangle/\sqrt{S(S+1)(S+2)} \quad (77)$$

in which the double bars denote reduced matrix elements. By using the explicit expressions for the 9-j symbols which occur in the evaluation of the reduced matrix elements, we are able to simplify these to

$$F^T = \frac{2T+3}{972} \quad (78)$$

and

$$G^T = \frac{10T-3}{324}. \quad (79)$$

The spin-spin term involves other spin-isospin matrix elements, A_{ij}^{ST}. The i,j subscripts label the generic term from which the A arises. In fact,

$$A_{ij}^{ST} = \tfrac{1}{4}\langle ST||\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j(36)||ST\rangle\sqrt{S(S+1)}. \quad (80)$$

By using standard recoupling techniques and explicit algebraic expressions for the 9-j symbols involved, these coefficients become

$$A_{12}^{ST} = \tfrac{1}{324}[-260ST + 150S + 150T - 99], \quad (81)$$

$$A_{14}^{ST} = \tfrac{1}{324}[-24ST + 72S + 42T - 99], \quad (82)$$

$$A_{36}^{ST} = \tfrac{1}{324}[-100ST + 66S + 102T - 9], \quad (83)$$

$$A_{23}^{ST} = A_{26}^{ST} = \tfrac{1}{324}[-20ST + 24S + 24T - 45]. \quad (84)$$

Finally, the last remaining symbols to be defined occur in the tensor term, the k_{ij}^T. These are reduced matrix elements of the tensor operator labeled again by the generic term from which they arise. After considerable algebra involving the 9-j symbols one arrives at deceptively simple expressions

$$k_{14}^T = \tfrac{1}{81}(T + \tfrac{3}{2}), \quad (85)$$

$$k_{36}^T = \tfrac{1}{81}(50T - 23), \quad (86)$$

$$k_{23}^T = k_{26}^T = \tfrac{1}{81}(-5T + 6). \quad (87)$$

The k_{12}^T is not displayed since that term vanishes. This concludes the essential results presented in this paper.

IX. CONCLUSION

The nonlocal Schrödinger equation for the two-nucleon system is now of the form

$$\int d^3r' \left(\frac{P^2}{M}\delta^3(\mathbf{r}-\mathbf{r}') + U_0(\mathbf{r},\mathbf{r}') + U_c(\mathbf{r},\mathbf{r}') + U(\mathbf{r},\mathbf{r}') - EK(\mathbf{r},\mathbf{r}')\right)\psi(\mathbf{r}') = 0, \quad (88)$$

where P is the momentum conjugate to $\mathbf{r}$; $U_0(\mathbf{r},\mathbf{r}')$, which arises out of the internal kinetic energy, is given by Eq. (63); $U_c(\mathbf{r},\mathbf{r}')$, which is the confinement contribution, is given by Eq. (67); and $U(\mathbf{r},\mathbf{r}')$ is the sum of all the contributions from OGE. The latter are given by Eq. (68). The kernel $K(\mathbf{r},\mathbf{r}')$ is given by Eqs. (56) and (60).

From the form of the OGE terms, it is clear that $U(\mathbf{r},\mathbf{r}')$ has the usual types of terms found in phenomenological nucleon-nucleon potentials: central, spin-orbit, spin-spin, and tensor. Because of the nonlocal nature of $U(\mathbf{r},\mathbf{r}')$ one cannot compare it directly with such phenomenological potentials. Instead, one must make a local potential approximation to Eq. (1) and for consistency it must be done through order P^2. Reed-Margetan[20] has developed such a local approximation. One can, however, immediately and directly apply Eq. (1) to the deuteron. We report the results of the calculation in the following paper.[10] Espinosa and Schmidt[21] have already used the deuteron numerical solution with good success in calculating the electron scattering form factors.

APPENDIX: THE V_{ij} INTEGRALS

As indicated in Sec. VIII, we give a few details of the V_{ij} integrals. We process the space dependence of the Hamiltonian terms by making the appropriate change of variables. First we change from $\boldsymbol{\rho}_A$, $\boldsymbol{\rho}_B$, $\boldsymbol{\lambda}_A$, $\boldsymbol{\lambda}_B$ to $\boldsymbol{\rho}_A$, $\boldsymbol{\rho}_B$, $\boldsymbol{\sigma}$, and $\mathbf{r}'$. The spin-orbit and momentum parts of V_{ij} involve momenta conjugate to the coordinates $\mathbf{r}_i$ and $\mathbf{r}_j$. The T_A and T_B involve momenta conjugate to the internal coordinates. To handle these we proceed as follows: (1) move the (36), where it occurs, to the left of the momentum term; (2) allow the *permuted* momenta to act on ϕ_A, ϕ_B, ψ, expressing the results in the form of the internal variables $\mathbf{r}$, and any form of $\mathbf{P}$ (conjugate with $\mathbf{r}$) acting on $\psi(\mathbf{r})$; (3) operate with (36) to express the results. This leaves a general form

$$\int d^3\rho_A\, d^3\rho_B\, d^3\sigma\, d^3\mathbf{r}'\, \phi_A\phi_B\mathcal{F}(36)\phi_A\phi_B\bar{\psi}(\mathbf{r})$$

in which $\mathcal{F}$ is a function of internal variables, $\mathbf{r}$, $\mathbf{r}'$ and $\bar{\psi}(\mathbf{r})$ is obtained from $\psi(\mathbf{r})$ by the operation of any $\mathbf{P}$ involved. For the generic terms, $\mathcal{F}$ will be a product of the results of any $\mathbf{p}$ operating on the internal functions (with the results expressed in terms of $\boldsymbol{\rho}_A$, $\boldsymbol{\rho}_B$, $\boldsymbol{\sigma}$, $\mathbf{r}$ and $\mathbf{r}'$) and $V_{ij}(\mathbf{r}_{ij})$. To perform the integrations we then make the further change of variables given in Table I for each of the generic terms. The guide is to require $\mathbf{z} = \mathbf{r}_{ij}$ to be one of the new variables and to choose a second variable $\mathbf{x}$ such that when $\rho_A^2 + \rho_B^2$ is written in terms of $\mathbf{x}$ and $\mathbf{z}$, no terms linear in $\mathbf{x}$ occur. We use

TABLE I. Defining equations, Jacobian, and change of variables.

Term	Defining equation for x	Jacobian	Variable change
$\mathbf{r}_{14}=\mathbf{z}$	$\rho_A=\mathbf{x}+\frac{\sqrt{2}}{2}[\mathbf{z}-\frac{3}{4}(\mathbf{r}+\mathbf{r}')]$	$2^{3/2}$	$\rho_A,\rho_B\rightarrow\mathbf{x},\mathbf{z}$
	$\rho_B=\mathbf{x}-\frac{\sqrt{2}}{2}[\mathbf{z}-\frac{3}{4}(\mathbf{r}+\mathbf{r}')]$		
$\mathbf{r}_{23}=\mathbf{z}$	$\rho_A=\sqrt{3}\,\mathbf{x}-\frac{2\sqrt{2}}{5}\mathbf{z}+\frac{3\sqrt{2}}{20}(3\mathbf{r}'-\mathbf{r})$	$2^{3/2}$	$\rho_A,\sigma\rightarrow\mathbf{x},\mathbf{z}$
	$\sigma=\mathbf{x}+\frac{\sqrt{6}}{5}\mathbf{z}-\frac{3\sqrt{6}}{40}(3\mathbf{r}'-\mathbf{r})$		
$\mathbf{r}_{26}=\mathbf{z}$	$\rho_A=\sqrt{3}\,\mathbf{x}-\frac{2\sqrt{2}}{5}\mathbf{z}+\frac{3\sqrt{2}}{2}(3\mathbf{r}-\mathbf{r}')$	$2^{3/2}$	$\rho_A,\sigma\rightarrow\mathbf{x},\mathbf{z}$
	$\sigma=\mathbf{x}+\frac{\sqrt{6}}{5}\mathbf{z}-\frac{3\sqrt{6}}{40}(3\mathbf{r}-\mathbf{r}')$		

$$\phi_A=\frac{\beta^3}{\pi^{3/2}}\exp\left(-\frac{\beta^2}{2}(\rho_A^2+\lambda_A^2)\right);$$
$$\phi_B=\frac{\beta^3}{\pi^{3/2}}\exp\left(-\frac{\beta^2}{2}(\rho_B^2+\lambda_B^2)\right) \tag{A1}$$

so that

$$\phi_A\phi_B(36)\phi_A\phi_B=\frac{\beta^{12}}{\pi^6}\exp[-\beta^2(\rho_A^2+\rho_B^2+2\sigma^2)]\times\exp[-\beta^2(\tfrac{15}{16}r^2+\tfrac{15}{16}r'^2-\tfrac{9}{8}\mathbf{r}\cdot\mathbf{r}')]. \tag{A2}$$

In all cases we are then led to an integral of the general form

$$\int d^3s\,s^p\exp(-as^2)\exp(b\mathbf{t}\cdot\mathbf{s})\,Y_{\lambda\nu}(\hat{s}).$$

To evaluate this, we first expand

$$\exp(b\mathbf{t}\cdot\mathbf{s})=4\pi\sum_{LM}I_L(bts)Y_{LM}(\hat{t})Y^*_{LM}(\hat{s}) \tag{A3}$$

in which

$$I_L(bts)=\frac{\Gamma(\frac{1}{2})}{2}\sum_k(bts/2)^{L+2k}k!\,\Gamma(L+k+\tfrac{3}{2})\,. \tag{A4}$$

We substitute Eq. (A4) into the integral and perform the angular integrations to find

$$\int d^3s\,s^p\exp(-as^2)\exp(b\mathbf{t}\cdot\mathbf{s})Y_{\lambda\nu}(\hat{s})=4\pi Y_{\lambda\nu}(\hat{t})\int ds\,s^{p+2}I_\lambda(bts)\exp(-as^2)$$
$$=Y_{\lambda\nu}(\hat{t})a^{-p+3/2}\left(\frac{bt}{2\sqrt{\alpha}}\right)^\lambda\frac{\pi\Gamma(\frac{1}{2})\Gamma[(\lambda+p+\frac{3}{2})]}{\Gamma(\lambda+\frac{3}{2})}\,{}_1F_1\left(\frac{\lambda+p+3}{2},\lambda+\tfrac{3}{2},\frac{b^2t^2}{4a}\right). \tag{A5}$$

When $b=0$ this reduces to a standard exponential integral and requires $\lambda=\nu=0$. That is

$$\int d^3s\,s^p\exp(-as^2)Y_{\lambda\nu}(\hat{s})=\frac{\sqrt{\pi}\,\Gamma[(p+3)/2]}{a^{(p+3)/2}}\delta_{\lambda0}\delta_{\nu0}. \tag{A6}$$

When $b\neq0$, it is convenient to rewrite the integral into the form

$$\int d^3\,s^{2n-2L-3}\exp[-(s^2-2\mathbf{s}\cdot\mathbf{t})]s^\lambda Y_{\lambda\nu}(\hat{s})=\frac{\pi\Gamma(\frac{1}{2})\Gamma(n)}{\Gamma(\lambda+3/2)}t^\lambda Y_{\lambda\nu}\,{}_1F_1(n,\lambda+\tfrac{3}{2};t^2). \tag{A7}$$

Equation (A7) is the standard form that we used throughout our derivation of $H(\mathbf{r},\mathbf{r}')$.

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