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Modified loop expansion for nuclear matter in a chiral model

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We consider loop expansions for the chirally symmetric linear σ model. The conventional loop expansion is reviewed, and the tachyon problems due to the large attractive $\sigma\pi$ and $\sigma\sigma$ couplings are investigated. We then study a modified loop expansion that sums the repulsive baryon loops to all orders at each order in meson loops. Renormalizability is explicitly demonstrated to modified one-loop order, which corresponds to the relativistic random-phase approximation. We show that tachyons present in the conventional loop expansion are avoided to a large extent, but ghosts arise due to the unphysical behavior of the vacuum loops in this field theory that is not asymptotically free. Implications and possible solutions to this problem are discussed.

I. INTRODUCTION

Reliable methods for calculations of nuclear phenomena at high energies or densities are necessary but not available at present. While traditional nonrelativistic nuclear physics based on nucleon and meson degrees of freedom is successful at energies small compared to the nucleon mass, the regime of perturbative quantum chromodynamics (QCD) sets in only at energies that are much larger than the nucleon mass. A large gap remains in between.

Renormalizable relativistic field theories with hadronic degrees of freedom can in principle provide a framework for systematic calculations with a minimal number of parameters. Quantum hadrodynamics[1] is a model of this type. To be realistic, any such model will contain large couplings, and it is essential to use nonperturbative techniques. The simplest of such techniques are the mean-field and relativistic Hartree approximations, which have been quite successful from a phenomenological point of view. These successes make it even more important to systematically improve these approximations.

Moreover, to properly connect traditional nuclear physics to the regime of perturbative QCD, hadronic models should maintain some symmetries of QCD. In spite of its phenomenological success, the Walecka model[2] of quantum hadrodynamics (QHD) lacks chiral symmetry, an important feature of both QCD and low-energy pion physics. In the present work, we consider the chirally symmetric linear σ model[3] and study its behavior using a loop expansion. Chiral symmetry imposes constraints on the form and magnitude of the nonlinear meson couplings, and pion dynamics is explicitly included. The relationship between the σ model and the Walecka model has recently been discussed in Ref. 4. The results indicate that the σ model can also provide a reasonable starting point for describing nuclear physics.

The relativistic Hartree approximation represents the first order of the familiar loop expansion.[5] Recently, the second-order ("two-loop") contribution to the energy density has been calculated in the Walecka model.[6] The loop expansion, however, appears not to be very useful: The two-loop results do not indicate any kind of convergence. One of the goals of the present work is to see if chiral symmetry can improve the situation. This is relevant since several existing calculations claim a successful description of nuclear matter in the σ model when certain loop contributions are included.[7-9]

In applying the loop expansion to the linear σ model, however, one immediately encounters a problem: The strong attractive $\sigma\pi$ interaction gives rise to a tachyonic pion ($m_\pi^2<0$) even at very low densities and in the one-loop approximation. This produces a complex effective potential. Strictly speaking, the one-loop effective potential is not defined at all, since the Legendre transformation from the generating functional to the effective potential is not possible. Although the loop expansion has been used previously to calculate the binding energy of nuclear matter in the linear σ model, the tachyon problem has traditionally been ignored[7,9] by omitting the pionic contributions completely. A similar problem occurs also for the σ meson, but at higher density. In Refs. 10 and 11 the tachyon problems associated with the pion and scalar propagators are recognized, but no satisfactory solution is given. Clearly, approximations that include σ loops but ignore π loops treat the σ and π asymmetrically and probably violate chiral symmetry.

A modified loop expansion was proposed by Weiss in Ref. 12. In this approach, the meson loops are included order-by-order in $\hbar$, but baryon loops (which provide repulsion) are summed to all orders. Weiss developed this technique for field theories with broken symmetry at finite temperature, where the effective potential can also be complex, and he argued that in some cases, the complex effective potential of the conventional loop expansion can be avoided. The modified loop expansion preserves both renormalizability and chiral symmetry. In Ref. 13, the σ model was considered in the zero-loop approximation to the modified loop expansion, which includes only baryon tadpoles. In this approximation, the tachyon problem does not occur.

Here we apply the modified loop expansion to the linear σ model. To one-loop order, which corresponds to a relativistic random-phase approximation, we demonstrate renormalizability at finite density and show that this expansion technique indeed moves the pionic tachyon pole to higher densities. Unfortunately, there is a price to pay: The summation of the baryon loops to all orders introduces additional ghost poles.

The ghost poles arise because the underlying hadronic model is not asymptotically free.[14–17] Ghost poles are also present in quantum electrodynamics, but they occur at energies above the Planck scale, because the electromagnetic coupling is small at zero momentum transfer. In the σ model, they occur at much lower energies and produce an imaginary part in the effective potential that is much larger than the real part. Thus, while the modified loop expansion can, to a certain extent, solve the tachyon problem, the resulting effective potential is dominated by its imaginary part, signifying an unstable ground state.[18] The occurrence of unwanted poles in the random-phase approximation is not unusual; indeed, similar poles produce an unphysical pion condensate in nonrelativistic nuclear models. In our case, the ghost poles come from the unphysical behavior of vacuum loops at high momenta.

The paper is organized as follows. In Sec. II we review the conventional loop expansion to first order for the chiral model and highlight the resulting problems. In Sec. III we describe the modified loop expansion and apply it to the chiral model. The renormalizability of the modified one-loop energy density is explicitly demonstrated in Sec. IV. In Sec. V we consider the special case of the conventional two-loop approximation for the Walecka model. We show how it can be obtained from the modified loop expansion and demonstrate agreement with the results obtained in Ref. 6. New results for the chiral model are presented in Sec. VI, and Sec. VII contains our conclusions.

II. ONE-LOOP APPROXIMATION

Systematic methods for calculating in strong-coupling field theories are rare because of the lack of a suitable expansion parameter. The loop expansion is an expansion in Planck's constant $\hbar$ or equivalently in the number of loops,[5] and it is nonperturbative in the mean fields. It can be obtained by systematic expansion of the exact path-integral representation of the generating functional. As applied to the Walecka model of quantum hadrodynamics,[1] keeping only the first-order terms is equivalent to the relativistic Hartree approximation. The two-loop energy density for this model has been evaluated in Ref. 6. The results do not indicate convergence of any kind for the loop expansion.

We now proceed to discuss the loop expansion for the linear σ model. The σ model is chirally symmetric and thus shares an important feature of QCD and correctly describes low-energy pion phenomenology. The Lagrangian is given by

$$\mathcal{L}_s=\bar{\Psi}[i\partial\!\!\!/-g(s+i\gamma_5\vec{\pi}\cdot\vec{\tau})]\Psi+\tfrac{1}{2}(\partial_\mu s\partial^\mu s+\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})-\tfrac{1}{4}\lambda(s^2+\vec{\pi}^2-v^2)^2+\epsilon s\ . \tag{2.1}$$

Here Ψ, s, and $\vec{\pi}$ are the nucleon, scalar meson, and pion fields, and g is the πN coupling constant. The parameters λ and v describe the strength of the meson self-couplings, and ϵ is a chiral symmetry breaking parameter related to the pion mass. This Lagrangian exhibits spontaneous symmetry breaking; the vacuum expectation value $\langle s\rangle$ of the classical scalar field, defined by minimizing the tree-level effective potential, is nonzero and provides the nucleon and the scalar meson with their masses. In the chiral limit ($\epsilon=0$) the pion is the massless Goldstone boson of the spontaneously broken symmetry.

In terms of the shifted scalar field

$$\phi\equiv\langle s\rangle-s \tag{2.2}$$

and the nucleon mass M, scalar meson mass m_s, and pion mass m_π defined by

$$M=g\langle s\rangle,\quad \epsilon=\frac{M}{g}m_\pi^2,\quad \lambda=\frac{m_s^2-m_\pi^2}{2M^2}g^2\ , \tag{2.3}$$

the Lagrangian reads

$$\mathcal{L}_\phi=\bar{\Psi}[i\partial\!\!\!/-(M-g\phi)-ig\gamma_5\vec{\pi}\cdot\vec{\tau}]\Psi+\tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi-m_s^2\phi^2)+\tfrac{1}{2}(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}-m_\pi^2\vec{\pi}^2)-V(\phi,\vec{\pi})\ , \tag{2.4}$$

where

$$V(\phi,\vec{\pi})=-g_1\phi(\phi^2+\vec{\pi}^2)+g_2(\phi^2+\vec{\pi}^2)^2\ , \tag{2.5}$$

with the abbreviations

$$g_1=g\frac{m_s^2-m_\pi^2}{2M},\quad g_2=g^2\frac{m_s^2-m_\pi^2}{8M^2}\ . \tag{2.6}$$

Since the nucleon mass, pion mass, and πN coupling are assumed known, the linear σ model has only one free parameter, the mass of the scalar meson. The coefficients of the nonlinear meson couplings are all related to each other and prescribed by chiral symmetry. It is possible to add a neutral, massive vector meson (ω) without destroying either the chiral symmetry or the renormalizability.

The generating functional $\mathcal{Z}$ with a source j for the scalar field is given by

$$\mathcal{Z}[j]=\exp\left[\frac{i}{\hbar}W[j]\right]=\mathcal{N}^{-1}\int\mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi\,\mathcal{D}\phi\,\mathcal{D}\vec{\pi}\exp\left[\frac{i}{\hbar}\int d^4x[\mathcal{L}_\phi+j\phi]\right]\ , \tag{2.7}$$

with the normalization constant

$$\mathcal{N}=\int\mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi\,\mathcal{D}\phi\,\mathcal{D}\vec{\pi}\exp\left[\frac{i}{\hbar}\int d^4x\,\mathcal{L}_\phi\right]_{\text{zero density}}\ . \tag{2.8}$$

The Lagrangian $\mathcal{L}_\phi$ [Eq. (2.4)] has to be supplemented by counterterms that are not written explicitly here and will be discussed later. Equation (2.7) is to be interpreted in the canonical ensemble at finite density ρ, which is achieved by imposing the appropriate boundary conditions on the baryon propagator used to evaluate the path integral. We will assume that all observable fields are uniform and that the ground state has well-defined parity, so that the expectation value $\langle \vec{\pi} \rangle$ of the pion field is zero to all orders in the coupling. The observed ground-state expectation value of the scalar field is given by

$$\langle \phi \rangle = \lim_{j \to 0} \left[\frac{\delta W}{\delta j} \right] . \tag{2.9}$$

The classical field equation for the scalar field in a uniform system (ϕ_0=const) reads

$$m_s^2 \phi_0 + V_\phi(\phi_0) = j , \tag{2.10}$$

with

$$V(\phi) \equiv V(\phi, \vec{\pi} = \vec{0}) = -g_1 \phi^3 + g_2 \phi^4 , \tag{2.11}$$

$$V_\phi(\phi) \equiv \frac{dV(\phi)}{d\phi} = -3 g_1 \phi^2 + 4 g_2 \phi^3 . \tag{2.12}$$

The loop expansion now proceeds by expanding the scalar field around its classical value,

$$\phi(x) = \phi_0 + \hbar^{1/2} \sigma(x) , \tag{2.13}$$

and by rescaling the fluctuating pion field:

$$\vec{\pi}(x) \to \hbar^{1/2} \vec{\pi}(x) . \tag{2.14}$$

Insertion into the generating functional Z produces

$$\begin{aligned} Z[j] = \mathcal{N}^{-1} \exp \left[\frac{i}{\hbar} \left[S[\phi_0] + \int d^4x \, j \phi_0 \right] \right] \\ \times \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}\sigma \mathcal{D}\vec{\pi} \exp \Big[i \int d^4x \{ \tfrac{1}{2} \sigma [-\Box - m_s^2 - V_{\phi\phi}(\phi_0)] \sigma + \bar{\Psi} [i \not{\partial} - (M - g\phi_0)] \Psi \\ + \tfrac{1}{2} \vec{\pi} [-\Box - m_\pi^2 - V_{\pi\pi}(\phi_0)] \vec{\pi} + \hbar^{1/2} g \bar{\Psi} \Psi \sigma - i \hbar^{1/2} g \bar{\Psi} \gamma_5 \vec{\tau} \Psi \vec{\pi} \\ + \hbar^{1/2} (g_1 - 4 g_2 \phi_0) \sigma (\sigma^2 + \vec{\pi}^2) - \hbar g_2 (\sigma^2 + \vec{\pi}^2)^2 \} \Big] , \end{aligned} \tag{2.15}$$

which is still exact. Here we have defined the derivatives of the potential as

$$V_{\phi\phi}(\phi) = \frac{d^2 V(\phi)}{d\phi^2} = 6(-g_1 \phi + 2 g_2 \phi^2) , \tag{2.16}$$

$$V_{\pi\pi}(\phi) = \left. \frac{\partial^2 V(\phi, \vec{\pi})}{\partial \vec{\pi}^2} \right|_{\vec{\pi} = \vec{0}} = -2 g_1 \phi + 4 g_2 \phi^2 , \tag{2.17}$$

and $S[\phi_0]$ is the classical action

$$S[\phi_0] = \int d^4x [-\tfrac{1}{2} m_s^2 \phi_0^2 - V(\phi_0)] \tag{2.18}$$

that gives rise to the classical or tree-level effective potential $U^{(0)}(\phi_0)$. For exact chiral symmetry,

$$U^{(0)}(\phi_0) = \frac{m_s^2}{8g^2} \frac{(M^2 - M^{*2})^2}{M^2} , \tag{2.19}$$

where the effective nucleon mass is defined by

$$M^* = M - g\phi_0 . \tag{2.20}$$

The physical minimum of $U^{(0)}(\phi_0)$ occurs at $\phi_0 = 0$.

The remaining terms in Eq. (2.15) describe the inverse scalar meson propagator as modified by the scalar meson self-interactions, the inverse baryon propagator in the presence of the classical scalar field, the modified inverse pion propagator, the interactions of the baryons with the fluctuating fields, and the boson field self-interactions. The last two groups of terms involve additional powers of $\hbar$ and therefore do not contribute to one-loop order. Thus, in the one-loop approximation, the path integrals can be evaluated separately for the scalar meson, baryon, and pion:

$$\mathcal{N}_\sigma^{-1} \int \mathcal{D}\sigma \exp \left[i \int d^4x \{ \tfrac{1}{2} \sigma [-\Box - m_s^2 - V_{\phi\phi}(\phi_0)] \sigma \} \right] = \exp \left[-\tfrac{1}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} \ln \left[1 - \frac{V_{\phi\phi}(\phi_0)}{k^2 - m_s^2 + i\eta} \right] \right] , \tag{2.21}$$

$$\mathcal{N}_B^{-1} \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \left[i \int d^4x \{ \bar{\Psi} [i \not{\partial} - (M - g\phi_0)] \Psi \} \right] = \exp \left[\int d^4x \int \frac{d^4k}{(2\pi)^4} \operatorname{tr} \ln \left[1 + \frac{g\phi_0}{\not{k} - M} \right] \right] , \tag{2.22}$$

$$\mathcal{N}_\pi^{-1} \int \mathcal{D}\vec{\pi} \exp \left[i \int d^4x \{ \tfrac{1}{2} \vec{\pi} [-\Box - m_\pi^2 - V_{\pi\pi}(\phi_0)] \vec{\pi} \} \right] = \exp \left[-\tfrac{3}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} \ln \left[1 - \frac{V_{\pi\pi}(\phi_0)}{k^2 - m_\pi^2 + i\eta} \right] \right] . \tag{2.23}$$

Here $\mathcal{N}_\sigma$, $\mathcal{N}_B$, and $\mathcal{N}_\pi$ are appropriate normalization constants determined from the noninteracting propagators. The boundary conditions on the fermion propagator are left unspecified and will be imposed later to generate the analytic structure appropriate for finite density.[1]

The one-loop effective action $\Gamma^{(1)}[\phi]$ is obtained from the connected generating function $W^{(1)}[j]$ by a Legendre transformation from the sources to the fields:

$$\Gamma^{(1)}[\phi]=W^{(1)}[j]-\int d^4x\, j\phi \; . \tag{2.24}$$

On the right-hand side, the source j drops out because the classical action is stationary about ϕ_0, which satisfies Eq. (2.10). The one-loop effective potential $U^{(1)}(\phi)$ is obtained from

$$\Gamma^{(1)}[\phi]=-\int d^4x\, U^{(1)}(\phi) \tag{2.25}$$

and in the linear σ model, apart from counterterms, is given by

$$\begin{aligned} U^{(1)}(\phi)=&\tfrac{1}{2}m_s^2\phi^2+V(\phi)\\ &-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left[1-\frac{V_{\phi\phi}(\phi)}{k^2-m_s^2+i\eta}\right]\\ &-3\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left[1-\frac{V_{\pi\pi}(\phi)}{k^2-m_\pi^2+i\eta}\right]\\ &+iN\hbar\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\ln\left[1+\frac{g\phi}{\not k-M}\right] \; . \end{aligned} \tag{2.26}$$

Here N denotes the baryon isospin degeneracy, and tr denotes a trace with respect to Dirac indices only.

The remaining integrals can be evaluated after dimensional regularization, and counterterms are introduced. As discussed in Ref. 13, the structure of the counterterms is restricted by chiral symmetry. We renormalize in the limit of exact chiral symmetry, thus minimizing the number of renormalization conditions. To order $\hbar$, the required counterterms in the original Lagrangian (in terms of the unshifted scalar field s) are

$$\begin{aligned} \delta\mathcal{L}_s=&\hbar[aM^2(s^2+\vec\pi^2)+b(s^2+\vec\pi^2)^2\\ &+c(\partial_\mu s\partial^\mu s+\partial_\mu\vec\pi\partial^\mu\vec\pi)] \; , \end{aligned} \tag{2.27}$$

where a, b, and c are constants to be determined by renormalization conditions. After shifting the scalar field by its vacuum expectation value, $\delta\mathcal{L}_s$ becomes

$$\begin{aligned} \delta\mathcal{L}_\phi=\hbar\Bigg[&\alpha_1\phi+\frac{\alpha_2}{2!}\phi^2+\frac{\alpha_3}{3!}\phi^3+\frac{\alpha_4}{4!}\phi^4+b\vec\pi^4\\ &+\frac{\beta_2}{2!}\vec\pi^2+\frac{\beta_3}{2!}\phi\vec\pi^2+\frac{\beta_4}{2!2!}\phi^2\vec\pi^2\\ &-\frac{1}{2!}\zeta_\pi\vec\pi\Box\vec\pi-\frac{1}{2!}\zeta_s\phi\Box\phi\Bigg] \; , \end{aligned} \tag{2.28}$$

where the constants α_i, β_i, and ζ are related to a, b, and c via[13]

$$\alpha_1=-\frac{2M^3}{g}\left[a+\frac{2b}{g^2}\right] , \tag{2.29}$$

$$\alpha_2=2M^2\left[a+\frac{6b}{g^2}\right] , \tag{2.30}$$

$$\alpha_3=-24\frac{Mb}{g} , \tag{2.31}$$

$$\alpha_4=24b , \tag{2.32}$$

$$\beta_2=2M^2\left[a+\frac{2b}{g^2}\right] , \tag{2.33}$$

$$\beta_3=-8\frac{Mb}{g} , \tag{2.34}$$

$$\beta_4=8b , \tag{2.35}$$

$$\zeta_\pi=\zeta_s=2c \; . \tag{2.36}$$

The finite parts of the counterterms are determined by renormalization conditions. We choose the first two conditions so that the pole of the pion propagator $\Delta_\pi(q)$ occurs at $q^2=0$ with unit residue in the vacuum. This determines the coefficients β_2 and ζ_π and automatically guarantees that the s-wave, isospin-averaged πN scattering length vanishes at threshold. We also observe that β_2, the coefficient of the pion mass counterterm, is proportional to α_1, the coefficient of the term linear in the scalar field ϕ. Thus proper renormalization of the pion mass automatically ensures, via chiral symmetry, that the minimum of the one-loop effective potential at zero density is still at $\phi=0$. The third renormalization condition is imposed on the scalar propagator $\Delta_s(q)$. For simplicity and since the scalar meson mass is not a physical observable, we require that

$$\Delta_s^{-1}(q^2=0)=-m_s^2 \; . \tag{2.37}$$

This condition guarantees an unchanged quadratic term $\frac{1}{2}m_s^2\phi^2$ in the effective potential (and energy density) and determines the coefficient α_2. Note that chiral symmetry does not allow us to independently impose unit residue at the pole; in fact, it is well known[3] that in the linear σ model, the residue at the pole of the scalar meson propagator is different from unity.

To one-loop order in uniform systems, only the polynomial terms in ϕ [the coefficients α_1 to α_4 in Eq. (2.28)] enter explicitly, since all the other terms involve an additional power of $\hbar$ from either $\vec\pi\to\hbar^{1/2}\vec\pi$ or $\phi=\phi_0+\hbar^{1/2}\sigma$ with $\phi_0=$const. Equations (2.29)–(2.32) imply that the α_i depend only on a and b, not on c. Therefore the meson mass renormalization conditions alone determine the renormalized one-loop effective potential. This is a nontrivial result, since the first *four* powers of ϕ generally have divergent coefficients.

Explicit evaluation[13] of the baryon contribution $U_B^{(1)}$ to the renormalized one-loop energy density at finite baryon density ρ yields

$$U_B^{(1)}(\phi,k_F)=\frac{N\hbar}{8\pi^2}\left[M^2(M^2-M^{*2})-\tfrac{3}{4}(M^4-M^{*4})-\tfrac{1}{2}M^{*4}\ln\left[\frac{M^{*2}}{M^2}\right]\right]+\frac{N\hbar}{\pi^2}\int_0^{k_F}k^2dk\sqrt{M^{*2}+k^2}\ , \quad (2.38)$$

where k_F is related to the density by

$$\rho=\frac{N}{3\pi^2}k_F^3\ , \quad (2.39)$$

and $M^*=M-g\phi$. This is the only one-loop contribution retained in Ref. 13. The contribution $U_s^{(1)}$ from the scalar meson is (see also Refs. 7 and 9)

$$U_s^{(1)}(\phi)=\frac{\hbar m_s^4}{64\pi^2}F\left[\frac{V_{\phi\phi}(\phi)}{m_s^2}\right]\ , \quad (2.40)$$

where

$$F(x)\equiv(1+x)^2\ln(1+x)-x-\tfrac{3}{2}x^2\ , \quad (2.41)$$

and $V_{\phi\phi}$ is defined in Eq. (2.16).

It is clear from this explicit formula that the scalar contribution becomes complex for $V_{\phi\phi}(\phi)<-m_s^2$. Therefore, in the limit of exact chiral symmetry, the contribution to the energy density from scalar loops is real only for

$$M^*/M>1/\sqrt{3}\ . \quad (2.42)$$

The value of M^* at any given density is determined by minimizing the energy density with respect to M^*. For example, if we keep only the baryonic and scalar contributions $U_B^{(1)}$ and $U^{(0)}+U_s^{(1)}$, and add a repulsive ω meson with parameters as determined in Ref. 13 for the chiral model, we find that the energy per nucleon

$$E=\frac{U^{(1)}}{\rho}-M \quad (2.43)$$

has a minimum in the range of real energy density only for Fermi momenta below

$$k_F^c=2.37\ \text{fm}^{-1}\ . \quad (2.44)$$

This is illustrated in Fig. 1, where the energy per nucleon of symmetric nuclear matter is given as a function of M^*/M for four different densities. At $k_F=1.3\ \text{fm}^{-1}$, there is a well-defined minimum (solid line). This minimum is already rather broad at $k_F=2\ \text{fm}^{-1}$ (dashed line), and for $k_F=k_F^c=2.37\ \text{fm}^{-1}$ (dotted line) the minimum occurs just at the critical value of M^*/M, where the effective potential acquires an imaginary part. At higher densities (the dot-dashed line is obtained for $k_F=3\ \text{fm}^{-1}$) the minimum occurs outside of the range of real effective potential.

The occurrence of a complex contribution to the energy density from the scalar meson is directly related to a tachyonic scalar propagator. From Eqs. (2.15) and (2.16), we see that the scalar meson propagator that enters in the one-loop approximation is

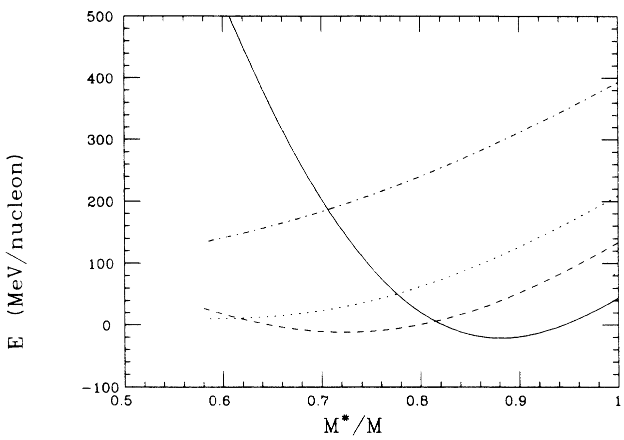


FIG. 1. The energy per nucleon $E=(U/\rho)-M$ as a function of M^*/M for nuclear matter in the conventional one-loop approximation with baryon and scalar meson loops. The solid line is obtained at $k_F=1.3\ \text{fm}^{-1}$, the dashed line at $k_F=2.0\ \text{fm}^{-1}$, the dotted line at $k_F=2.37\ \text{fm}^{-1}$, and the dot-dashed line at $k_F=3.0\ \text{fm}^{-1}$.

$$\Delta_s^{-1}(k,\phi)=k^2-[m_s^2+V_{\phi\phi}(\phi)]\ , \quad (2.45)$$

and therefore the squared effective scalar meson mass becomes negative, i.e., the scalar meson becomes tachyonic, if $V_{\phi\phi}(\phi)<-m_s^2$.

We can view the same problem in yet another way. To make the Legendre transformation of Eq. (2.24), it is necessary to invert the function $\phi_0(j)$, where $\phi_0(j)$ is determined from the field equation (2.10) or equivalently from finding the minimum of the function

$$\tilde{V}(\phi_0)=\tfrac{1}{2}m_s^2\phi_0^2+V(\phi_0)-j\phi_0\ . \quad (2.46)$$

Solutions $\phi_0(j)$ with $\tilde{V}_{\phi\phi}(\phi_0)<0$ correspond to unstable configurations that must be rejected, since the function $j(\phi_0)$ is not defined there and the Legendre transformation is not possible. The condition $\tilde{V}_{\phi\phi}(\phi)<0$ for a well-defined Legendre transformation is exactly the same as that derived before for a real effective potential.

We now consider the one-loop pionic contribution $U_\pi^{(1)}$ to the energy density, as obtained from Eq. (2.23). It can also be written in terms of the function F defined in Eq. (2.41):

$$U_\pi^{(1)}(\phi)=\frac{3\hbar m_\pi^4}{64\pi^2}F\left[\frac{V_{\phi\phi}(\phi)}{3m_\pi^2}\right]\ . \quad (2.47)$$

Thus, in the limit of small pion mass, the argument of the logarithm in the function F is negative even for a very small scalar field. We find, with

$$r\equiv\frac{m_\pi^2}{m_s^2-m_\pi^2}\ll 1\ , \quad (2.48)$$

a real effective potential only for $M^*/M>(1-r)$. A typical value for the scalar meson mass[13] is $m_s=769$ MeV, which yields $r\approx 0.03$. As for the scalar meson, this problem is already apparent in the pion propagator of Eq. (2.15):

$$\Delta_\pi^{-1}(k,\phi)=k^2-\tilde{m}_\pi^2,\quad \tilde{m}_\pi^2=m_\pi^2+\frac{\lambda}{g^2}(M^{*2}-M^2)\ . \tag{2.49}$$

Because $m_\pi^2 \ll m_s^2$, the pion becomes tachyonic for much smaller values of the scalar field than does the scalar meson.[19] *In the exact chiral limit, the pion becomes tachyonic for any positive value of the scalar field.*

In Fig. 2 we show the real and imaginary contributions from the scalar meson (solid and dashed lines) and from the pion (dot-dashed and dotted lines) to the one-loop effective potential as a function of the ratio M^*/M. The parameters are taken from Ref. 13. As discussed, the imaginary part from the pion occurs even at very small values of the scalar field. For both the scalar meson and the pion, the imaginary parts are of the same order of magnitude as the real parts and certainly not negligible.

This feature of the one-loop approximation to the linear σ model raises doubts about its usefulness. The physical origin of the problem is the strong attractive interaction between the mesons and the classical scalar field, while the potentially stabilizing repulsive baryon interactions are not included. In general, an imaginary part of the effective potential may be interpreted either as an actual dynamical instability of the ground state or as a shortcoming of the model or approximation used. In the conventional loop expansion for the linear σ model, however, it is quite clear that the imaginary part of the effective potential indicates an inappropriate approximation in light of the strong meson self-couplings. For the scalar meson, there have been attempts to solve the problem[7,9] by increasing the scalar mass by inclusion of parts of higher-loop contributions. This procedure, however, is rather *ad hoc* and unsatisfactory, since it does not correspond to a well-defined, systematic approximation scheme. The pion contribution to the one-loop effective potential is typically ignored completely (both the real part and the imaginary part). Thus π and σ are not treated symmetrically, as they should be in a chirally symmetric model.

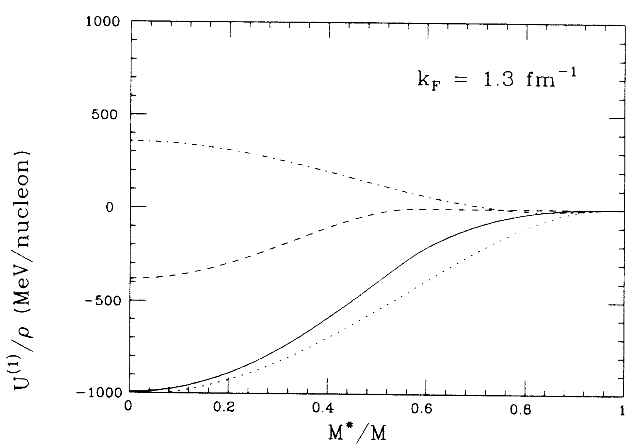


FIG. 2. Meson contributions to the energy per nucleon as a function of M^*/M at $k_F=1.3$ fm^{-1} in the conventional one-loop approximation. The solid and dashed lines are the real and imaginary parts of the scalar meson contribution $U_s^{(1)}(\phi,k_F)$. The dot-dashed and dotted lines are the real and imaginary parts of the pion contribution $U_\pi^{(1)}(\phi,k_F)$.

III. MODIFIED LOOP EXPANSION: FORMALISM

In Ref. 12 Weiss proposed a modified loop expansion that, in n-loop approximation, sums the meson loops to finite order n but includes iterated baryon loops to all orders. In view of the origin of the tachyon problems discussed at the end of Sec. II, the modified loop expansion certainly has the potential to solve these problems (or at least shift them to larger values of the scalar field, thus allowing for a real energy density at larger densities). We therefore briefly review this technique as applied to the linear σ model at zero temperature.

Our starting point is the exact path-integral representation of the generating functional $\mathcal{Z}$ given in Eq. (2.7). For simplicity, we suppress the scalar source, since we will need only the modified one-loop result, for which the Legendre transformation is immediate. The baryonic path integral can be integrated formally to yield

$$\mathcal{Z}[j=0]=\mathcal{N}_Z^{-1}\int\mathcal{D}\phi\,\mathcal{D}\vec{\pi}\,\det(G_0G^{-1})\exp\left[\frac{i}{\hbar}S[\phi,\vec{\pi}]\right]\ , \tag{3.1}$$

where

$$S[\phi,\vec{\pi}]=\int d^4x\left[\tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi-m_s^2\phi^2)+\tfrac{1}{2}(\partial_\mu\vec{\pi}\partial^\mu\vec{\pi}-m_\pi^2\vec{\pi}^2)-V(\phi,\vec{\pi})\right]\ , \tag{3.2}$$

and the normalization is provided by

$$\mathcal{N}_Z=\int\mathcal{D}\phi\,\mathcal{D}\vec{\pi}\,\det(G_0G^{-1})\times\exp\left[\frac{i}{\hbar}S[\phi,\vec{\pi}]\right]\Bigg|_{\text{zero density}}\ . \tag{3.3}$$

The inverses of the noninteracting and interacting baryon propagators are

$$G_0^{-1}(x,y)=(i\partial\!\!\!/_x-M)\delta^{(4)}(x-y)\ , \tag{3.4}$$

$$G^{-1}(x,y)=G_0^{-1}(x,y)+[g\phi(x)-ig\gamma_5\vec{\tau}\cdot\vec{\pi}(x)]\delta^{(4)}(x-y)\ , \tag{3.5}$$

and the compactly written convolution G_0G^{-1} is defined by

$$(G_0G^{-1})(x,y)\equiv\int d^4z\,G_0(x,z)G^{-1}(z,y) \tag{3.6}$$

$$=\delta^{(4)}(x-y)+gG_0(x,y)\phi(y)-igG_0(x,y)\gamma_5\vec{\tau}\cdot\vec{\pi}(y)\ . \tag{3.7}$$

The determinant of a matrix can be written as the exponential of the trace (Tr) with respect to space-time, spin, and isospin indices,

$$\mathrm{Tr}f(x,y)=\int d^4x\,\mathrm{tr}f(x,x)\ , \tag{3.8}$$

of the logarithm of the operator. Thus

$$\mathcal{Z}=\mathcal{N}_Z^{-1}\int\mathcal{D}\phi\,\mathcal{D}\vec{\pi}\exp\left[\frac{i}{\hbar}\tilde{S}[\phi,\vec{\pi}]\right]\ , \tag{3.9}$$

with

$$\tilde{S}[\phi,\vec{\pi}]=S[\phi,\vec{\pi}]-i\hbar\,\mathrm{Tr}\ln[\delta^{(4)}(x-y)+gG_0(x,y)\phi(y)-igG_0(x,y)\gamma_5\vec{\tau}\cdot\vec{\pi}(y)]\ . \tag{3.10}$$

This modified action $\tilde{S}$ contains all baryon contributions.

In the conventional loop expansion, the zeroth-order effective potential is obtained by minimizing $S[\phi_0,\vec{\pi}=\vec{0}]$ with respect to the background scalar field ϕ_0. The modified loop expansion defines a modified background field by minimizing instead $\tilde{S}[\phi_0,\vec{\pi}=\vec{0}]$. Thus, the modified classical field involves no meson loops, but iterated baryon loops are included to all orders. Any model with mesons and N identical species of baryons really contains two expansion parameters: a mesonic $\hbar$ and a baryonic $N\hbar$, each counting the number of the corresponding loops. While both are treated on equal footing in the conventional loop expansion, the modified n-loop approximation exploits the presence of two different expansion parameters and includes all powers of $N\hbar$ at each order $\hbar^n$ of meson loops, while retaining renormalizability and chiral symmetry.

Therefore, apart from counterterms, the modified zero-loop result is given by [compare Eq. (2.26)]

$$\tilde{U}^{(0)}(\phi,k_F)=\tfrac{1}{2}m_s^2\phi^2+V(\phi)+i(N\hbar)\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\ln\left[1+\frac{g\phi}{\not{k}-M}\right]\ . \tag{3.11}$$

Here ϕ_0 has been replaced by ϕ through the Legendre transformation, and the trace is now over Dirac indices only. The density dependence of $\tilde{U}^{(0)}$ is contained in the as yet unspecified boundary conditions on the baryon propagator.

After imposing the pole structure corresponding to the density ρ, which is related to k_F as in Eq. (2.39), and after renormalization, we obtain

$$\begin{aligned}\tilde{U}^{(0)}(\phi,k_F)&=U^{(0)}(\phi)+U_B^{(1)}(\phi,k_F)\\&=\frac{m_s^2}{8g^2}\frac{(M^2-M^{*2})^2}{M^2}+\frac{N\hbar}{8\pi^2}\left[M^2(M^2-M^{*2})-\tfrac{3}{4}(M^4-M^{*4})-\tfrac{1}{2}M^{*4}\ln\left[\frac{M^{*2}}{M^2}\right]\right]\\&\quad+\frac{N\hbar}{\pi^2}\int_0^{k_F}k^2dk\sqrt{M^{*2}+k^2}\ ,\end{aligned} \tag{3.12}$$

with $U^{(0)}$ from Eq. (2.19) and $U_B^{(1)}$ from Eq. (2.38). This is exactly the result obtained in Ref. 13, which has here been identified with the zeroth-order modified loop approximation. Notice that $\tilde{U}^{(0)}$ is an even function of M^*, as required by chiral symmetry.[3] Thus the asymmetric treatment of baryon and meson loops in the modified expansion does not violate chiral symmetry. This conclusion is also apparent in the conventional one-loop result, since the baryon, scalar, and pion vacuum contributions each scale as the fourth power of a different (independent) mass parameter: M^4, m_s^4, or m_π^4.

To one-loop order, we expand around the modified zeroth-order fields as in Eqs. (2.13) and (2.14) and use this expansion in the modified action. After expansion of the logarithm in Eq. (3.10), we obtain

$$\begin{aligned}\tilde{S}^{(1)}[\phi]=&-\int d^4x\,\tilde{U}^{(0)}(\phi,k_F)+\hbar\int d^4x\int d^4y\tfrac{1}{2}\sigma(x)\{[-\Box_x-m_s^2-V_{\phi\phi}(\phi)]\delta^{(4)}(x-y)-\Pi_s(x-y,\phi)\}\sigma(y)\\&+\hbar\int d^4x\int d^4y\tfrac{1}{2}\vec{\pi}(x)\{[-\Box_x-m_\pi^2-V_{\pi\pi}(\phi)]\delta^{(4)}(x-y)-\Pi_\pi(x-y,\phi)\}\vec{\pi}(y)\ ,\end{aligned} \tag{3.13}$$

with Π_s and Π_π the (baryonic) polarization insertions in the presence of the uniform background scalar field ϕ:

$$\Pi_s(x-y,\phi)=-ig^2(N\hbar)\mathrm{tr}[G_\phi(x-y)G_\phi(y-x)]\ , \tag{3.14}$$

$$\Pi_\pi(x-y,\phi)=ig^2(N\hbar)\mathrm{tr}[G_\phi(x-y)\gamma_5G_\phi(y-x)\gamma_5]\ . \tag{3.15}$$

The baryon propagator is

$$G_\phi^{-1}(x,y)=(i\not{\partial}_x-M+g\phi)\delta^{(4)}(x-y)\ , \tag{3.16}$$

with boundary conditions appropriate for finite density (see the discussion below). We recognize the modified scalar and pion propagators as

$$\Delta_s^{-1}(x-y)=[-\Box_x-m_s^2-V_{\phi\phi}(\phi)]\delta^{(4)}(x-y)-\Pi_s(x-y,\phi)\ , \tag{3.17}$$

$$\Delta_\pi^{-1}(x-y)=[-\Box_x-m_\pi^2-V_{\pi\pi}(\phi)]\delta^{(4)}(x-y)-\Pi_\pi(x-y,\phi)\ . \tag{3.18}$$

After replacing $\tilde{S}$ with $\tilde{S}^{(1)}$ in Eq. (3.9), it is possible to evaluate the remaining path integrals in Eq. (3.9) with respect to the scalar and pion fields, yielding an energy density

$$\tilde{U}^{(1)}(\phi,k_F)=\tilde{U}^{(0)}(\phi,k_F)+\tilde{U}_s^{(1)}(\phi,k_F)+\tilde{U}_\pi^{(1)}(\phi,k_F) \tag{3.19}$$

$$=\tilde{U}^{(0)}(\phi,k_F)-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left[\frac{k^2-m_s^2-V_{\phi\phi}(\phi)-\Pi_s(k,\phi)}{k^2-m_s^2-\Pi_s(k,0)}\right]$$
$$-\frac{3i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left[\frac{k^2-m_\pi^2-V_{\pi\pi}(\phi)-\Pi_\pi(k,\phi)}{k^2-m_\pi^2-\Pi_\pi(k,0)}\right]. \tag{3.20}$$

Here the denominators of the logarithms are the vacuum subtractions arising from $\mathcal{N}_Z^{-1}$. Note that $\Pi_s(k,0)$ and $\Pi_\pi(k,0)$ are to be evaluated at zero classical field and zero density. The polarization insertions are proportional to $N\hbar$, and the energy density therefore explicitly includes all powers of $N\hbar$. In the Walecka model, the corresponding result from the first-order modified loop expansion is the ring energy as investigated by Chin in Ref. 14, *generalized to include the vacuum contributions.* We have thus identified the first order of the modified loop expansion as the relativistic random-phase approximation (RPA). The modified scalar and pion propagators differ from those in the conventional loop expansion, as given in Eqs. (2.45) and (2.49), by the presence of the polarization insertions. We will see in Sec. IV how this stabilizes the result to a certain extent against the tachyon poles present in the conventional one-loop approximation.

To evaluate the integrals in Eq. (3.20), it is useful to perform a Wick rotation to Euclidean space. This is a straightforward procedure if all masses are real, but in the present calculation, we will have both tachyon and ghost poles (or branch points) to consider, even after the rotation. We will include all singularities that occur on the imaginary frequency axis, and we assume that no other singularities arise in the first and the third quadrants of the complex frequency plane. It is possible that there are additional poles at complex frequencies, but we will leave this as a topic for future study.

The constraint of finite density is implemented by replacing q_0 by $q_0-i\mu$ in the baryon propagators that appear in the Euclidean loop integrals for the meson polarizations. The Euclidean baryon propagator in momentum space is thus given by

$$iG_\phi^E(q^0,\mathbf{q})=\frac{i\gamma_0(q^0-i\mu)-\gamma\cdot\mathbf{q}+M^*}{\mathbf{q}^2+M^{*2}+(q^0-i\mu)^2}. \tag{3.21}$$

This is equivalent to working in the grand canonical ensemble, and now the effective action Γ is directly proportional to the grand canonical (thermodynamic) potential.

The modified one-loop result for the grand canonical potential density Ω at fixed chemical potential μ in the relativistic RPA becomes

$$\tilde{\Omega}^{(1)}(\phi,\mu)=\tfrac{1}{2}m_s^2\phi^2+V(\phi)+U_B^{(1)}(\phi,\mu)+\frac{\hbar}{2}\int\frac{d^4k_E}{(2\pi)^4}\ln\left[\frac{k_E^2+m_s^2+V_{\phi\phi}(\phi)+\Pi_s(k_E,\phi)}{k_E^2+m_s^2+\Pi_s(k_E,0)}\right]$$
$$+\frac{3\hbar}{2}\int\frac{d^4k_E}{(2\pi)^4}\ln\left[\frac{k_E^2+m_\pi^2+V_{\pi\pi}(\phi)+\Pi_\pi(k_E,\phi)}{k_E^2+m_\pi^2+\Pi_\pi(k_E,0)}\right], \tag{3.22}$$

apart from counterterms, which will be discussed in the following section. The squared Euclidean four-momentum k_E^2 is here defined by

$$k_E^2=k_0^2+\mathbf{k}^2, \tag{3.23}$$

and since we will henceforth always work in Euclidean space, we omit the corresponding index on the four-momentum variables for brevity.

The chemical potential μ corresponding to the density ρ can be evaluated from the thermodynamic relation

$$\rho=-\left[\frac{\partial\Omega}{\partial\mu}\right], \tag{3.24}$$

and the energy density at fixed density is then obtained by

$$U(\rho)=\Omega(\mu)+\mu\rho. \tag{3.25}$$

To order $\hbar$ this procedure can be simplified considerably.[20] We expand the thermodynamic potential $\tilde{\Omega}^{(1)}$ and chemical potential $\mu^{(1)}$ in powers of $\hbar$,

$$\tilde{\Omega}^{(1)}(\mu)=\tilde{\Omega}^{(0)}(\mu)+\hbar\delta\tilde{\Omega}^{(1)}(\mu), \tag{3.26}$$

$$\mu^{(1)}=\mu^{(0)}+\hbar\delta\mu^{(1)}, \tag{3.27}$$

and note that $\mu^{(0)}$ is implicitly defined in terms of the density by

$$\rho = -\left.\frac{\partial\tilde{\Omega}^{(0)}}{\partial\mu}\right|_{\mu^{(0)}} . \tag{3.28}$$

The energy density U at fixed density can then be obtained from

$$\tilde{U}^{(1)}(\rho) = \tilde{\Omega}^{(1)}(\mu^{(1)}) + \mu^{(1)}\rho = \tilde{\Omega}^{(0)}(\mu^{(0)}) + \hbar\delta\mu^{(1)}\left.\frac{\partial\tilde{\Omega}^{(0)}}{\partial\mu}\right|_{\mu^{(0)}} + \hbar\delta\tilde{\Omega}^{(1)}(\mu^{(0)}) + \rho(\mu^{(0)} + \hbar\delta\mu^{(1)}) + O(\hbar^2)$$

$$= \tilde{\Omega}^{(0)}(\mu^{(0)}) + \hbar\delta\tilde{\Omega}^{(1)}(\mu^{(0)}) + \rho\mu^{(0)} + O(\hbar^2) , \tag{3.29}$$

$$= \tilde{U}^{(0)}(\rho) + \hbar\delta\tilde{\Omega}^{(1)}(\mu^{(0)}) + O(\hbar^2) . \tag{3.30}$$

Evidently, to order $\hbar$ it is sufficient to evaluate the modified one-loop grand potential $\tilde{\Omega}^{(1)}$ at the chemical potential $\mu^{(0)}$ obtained in the modified tree-level calculation. The chemical potential $\mu^{(0)}$ is simply related to the baryon density via

$$\rho = \frac{N}{3\pi^2}k_F^3 = \frac{N}{3\pi^2}[(\mu^{(0)})^2 - M^{*2}]^{3/2} . \tag{3.31}$$

Each meson polarization can be decomposed into a density-independent Feynman part Π^F and a density-dependent part Π^D. The Feynman part is renormalized by imposing the renormalization conditions discussed after Eq. (2.36). For the pion we obtain

$$\Pi_\pi^F(k,\phi) = \left[\frac{N\hbar g^2}{4\pi^2}\right]\left\{\left[1 - \ln\left[\frac{M^*}{M}\right]\right]k^2 - \left[M^2 - M^{*2} + M^{*2}\ln\left[\frac{M^{*2}}{M^2}\right]\right] - \tfrac{1}{2}k^2\eta^*\ln\left[\frac{\eta^*+1}{\eta^*-1}\right]\right\} , \tag{3.32}$$

and the corresponding result for the scalar meson reads

$$\Pi_s^F(k,\phi) = \left[\frac{N\hbar g^2}{4\pi^2}\right]\left\{\left[1 - \ln\left[\frac{M^*}{M}\right]\right]k^2 - \left[M^2 - 5M^{*2} + 3M^{*2}\ln\left[\frac{M^{*2}}{M^2}\right]\right] - \tfrac{1}{2}k^2\eta^{*3}\ln\left[\frac{\eta^*+1}{\eta^*-1}\right]\right\} , \tag{3.33}$$

where

$$\eta^* \equiv \left[1 + \frac{4M^{*2}}{k^2}\right]^{1/2} . \tag{3.34}$$

For the density-dependent contributions, we find

$$\Pi_\pi^D(k,\phi) = \frac{N\hbar g^2}{\pi^2}\int_0^{k_F}\frac{q^2dq}{E_q^*} + \frac{N\hbar g^2k^2}{8\pi^2|\mathbf{k}|}\int_0^{k_F}\frac{q\,dq}{E_q^*}\ln\left[\frac{4k_0^2E_q^{*2} + (k^2 - 2q|\mathbf{k}|)^2}{4k_0^2E_q^{*2} + (k^2 + 2q|\mathbf{k}|)^2}\right] \tag{3.35}$$

and

$$\Pi_s^D(k,\phi) = \frac{N\hbar g^2}{\pi^2}\int_0^{k_F}\frac{q^2dq}{E_q^*} + \frac{N\hbar g^2(k^2 + 4M^{*2})}{8\pi^2|\mathbf{k}|}\int_0^{k_F}\frac{q\,dq}{E_q^*}\ln\left[\frac{4k_0^2E_q^{*2} + (k^2 - 2q|\mathbf{k}|)^2}{4k_0^2E_q^{*2} + (k^2 + 2q|\mathbf{k}|)^2}\right] , \tag{3.36}$$

with $E_q^* \equiv \sqrt{M^{*2} + q^2}$. Here $k^\mu = (k^0, \mathbf{k})$ is a Euclidean four-momentum. The remaining integrals can be evaluated analytically, but the results are rather lengthy.

IV. RENORMALIZATION OF THE RELATIVISTIC RPA

While the reordering of the standard loop expansion allows us to treat the baryon contribution to the energy density exactly for each order of $\hbar$ associated with meson fluctuations, a practical calculation requires that finite results can be obtained in each order of the modified loop expansion with a renormalization procedure that preserves chiral symmetry. Here renormalizability will be demonstrated explicitly for the first-order pion fluctuation contribution to the energy density.

In the spirit of Ref. 12, we isolate divergent terms by first expanding the inverse propagator in powers of $1/k^2$ to find

$$\Delta_\pi^{-1}(k,\phi,k_F) = k^2 + m_\pi^2 + V_{\pi\pi}(\phi) + \Pi_\pi(k,\phi)$$

$$= Z(k^2)k^2 + m_\pi^2 - \frac{\lambda}{g^2}(M^2 - M^{*2}) - \frac{N\hbar g^2}{4\pi^2}\left[M^2 + M^{*2}\ln\frac{k^2}{M^2}\right] - \frac{N\hbar g^2}{8\pi^2}\frac{1}{k^2}\left[M^{*4} + 2M^{*4}\ln\frac{M^{*2}}{k^2}\right]$$

$$+ \frac{4N\hbar g^2}{\pi^2}\frac{1}{k^2}\int_0^{k_F}\frac{p^2dp}{E_p^*}\left[M^{*2}\frac{k_0^2}{k^2} + p^2\left[1 - \frac{4\mathbf{k}^2}{3k^2}\right]\right] + O\left[\frac{1}{k^4}\right] , \tag{4.1}$$

where

$$Z(k^2)\equiv 1+\frac{N\hbar g^2}{4\pi^2}-\frac{N\hbar g^2}{8\pi^2}\ln\frac{k^2}{M^2} \ . \tag{4.2}$$

For the expansion of the pion propagator [Eq. (4.1)], the polarization insertion Π_π is taken from Eqs. (3.32) and (3.35). Using this expansion, the first-order pion contribution $\tilde{U}_\pi^{(1)}(\phi,k_F)$ is proportional to

$$\begin{aligned}\int\frac{d^nk}{(2\pi)^n}\ln\Delta_\pi^{-1}(k,\phi,k_F)=\int\frac{d^nk}{(2\pi)^n}&\left\{\ln[Z(k^2)k^2+m_\pi^2]-\frac{1}{Z(k^2)k^2+m_\pi^2}\left[\frac{\lambda}{g^2}(M^2-M^{*2})+\frac{N\hbar g^2}{4\pi^2}\left[M^2+M^{*2}\ln\frac{k^2}{M^2}\right]\right]\right.\\&-\frac{1}{2[Z(k^2)k^2+m_\pi^2]^2}\left[\frac{\lambda}{g^2}(M^2-M^{*2})+\frac{N\hbar g^2}{4\pi^2}\left[M^2+M^{*2}\ln\frac{k^2}{M^2}\right]\right]^2\\&-\frac{N\hbar g^2}{8\pi^2}\frac{M^{*4}}{[Z(k^2)k^2+m_\pi^2]k^2}\left[1+2\ln\frac{M^2}{k^2}\right]\\&\left.-\frac{N\hbar g^2}{8\pi^2}\frac{1}{[Z(k^2)k^2+m_\pi^2]k^2}\left[2M^{*4}\ln\frac{M^{*2}}{M^2}-8M^{*2}\int_0^{k_F}\frac{p^2dp}{E_p^*}\right]+O\left[\frac{1}{k^6}\right]\right\} ,\end{aligned} \tag{4.3}$$

where all terms written explicitly are divergent as the number of dimensions $n\to 4$. We must now demonstrate that all divergences in Eq. (4.3) can be removed using counterterms that are consistent with chiral symmetry and that are defined in terms of vacuum amplitudes.

The first term in the integrand is independent of the scalar field and the density and is removed by a vacuum subtraction. The next three terms are independent of k_F and produce a divergent fourth-order polynomial in the scalar field ϕ. Due to chiral symmetry, these four divergent terms can be removed by a second-order polynomial in M^{*2},

$$\delta U_{\text{poly}}=c_0+c_1M^{*2}+c_2M^{*4} \ . \tag{4.4}$$

The constant term c_0 represents a vacuum subtraction, while c_1 and c_2 are chosen to fix the ϕ and ϕ^2 coefficients in the effective potential.

The fifth term, which is density dependent and which contains divergences (as $n\to 4$) *in all powers of ϕ*, can be removed by defining a bare mass (M_0) and coupling constant (g_0) by setting

$$\begin{bmatrix}g_0^2\\M_0^2\end{bmatrix}=\begin{bmatrix}g^2\\M^2\end{bmatrix}\left[1-3\hbar g^2\int\frac{d^nk}{(2\pi)^n}\frac{1}{[Z(k^2)k^2+m_\pi^2]k^2}+\text{finite terms}\right] \tag{4.5}$$

to $O(\hbar)$. Using the bare parameters to $O(\hbar)$ in the zeroth-order result [Eq. (3.12)] then produces $O(\hbar)$ corrections that precisely cancel the fifth term in Eq. (4.3). We emphasize, however, that in general, the finite contributions arising from the appropriate renormalization of the baryon mass and vertex make quantitative contributions to the energy density. The redefinition of g_0^2 in Eq. (4.5) is precisely what is needed to make the pion-exchange contribution to the baryon scalar vertex function finite to first order in $\hbar$ and to all orders of $N\hbar$. Chiral symmetry ensures that this redefinition applied to M_0^2 also renders the baryon self-energy finite to this order. (An independent wave-function counterterm is also needed.) Additional finite contributions to these redefinitions are prescribed by fixing the pole position of the baryon propagator or the strength of the scalar coupling. Therefore, to fully specify g_0 and M_0 (and hence the modified one-loop energy density), one must compute the baryon self-energy and vertex function to the relevant order in $\hbar$. In what follows, we omit the "finite terms" in Eq. (4.5) for the modified one-loop results because it is difficult to determine the quantitative relationship between bare and physical parameters, and finite corrections will not affect our conclusions.

Finally, a finite expression for the pion contribution to the energy density can be written in terms of the integral of Eq. (4.3), along with chiral counterterms, as

$$\begin{aligned}\tilde{U}_\pi^{(1)}(\phi,k_F)=\frac{3\hbar}{2}\int\frac{d^4k}{(2\pi)^4}&\left\{\ln\frac{\Delta_\pi^{-1}(k,\phi,k_F)}{Z(k^2)k^2+m_\pi^2}+\frac{1}{Z(k^2)k^2+m_\pi^2}\left[\frac{\lambda}{g^2}(M^2-M^{*2})+\frac{N\hbar g^2}{4\pi^2}\left[M^2+M^{*2}\ln\frac{k^2}{M^2}\right]\right]\right.\\&+\frac{1}{2[Z(k^2)k^2+m_\pi^2]^2}\left[\frac{\lambda}{g^2}(M^2-M^{*2})+\frac{N\hbar g^2}{4\pi^2}\left[M^2+M^{*2}\ln\frac{k^2}{M^2}\right]\right]^2\\&+\frac{N\hbar g^2}{8\pi^2}\frac{M^{*4}}{[Z(k^2)k^2+m_\pi^2]k^2}\left[1+2\ln\frac{M^2}{k^2}\right]+\frac{N\hbar g^2}{8\pi^2}\frac{1}{[Z(k^2)k^2+m_\pi^2]k^2}\\&\left.\times\left[2M^{*4}\ln\frac{M^{*2}}{M^2}-8M^{*2}\int_0^{k_F}\frac{p^2dp}{E_p^*}\right]\right\}+c_0'+c_1'M^{*2}+c_2'M^{*4} ,\end{aligned} \tag{4.6}$$

where Δ_π^{-1} uses the full polarizations in Eqs. (3.32) and (3.35). Although the integral of Eq. (4.6) is now finite in four dimensions, we must still determine the *finite* counterterms c_0', c_1', and c_2' numerically. Thus the integral of Eq. (4.6) is evaluated numerically, and the finite counterterms c_i' are chosen to satisfy the three conditions

$$\begin{aligned} &\tilde{U}_\pi^{(1)}(\phi=0,k_F=0)=0\ ,\\ &\mathrm{Re}\frac{d}{d\phi}\tilde{U}_\pi^{(1)}(\phi,k_F=0)\bigg|_{\phi=0}=0\ ,\\ &\mathrm{Re}\frac{d^2}{d\phi^2}\tilde{U}_\pi^{(1)}(\phi,k_F=0)\bigg|_{\phi=0}=0\ . \end{aligned} \tag{4.7}$$

Similar considerations allow the renormalization of the scalar meson contribution to first order in the modified loop expansion. The analogous expression for the scalar contribution is

$$\begin{aligned} \tilde{U}_s^{(1)}(\phi,k_F)=\frac{\hbar}{2}\int\frac{d^4k}{(2\pi)^4}&\Bigg\{\ln\frac{\Delta_s^{-1}(k,\phi,k_F)}{Z(k^2)k^2+m_s^2}+\frac{1}{Z(k^2)k^2+m_s^2}\left[\frac{3\lambda}{g^2}(M^2-M^{*2})+\frac{N\hbar g^2}{4\pi^2}\left[M^2-4M^{*2}+3M^{*2}\ln\frac{k^2}{M^2}\right]\right]\\ &+\frac{1}{2[Z(k^2)k^2+m_s^2]^2}\left[\frac{3\lambda}{g^2}(M^2-M^{*2})+\frac{N\hbar g^2}{4\pi^2}\left[M^2-4M^{*2}+3M^{*2}\ln\frac{k^2}{M^2}\right]\right]^2\\ &+\frac{N\hbar g^2}{8\pi^2}\frac{3M^{*4}}{[Z(k^2)k^2+m_s^2]k^2}\left[3-2\ln\frac{M^2}{k^2}\right]-\frac{N\hbar g^2}{8\pi^2}\frac{3}{[Z(k^2)k^2+m_s^2]k^2}\\ &\times\left[2M^{*4}\ln\frac{M^{*2}}{M^2}-8M^{*2}\int_0^{k_F}\frac{p^2dp}{E_p^*}\right]\Bigg\}+c_0''+c_1''M^{*2}+c_2''M^{*4}\ . \end{aligned} \tag{4.8}$$

The finite constants $c_{0,1,2}''$ are determined numerically from the conditions of Eq. (4.7) with the pionic contribution $\tilde{U}_\pi^{(1)}$ replaced by the scalar meson contribution $\tilde{U}_s^{(1)}$. A comparison of the final term in curly braces in Eqs. (4.6) and (4.8) reveals that the divergent parts of the subtractions generated by the bare mass and coupling cancel when the pion and scalar contributions are added. Similarly, due to chiral symmetry, the divergent contributions to the baryon self-energy from pion and scalar exchange cancel, and only finite renormalizations are actually needed for the full one-loop self-energy.

V. REPRODUCTION OF THE TWO-LOOP RESULT

Before proceeding, we illustrate the new renormalization technique developed in the preceding section by applying it to the conventional loop expansion in the Walecka model. We calculate the two-loop contribution to the energy density for a special case where nuclear matter saturation is obtained without including the ω vector meson (parameter set B of Ref. 6). This demonstrates that our technique, which represents a very different numerical realization of renormalization, can reproduce the results obtained in Ref. 6.

The Walecka model result can be obtained from the chiral model by the following modifications: (1) remove the pion contribution; (2) turn off all scalar meson self-couplings; (3) allow counterterms multiplying the first four powers of ϕ to be adjusted independently; and (4) expand the modified one-loop result (which contains all orders in $N\hbar$) to order $\hbar(N\hbar)$ and set $N=2$ for symmetric nuclear matter. Thus, the conventional two-loop term can be identified as the part of the modified one-loop term that is explicitly of order $\hbar(N\hbar)$.

The zeroth-order term follows directly from the classical part of the Lagrangian and is given by

$$U_W^{(0)}(\phi)=\tfrac{1}{2}m_s^2\phi^2=\frac{m_s^2}{2g^2}(M^*-M)^2\ . \tag{5.1}$$

Since there are no meson self-couplings in the Walecka model, the term of $O(\hbar)$ is given entirely by the one-baryon-loop graphs. The one-loop contribution is then[14,6]

$$\begin{aligned} U_W^{(1)}(\phi,k_F)=&-\frac{\hbar}{8\pi^2}\left[M^{*4}\ln\frac{M^{*2}}{M^2}-2M^3(M^*-M)-7M^2(M^*-M)^2-\tfrac{26}{3}M(M^*-M)^3-\tfrac{25}{6}(M^*-M)^4\right]\\ &+\frac{2\hbar}{\pi^2}\int_0^{k_F}p^2E_p^*dp\ , \end{aligned} \tag{5.2}$$

which differs from the result in Eq. (3.12) only in the choice of the finite parts of the counterterms. In Eq. (5.2) *all* density-independent terms behaving as a quartic polynomial in ϕ have been removed by renormalization conditions.

To make contact with the two-loop contribution of Ref. 6, we expand the modified scalar one-loop term in Eq. (3.22) to order $\hbar(N\hbar)$ and set $N=2$. This gives

$$U_W^{(2)}(\phi,k_F)=\frac{\hbar}{2}\int\frac{d^nk}{(2\pi)^n}\frac{1}{k^2+m_s^2}[\Pi_s(k,\phi)-\Pi_s(k,0)]+\text{counterterms}\ , \tag{5.3}$$

where the counterterms are of two types. The first type is a fourth-order polynomial in ϕ that is chosen to remove the first four powers of ϕ from $U_W^{(2)}(\phi,k_F=0)$. In addition, we must interpret the parameters in $U_W^{(1)}(\phi,k_F)$ as bare parameters and reexpress them in terms of the physical mass (M) and coupling (g) to first order in $\hbar$, which generates additional counterterms of order $\hbar^2$. These terms will be density dependent and contain all powers of the background field ϕ. Both M and g in $U_W^{(1)}(\phi,k_F)$ are to be replaced by M_0 and g_0, where the parameters M_0 and g_0 are determined by examining the baryon propagator G given in Minkowski space by

$$Z^{-1}G^{-1}(\not p,\phi)=\not p-M_0+g_0\phi-\Sigma(\not p,\phi)\ . \tag{5.4}$$

As in Ref. 6, the baryon self-energy Σ is needed only to first order in $\hbar$. In Minkowski space, the unrenormalized self-energy is

$$\Sigma(\not p,\phi)=i\hbar g^2\int\frac{d^nk}{(2\pi)^n}\left[\frac{1}{k^2-m_s^2+i\eta}\right]\left[\frac{1}{\not p-\not k-M^*+i\eta}\right]\ . \tag{5.5}$$

The parameters M_0, g_0, and Z are determined by the three renormalization conditions:

$$G^{-1}(\not p=M,\phi=0)=0\ , \tag{5.6}$$

$$\left.\frac{d}{d\not p}G^{-1}(\not p,\phi=0)\right|_{\not p=M}=1\ , \tag{5.7}$$

$$\left.\frac{d}{d\phi}G^{-1}(\not p=M,\phi)\right|_{\phi=0}=g\ . \tag{5.8}$$

We find

$$\begin{aligned}M_0&=M\left[1+\frac{\hbar g^2}{16\pi^2}\Gamma\left[2-\frac{n}{2}\right]\int_0^1 d\alpha(2-\alpha)[M^2\alpha^2+m_s^2(1-\alpha)]^{(n/2)-2}\right]\\&\xrightarrow{n\to4}M\left[1+\frac{3\hbar}{2}g^2\int\frac{d^nk}{(2\pi)^n}\frac{1}{(k^2+m_s^2)k^2}-\frac{3\hbar g^2}{32\pi^2}-\frac{\hbar g^2}{16\pi^2}\int_0^1 d\alpha(2-\alpha)\ln\frac{M^2\alpha^2+m_s^2(1-\alpha)}{m_s^2}\right]\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}g_0&=g\left[1+\frac{\hbar g^2}{16\pi^2}\Gamma\left[2-\frac{n}{2}\right]\int_0^1 d\alpha(2-\alpha)[M^2\alpha^2+m_s^2(1-\alpha)]^{(n/2)-2}\right.\\&\qquad\left.-\frac{\hbar g^2}{16\pi^2}\Gamma\left[3-\frac{n}{2}\right]2M^2\int_0^1 d\alpha\,\alpha^2(2-\alpha)[M^2\alpha^2+m_s^2(1-\alpha)]^{(n/2)-3}\right]\\&\xrightarrow{n\to4}g\left[\frac{M_0}{M}-\frac{\hbar g^2}{16\pi^2}2M^2\int_0^1 d\alpha\frac{\alpha^2(2-\alpha)}{M^2\alpha^2+m_s^2(1-\alpha)}\right]\end{aligned} \tag{5.10}$$

to $O(\hbar)$, which yield the same renormalized self-energy as in Ref. 6. Here both the infinite and the finite pieces of the self-energy can be evaluated explicitly. This is in contrast to Eq. (4.5), where the presence of the baryon polarization insertion Π_s in the meson propagator makes the evaluation of the finite pieces of the integral much more difficult. As in Eq. (4.5), the divergent parts of the bare parameters are here expressed as an n-dimensional *Euclidean* integral that diverges as $n\to4$. This is a convenient way to generate additional subtractions in the integrand of Eq. (5.3) that make the overall integral finite as $n\to4$. The finite parts of Eqs. (5.9) and (5.10) must be retained, as they produce additional finite contributions to the two-loop result.

The density-independent contribution to Π_s that decays at least as fast as k^{-4} follows from Eq. (3.33) and is given by

$$\left[\frac{\hbar g^2}{2\pi^2}\right]f(M^*,k^2)\equiv\left[\frac{\hbar g^2}{2\pi^2}\right]\left[M^{*2}+3M^{*2}\ln\frac{k^2}{M^{*2}}+\frac{1}{2k^2}\left[9M^{*4}+6M^{*4}\ln\frac{k^2}{M^{*2}}\right]-\frac{k^2}{2}\left[\eta^{*3}\ln\frac{\eta^*+1}{\eta^*-1}+\ln\frac{M^{*2}}{k^2}\right]\right]\ . \tag{5.11}$$

All terms that decay slower than k^{-4} are removed by either a vacuum subtraction, or by fourth-order polynomial counterterms $\Sigma_{i=1}^4 C_i\phi^i$, or by the divergent contributions from the redefinitions of Eqs. (5.9) and (5.10) used in $U_W^{(1)}$. Insertion of Eq. (5.11) into Eq. (5.3) produces a finite result. However, the conventional renormalization conditions for the Walecka model must be enforced by removing the first four derivatives of $f(M^*,k^2)$ with respect to M^*. The derivatives are

$$f'(M,k^2)=6M\left[\ln\frac{k^2}{M^2}+\frac{2}{k^2}\left[M^2+M^2\ln\frac{k^2}{M^2}\right]-\eta\ln\frac{\eta+1}{\eta-1}\right], \quad (5.12)$$

$$f''(M,k^2)=6\left[\eta^2\ln\frac{k^2}{M^2}+\frac{2}{k^2}\left[M^2+M^2\ln\frac{k^2}{M^2}\right]-(2\eta-\eta^{-1})\ln\frac{\eta+1}{\eta-1}\right], \quad (5.13)$$

$$f^{(3)}(M,k^2)=-\frac{24M}{k^2}\left[\frac{8M^2}{k^2}\eta^{-2}-3\ln\frac{k^2}{M^2}+(2\eta^{-1}+\eta^{-3})\ln\frac{\eta+1}{\eta-1}\right], \quad (5.14)$$

$$f^{(4)}(M,k^2)=-\frac{24}{k^2}\left[8-2\eta^{-2}-6\eta^{-4}-3\ln\frac{k^2}{M^2}+3\eta^{-5}\ln\frac{\eta+1}{\eta-1}\right]. \quad (5.15)$$

The density-dependent part of Π_s has exactly the same form as in the chiral model and again produces divergences to all orders in ϕ. As before, these divergences are removed by subtractions that arise when bare parameters are used in the density-dependent part of $U_W^{(1)}$.

Therefore, the complete two-loop contribution to the effective potential at finite density can be written as

$$\begin{aligned}U_W^{(2)}(\phi,k_F)=&\frac{\hbar^2g^2}{4\pi^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m_s^2}\left[f(M^*,k^2)-\sum_{j=0}^{4}f^{(j)}(M,k^2)(M^*-M)^j\right]\\&+\frac{\hbar^2g^2}{\pi^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m_s^2}\int_0^{k_F}\frac{p^2dp}{E_p^*}\left\{\left[1+\frac{k^2+4M^{*2}}{8|\mathbf{k}|p}\ln\frac{(k^2-2|\mathbf{k}|p)^2+4k_0^2E_p^{*2}}{(k^2+2|\mathbf{k}|p)^2+4k_0^2E_p^{*2}}\right]\right.\\&\left.-\frac{4}{k^2}\left[M^{*2}\left[\frac{k_0^2}{k^2}-1\right]+p^2\left[1-\frac{4\mathbf{k}^2}{3k^2}\right]\right]\right\}\\&+\frac{8\hbar^2g^2}{(4\pi)^4}\left[\tfrac{3}{2}+\int_0^1d\alpha(2-\alpha)\ln\frac{M^2\alpha^2+m_s^2(1-\alpha)}{m_s^2}+2M^2\int_0^1d\alpha\frac{\alpha^2(2-\alpha)}{M^2\alpha^2+m_s^2(1-\alpha)}\right]C_1(\phi,k_F)\\&-\frac{16\hbar^2g^2}{(4\pi)^4}\left[M^2\int_0^1d\alpha\frac{\alpha^2(2-\alpha)}{M^2\alpha^2+m_s^2(1-\alpha)}\right]C_2(\phi,k_F)-\frac{2\hbar^2g^2}{(4\pi)^4}C_1(\phi,k_F).\end{aligned} \quad (5.16)$$

All integrals are now explicitly finite and can be evaluated numerically. The first term of Eq. (5.16) contributes to the vacuum energy and is easily evaluated as a one-dimensional integral over k^2. The second term follows directly from the density-dependent part of Π_s in Eq. (5.3) and from the divergent part of the mass and coupling constant redefinition in $U_W^{(1)}$. The integration over p is done analytically, leaving a two-dimensional Euclidean integration over k^2 and k_0. The third and fourth terms of Eq. (5.16) arise from the finite parts of the mass and coupling redefinitions in Eqs. (5.9) and (5.10). They contain both vacuum terms and density-dependent parts in the functions C_1 and C_2 defined by

$$C_1(\phi,k_F)=M^{*4}\ln\frac{M^{*2}}{M^2}-2M^3(M^*-M)-7M^2(M^*-M)^2-\tfrac{26}{3}M(M^*-M)^3-\tfrac{25}{6}(M^*-M)^4-4M^{*2}\int_0^{k_F}\frac{p^2dp}{E_p^*}, \quad (5.17)$$

$$C_2(\phi,k_F)=M^{*3}M\ln\frac{M^{*2}}{M^2}-2M^3(M^*-M)-5M^2(M^*-M)^2-\tfrac{11}{3}M(M^*-M)^3-\tfrac{1}{2}(M^*-M)^4-4M^*M\int_0^{k_F}\frac{p^2dp}{E_p^*}. \quad (5.18)$$

Notice that the density-independent parts of both C_1 and C_2 start at order $(M^*-M)^5$.

The final term of Eq. (5.16) arises because in Ref. 6, all integrals (including the one contained in Π_s) are taken to four dimensions simultaneously, whereas here we insert the *renormalized* four-dimensional vacuum polarization Π_s into the Euclidean integral (5.3) and take the number of dimensions to four at the end. Let us illustrate this subtlety with a simple example. For some function u of two variables, consider the integral

$$\Pi(k;\tau)=\int d^\tau q\,u(k,q)\equiv\alpha(k)+\beta(k)\left[2-\frac{\tau}{2}\right] \quad (5.19)$$

in τ dimensions that is finite and equal to $\alpha(k)$ as $\tau\to4$. Then the renormalized n-dimensional integral $I(n,\tau)$ of $\Pi(k;\tau)$, with a weight function $f(k)$, is given by

$$I(n,\tau)=\int d^nk\, f(k)\Pi(k;\tau)-a\Gamma\left[2-\frac{n}{2}\right] \tag{5.20}$$

$$=\left[2-\frac{\tau}{2}\right]\Gamma\left[2-\frac{n}{2}\right]b+c+d\left[2-\frac{\tau}{2}\right], \tag{5.21}$$

where the constants a, b, c, and d are defined by

$$\int d^nk\, f(k)\alpha(k)=a\Gamma\left[2-\frac{n}{2}\right]+c\ , \tag{5.22}$$

$$\int d^nk\, f(k)\beta(k)=b\Gamma\left[2-\frac{n}{2}\right]+d\ . \tag{5.23}$$

From Eq. (5.21), it is clear that the finite value $I(4,4)$ depends on how the limits $\tau\to 4$ and $n\to 4$ are taken. For this example, we obtain

$$\lim_{n\to 4}\lim_{\tau\to 4} I(n,\tau)=c\ , \tag{5.24}$$

$$\lim_{n\to 4} I(n,n)=c+b\ . \tag{5.25}$$

Our numerical renormalization procedure corresponds to the first way of taking the limit, while in dimensional regularization (as used in Ref. 6), one must consider the second way.[21]

For the conventional loop expansion discussed in this section, the additional finite piece b [which is contained in Eq. (5.16) as the last term] can be evaluated analytically by expressing Π_s^F as an n-dimensional integral and inserting this expression into Eq. (5.3). In the relativistic RPA calculation of Sec. IV, however, this difficult calculation is omitted, because we have already dropped the finite terms in Eq. (4.5). [These would produce finite contributions analogous to the third and fourth terms in Eq. (5.16).] This omission from the modified one-loop results will not qualitatively affect our conclusions, as we discuss in Sec. VI, but all the contributions must be computed here to compare with the results of Ref. 6.

We can now use Eqs. (5.1), (5.2), and (5.16) to reproduce the two-loop result of Ref. 6 by calculating the energy per nucleon

$$E=\frac{1}{\rho}[U_W^{(0)}(\phi)+U_W^{(1)}(\phi,k_F)+U_W^{(2)}(\phi,k_F)]-M \tag{5.26}$$

with parameter set B of Ref. 6 at saturation density $k_F=1.3\ \text{fm}^{-1}$. In Fig. 3, we show E as a function of M^*/M. It has a minimum at $M^*/M=0.875$ with the expected value of $E=-15.75$ MeV. Similar agreement is obtained at other values of k_F. We emphasize that Eq. (5.26) has been computed here with the techniques developed for the renormalization of the relativistic RPA, and it represents a very different numerical realization of the two-loop contribution than that used in Ref. 6.

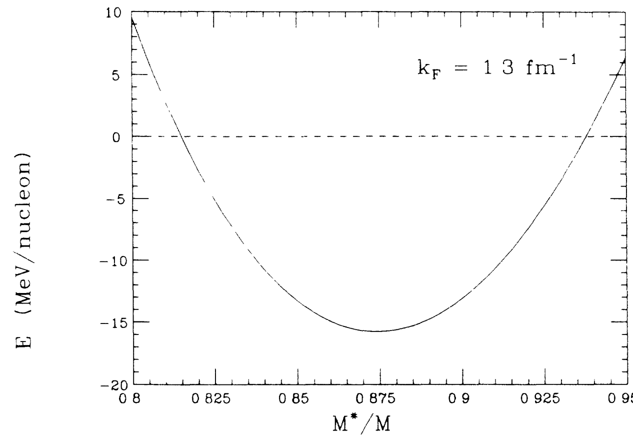


FIG. 3. The energy per nucleon at saturation density as a function of M^*/M in the conventional two-loop approximation to the Walecka model.

VI. MODIFIED LOOP EXPANSION: RESULTS

In Sec. III we derived expressions for the energy density in the first order of the modified loop expansion of the linear σ model. The contribution from the scalar meson, apart from the counterterms discussed in Sec. IV, is given in Eq. (3.20) as a four-dimensional integral over a Euclidean momentum. The integrand is the logarithm of the inverse scalar meson propagator at Fermi momentum k_F and classical scalar field ϕ, normalized by the propagator at zero density and zero scalar field (vacuum subtraction). After introducing four-dimensional polar coordinates, the two angular integrals corresponding to the direction of the three-momentum are trivial and give a factor of 4π. The remaining two-dimensional integral with respect to k^2 and the angular variable

$$x=\frac{k_0}{\sqrt{k^2}} \tag{6.1}$$

is evaluated numerically and is finite when the counterterms are included as discussed in Sec. IV.

At zero density, the integrand does not depend on x, and its real part is shown in Fig. 4 as a function of k^2/M^2 for $g\phi/M=0.15$, $m_\pi=135$ MeV, and $m_s=550$

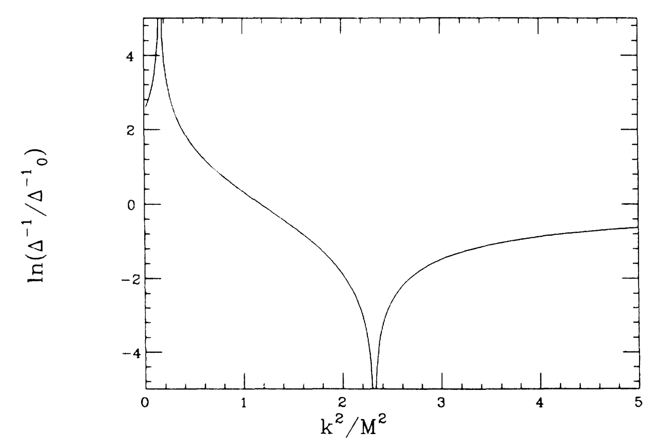


FIG. 4. The real part of the integrand for the modified one-loop contribution from the scalar meson for $g\phi/M=0.15$ at zero density.

MeV. There is a logarithmic singularity whenever the numerator or denominator of the argument of the logarithm passes through zero. We will refer to these new singularities, which arise from poles in the modified one-loop propagator at Euclidean momenta, as (Landau) ghost poles, and reserve the name "tachyons" for poles present already in the tree-level propagator. The denominator (the vacuum subtraction) vanishes at $k^2/M^2 \approx 0.16$ and generates the first pole, while the numerator vanishes at $k^2/M^2 \approx 2.31$ and causes the second pole. The pole at zero density occurs at a rather low value of (k^2/M^2) because in the chiral model, as already remarked after Eq. (2.37), the wave function renormalization for the scalar meson propagator cannot be adjusted independently, but is prescribed by chiral symmetry. The argument of the logarithm is negative between these two poles, giving rise to an imaginary part of the effective potential. The sign of the imaginary part is chosen to be negative, corresponding to a decaying ground state.

The poles of the real part of the integrand are evaluated by finding the pole positions numerically and subtracting the (integrable) singularity from the integrand. The singular part of the integrand is then computed analytically, and the remaining smooth integrand can be easily evaluated numerically. In the case of a single zero x_0 of a function $f(x)$, this procedure yields

$$\int_a^b dx \ln f(x) = \int_a^b dx \ln\left[\frac{f(x)}{x-x_0}\right] + \int_a^b dx \ln(x-x_0) \ . \tag{6.2}$$

Alternatively, the real part of the integral can be obtained by smoothing the pole with a small parameter ϵ,

$$\mathrm{Re}\int_a^b dx \ln f(x) = \lim_{\epsilon\to 0}\tfrac{1}{2}\int_a^b dx \ln[f^2(x)+\epsilon^2] \ . \tag{6.3}$$

In our calculations, these two methods agreed to at least four digits. The imaginary part of the integral is simply proportional to the difference of the pole positions to the fourth power.

At finite density, the integrand depends on k^2 and the angular variable x [see Eq. (6.1)] separately. Its real part is shown in Fig. 5 for $k_F = 1.3\ \mathrm{fm}^{-1}$ and $g\phi/M = 0.15$ at the angular variable $x = 0$. The argument of the logarithm here is negative at $k^2 = 0$ and the denominator is zero at $k^2/M^2 = 0.16$, thus resulting in the first pole at the same position as at zero density. (This pole arises from the vacuum subtraction and is thus independent of the density and the scalar field.) The other ghost poles are caused by zeros of the numerator, and thus their positions depend on both the density and the scalar field strength. The imaginary part of the effective potential here arises from the region between zero and the first pole, and from the region between the second and third poles.

A better overview of the situation at finite density can be obtained from Fig. 6, which shows the pole positions for the scalar meson integrand as a function of k^2 and x for different values of the scalar field, at fixed Fermi momentum $k_F = 1.3\ \mathrm{fm}^{-1}$. The pole from the vacuum subtraction is not shown, as it always occurs at $k^2/M^2 = 0.16$. The solid and dashed lines correspond to $g\phi/M = 0.1$ and $g\phi/M = 0.2$, respectively, and the dot-dashed line is obtained for $g\phi/M = 0.025$. The pole at small angles disappears as ϕ decreases, and the second ghost moves to smaller values of k^2.

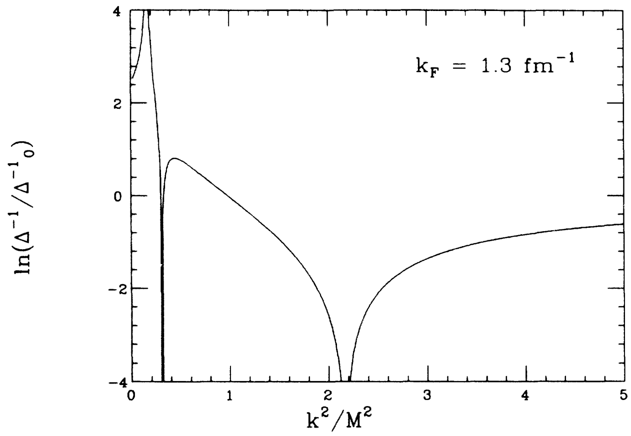


FIG. 5. The real part of the integrand for the modified one-loop contribution from the scalar meson for $g\phi/M = 0.15$, $k_F = 1.3\ \mathrm{fm}^{-1}$, and $x = 0$.

In Fig. 7, we show the pole positions as a function of k_F at fixed scalar field $g\phi/M = 0.15$. The solid lines are obtained for $k_F = 1.3\ \mathrm{fm}^{-1}$. The two poles at small angles correspond to the second two poles in Fig. 5. (The first pole in Fig. 5 results from the vacuum subtraction and is not shown in the present Fig. 7.) At $k_F = 0.5\ \mathrm{fm}^{-1}$ (dashed line), there is only one pole, and its dependence on x is even weaker than for the corresponding pole at $k_F = 1.3\ \mathrm{fm}^{-1}$. (As noted before, the position of this pole at zero density is independent of x.) As we increase the Fermi momentum to $k_F = 2.0\ \mathrm{fm}^{-1}$ (dot-dashed line), the angular dependence becomes strong, and no pole is present below $x \approx 0.08$, two poles exist for $0.08 \lesssim x \lesssim 0.26$, and one pole exists for $x \gtrsim 0.26$.

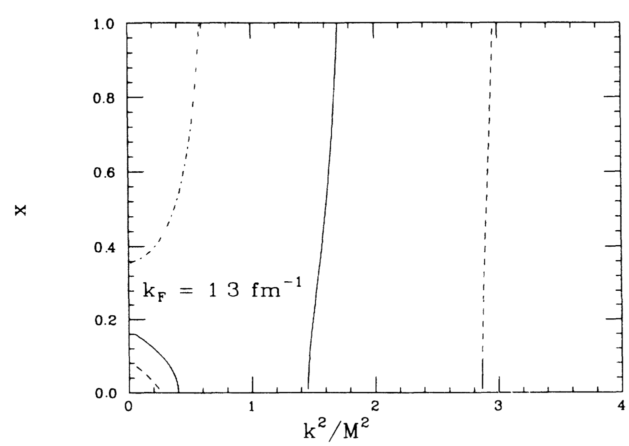


FIG. 6. The pole positions in the integrand for the scalar meson contribution as a function of k^2 and x at $k_F = 1.3\ \mathrm{fm}^{-1}$. The solid lines are obtained for $g\phi/M = 0.1$, the dashed lines for $g\phi/M = 0.2$, and the dot-dashed line for $g\phi/M = 0.025$.

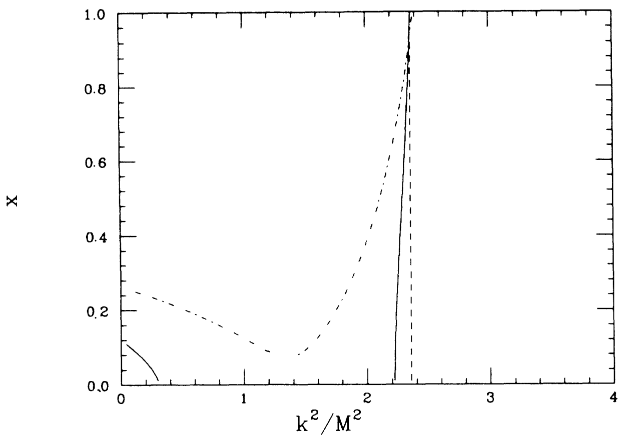


FIG. 7. The pole positions in the integrand for the scalar meson contribution as a function of k^2 and x at $g\phi/M=0.15$. The solid lines are obtained for $k_F=1.3$ fm^{-1}, the dashed line for $k_F=0.5$ fm^{-1}, and the dot-dashed line for $k_F=2.0$ fm^{-1}.

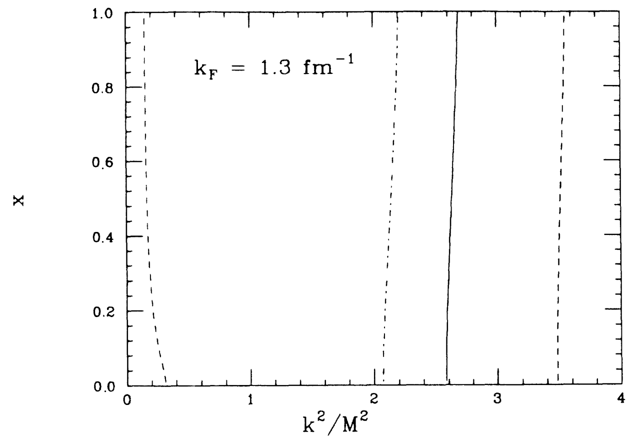


FIG. 8. The pole positions in the integrand for the pion contribution as a function of k^2 and x at $k_F=1.3$ fm^{-1}. The solid line is obtained for $g\phi/M=0.1$, the dashed lines for $g\phi/M=0.2$, and the dot-dashed line for $g\phi/M=0.05$.

So far we have discussed the pole positions for the scalar meson contribution. In Fig. 8 we show the pole positions for the pion contribution to the energy per nucleon, as a function of the scalar field ϕ for $k_F=1.3$ fm^{-1} and pion mass $m_\pi=135$ MeV. (This figure corresponds to Fig. 6 for the scalar meson.) For $g\phi/M=0.2$ (dashed lines), there are two poles present for all values of x (again, we do not show the additional pole from vacuum subtraction). The first pole corresponds to the tachyon pole in the conventional first-order loop expansion. It disappears for $g\phi/M\leq 0.115$ (solid line: $g\phi/M=0.1$; dot-dashed line: $g\phi/M=0.05$). This demonstrates that the modified loop expansion can improve, to a certain extent, the tachyon problem that plagues the conventional loop expansion. (Recall that at conventional one-loop order, the tachyon was present even for very small values of ϕ.) At each density, the self-consistent value of the scalar field is determined by minimizing the energy density with respect to ϕ. At densities for which the self-consistent value of ϕ is below the critical value of $g\phi/M=0.115$, the pion is no longer tachyonic. It is interesting to note that for $m_\pi=0$, this critical value is just the self-consistent value of the scalar field in the modified zeroth-order loop expansion.

Figure 9 shows the pion pole positions at the fixed value of $g\phi/M=0.15$ for different values of k_F. As for the scalar meson poles, one of the ghost poles depends weakly on x for small k_F (dashed lines: $k_F=0.5$ fm^{-1}) and increasingly stronger for $k_F=1.3$ fm^{-1} (solid lines) and $k_F=2.0$ fm^{-1} (dot-dashed line). The tachyon pole disappears with increasing density at fixed $g\phi/M$.

Having analyzed the pole structure of the integrand for the contributions from the scalar mesons and pions, we now show results for the real and imaginary parts of the energy density. They are obtained by numerical evaluation of the integrand as discussed previously, but after renormalization. In practice, as discussed in Sec. IV, this amounts to subtracting from the integrand terms that decay slower than k^{-6}. Furthermore, we numerically subtract the first two powers of M^{*2} from the finite real part, thus ensuring an unshifted minimum at zero density and an unmodified coefficient m_s^2 of the quadratic term in ϕ. Since the subtractions generate an additional unphysical pole at $k^2/M^2\approx 9$ [from a zero of the denominator in Eq. (4.5)], we invoke this subtraction only for k^2 above some value $\Lambda > 9M^2$.

Ideally, we would also evaluate the bare parameters M_0 and g_0 to modified one-loop order in terms of the physical baryon mass M and meson coupling g and thus determine the precise values of M_0 and g_0 to be inserted in the zeroth-order energy density. In Sec. V we showed how this identification can be achieved in the second order of the conventional loop expansion, as obtained by expanding our modified first-order result to $O(\hbar(N\hbar))$. In the present calculation, we simply drop the finite terms from Eq. (4.5). The ambiguity of the finite result is reflected in the arbitrariness in the choice of the subtrac-

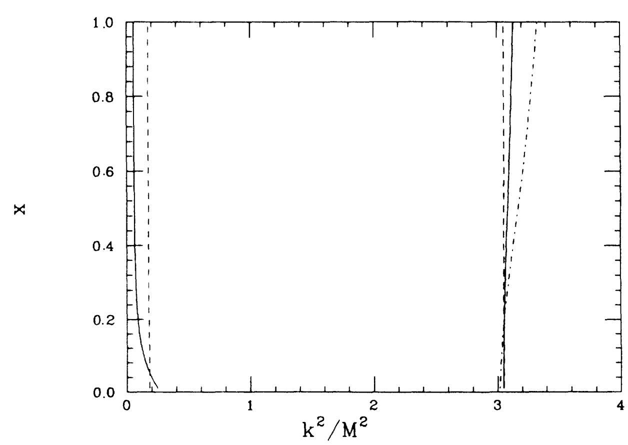
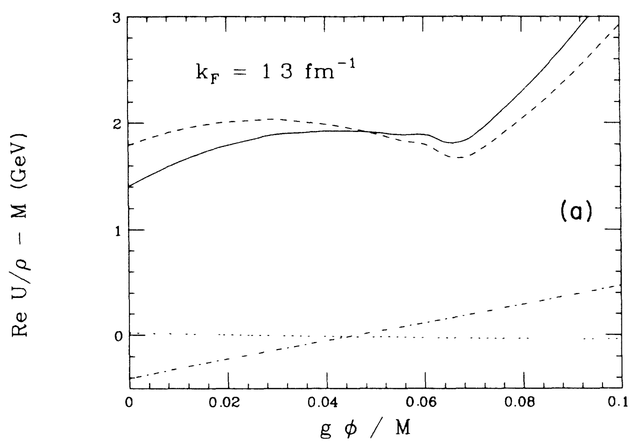


FIG. 9. The pole positions in the integrand for the pion contribution as a function of k^2 and x at $g\phi/M=0.15$. The solid lines are obtained for $k_F=1.3$ fm^{-1}, the dashed lines for $k_F=0.5$ fm^{-1}, and the dot-dashed line for $k_F=2.0$ fm^{-1}.

tion point Λ. We find a smooth dependence of the resulting real part on Λ. We expect that a more elaborate calculation that determines M_0 and g_0 completely[23] would give qualitatively similar results. In particular, the imaginary part of the energy is determined solely from the pole positions of the integrand and is thus *independent* of Λ. (It does depend, however, on the choice of M and g.)

Figures 10(a) and (b) show the real part of the energy per nucleon as a function of $g\phi/M$ at $k_F=1.3\ \text{fm}^{-1}$, for two values of the scalar meson mass: $m_s=769$ MeV [Fig. 10(a)] and $m_s=900$ MeV [Fig. 10(b)]. The subtraction point is $\Lambda=20M^2$. The dotted line shows the modified zeroth-order result (after subtraction of the nucleon mass), which is on the order of MeV. The dot-dashed line shows the pion contribution to the energy, which is approximately a linear function of the scalar field ϕ. In contrast, the contribution from the σ meson (dashed line) has a minimum. Therefore the sum of all three contributions (solid line), i.e., the total energy per nucleon in the renormalized relativistic RPA, has a (local) minimum.



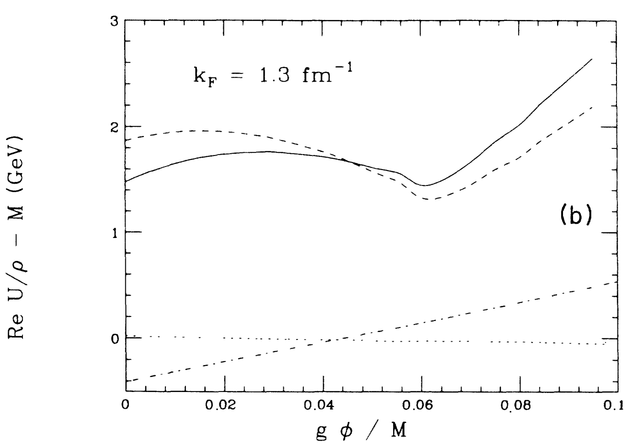


FIG. 10. The real part of the energy per nucleon as a function of $g\phi/M$ at $k_F=1.3\ \text{fm}^{-1}$, for $m_\pi=135$ MeV and $m_s=769$ MeV [part 10(a)] or $m_s=900$ MeV [part 10(b)]. The dotted line shows the result from the zeroth-order modified loop expansion. The dot-dashed and dashed lines show the contributions to the energy from the pion and scalar meson, respectively. The solid line shows the sum of all three contributions.

While the exact position and depth of the minimum depend on the scalar meson mass m_s and the parameter Λ, it is important to note that the minimum typically occurs at $g\phi/M\approx 0.06$, well below the onset of the tachyon pole at $g\phi/M\approx 0.11$. This result indicates that the modified loop expansion can indeed solve the tachyon problem. Note, however, that the magnitude of the first-order corrections relative to the zeroth-order contribution does not indicate convergence of the modified loop expansion for the linear σ model.

In Fig. 11 we show the imaginary part of the energy per nucleon at $k_F=1.3\ \text{fm}^{-1}$ for $m_s=769$ MeV. The dot-dashed line is the contribution from the pion, the dashed line gives the scalar meson contribution, and the solid line is the sum. At the minimum of the real part of the binding energy, the imaginary part is approximately -30 GeV/nucleon, which is roughly one order of magnitude bigger than the real part. Following the interpretation of the imaginary part by Weinberg and Wu[18] as half the decay rate per volume, this signifies a highly unstable ground state in the modified loop expansion to first order.

The different orders of magnitude for the real and imaginary parts of the energy arise because we have enforced the renormalization conditions from Eq. (4.7) for the real part of the effective potential, but not for the imaginary part. The latter would correspond to inserting *complex* counterterms in the Lagrangian defined so that the meson masses are real in the vacuum. The vacuum is now stable by construction, although nuclear matter may still be unstable at finite density. In Fig. 12 we show the results for the imaginary part of the energy after generalizing the renormalization subtractions. The contributions from the pion (dot-dashed line) and the scalar meson (dashed line) are now of the same order of magnitude as the real part, but they have opposite sign and result in a comparatively small total imaginary part (solid line). Note that this approximate equality of the real and

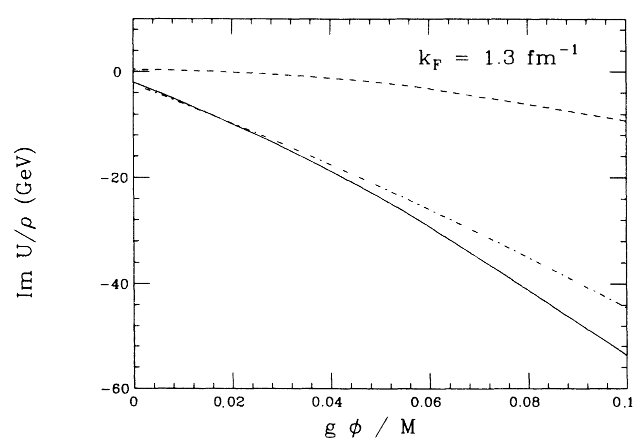


FIG. 11. The imaginary part of the energy per nucleon as a function of $g\phi/M$ at $k_F=1.3\ \text{fm}^{-1}$ in the modified one-loop approximation. The dot-dashed line shows the contribution from the pion, the dashed line gives the contribution from the scalar meson, and the solid line shows the sum of these contributions.

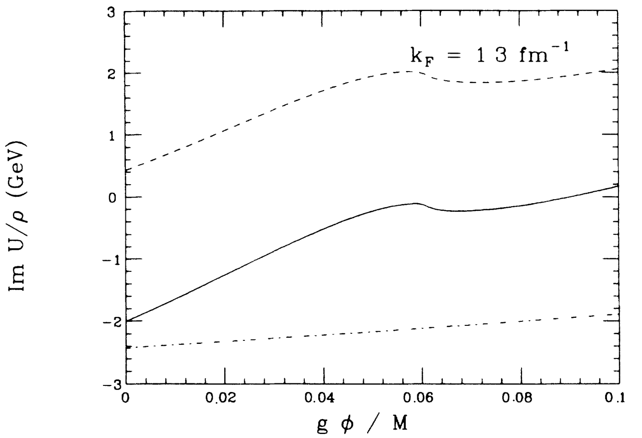


FIG. 12. The same as Fig. 11, but with the subtractions from the renormalization conditions (4.7) also applied to the imaginary part.

imaginary parts of the energy density is reminiscent of the conventional one-loop result (Fig. 2).

VII. CONCLUSION

We have studied loop expansion techniques for the linear σ model. While the conventional loop expansion for the Walecka model of quantum hadrodynamics, as carried out to first and second order, does not show any signs of convergence, the additional constraints due to chiral symmetry might change the situation. We have reviewed the situation for the linear σ model: The conventional loop expansion gives rise to a complex effective potential, even to first order in $\hbar$. Due to the strong attractive interactions with the classical scalar field, the pion becomes tachyonic even at very small densities. The resulting imaginary part of the effective potential is of the same order of magnitude as its real part and cannot simply be disregarded as negligible or irrelevant. This problem has traditionally been ignored or at best, mentioned only in passing.[9–11]

The modified loop expansion developed by Weiss[12] offers the prospect to solve these problems, since the repulsive baryon loops are summed to all orders, while the mesonic loops are truncated at some finite order n. We have identified the first order of the modified loop expansion with the renormalized relativistic random-phase approximation. Our results indicate that at least near saturation density, the tachyon problem does not occur in the renormalized RPA. Unfortunately, the infinite summation of baryon loops changes the analytic structure of the propagator, and we have traded the tachyons for ghost poles. These new poles of the propagators in Euclidean space are present for any value of the classical scalar field or density, they occur at rather low values of the momentum, and they give rise to a very large imaginary part of the effective potential.

Ghost poles can be expected in any model that is not asymptotically free, and similar poles are also present in comparable nonrelativistic calculations, producing, for example, an unphysical pion condensate. An expedient solution would be simply to disregard the offending vacuum loops, but this would violate unitarity, Lorentz covariance, and electromagnetic current conservation.[22] In the framework of relativistic approaches to nuclear physics, the challenge is to construct nonperturbative, systematic approximations that avoid these ghost poles or shift them to momenta in the range of validity of perturbative quantum chromodynamics. Methods must be found to eliminate the ghost poles by modifying the high-momentum structure of the vacuum loops. This might be achieved by including short-range correlations and vertex corrections in the hadronic framework, or perhaps by matching the behavior of loop integrands to that predicted by asymptotic QCD.

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