Integrodifferential equation approach. I. Triton and α-particle binding energies

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A single integrodifferential equation in two variables, valid for A nucleons interacting by pure Wigner forces, which has previously only been solved in the extreme and uncoupled adiabatic approximations, is now solved exactly for three- and four-nucleon systems. The results are in good agreement with the values obtained for the binding energies by means of an empirical interpolation formula. This validates all our previous conclusions, in particular that the omission of higher (than two) order correlations in our four-body equation only produces a rather small underbinding. The integrodifferential equation approach is here also extended to spin-dependent forces of the Malfliet-Tjon type, resulting in two coupled integrodifferential equations in two variables. The exact solution and the interpolated adiabatic approximation are again in good agreement. The inclusion of the hypercentral part of the two-body interaction in the definition of the Faddeev-type components again leads to substantial improvement for fully local potentials, acting in all partial waves.

I. INTRODUCTION

In this first paper of a series of applications of the newly developed integrodifferential equation approach (IDEA) to A-body bound-state problems,[1–3] we consider the ground state of $A=3$ and 4 nuclei with nuclear forces of the Malfliet-Tjon type and a few Gaussian-type forces. It has been previously shown that the IDEA is exact for three bosons interacting in S states and reduces to the integrodifferential form of the Faddeev equation in coordinate space.[4] Otherwise, the IDEA has been shown to take all the two-body correlations into account exactly[3] even for $A \geq 4$. Because of the restriction to two-body correlations, the IDEA does not increase rapidly in complexity with A and can even be successfully applied to 16-fermion bound systems.[5] Even for $A \geq 4$ the integrodifferential equations of the IDEA remain two-variable equations. The only significant increase in complexity is the step from $A=3$ to 4, when the integral kernel also includes disconnected pairs, e.g., the pairs (1,2) and (3,4). On the other hand, the adiabatic approximations to the integrodifferential equation reduce it to two one-variable equations, which improve in accuracy with increasing A. For larger nuclei the major difficulty is the calculation of the weight functions which occur in the integrodifferential equation from the nuclear structure. For $A=3$ and 4 this is no problem at all, but the accuracy of the extreme and uncoupled adiabatic approximations, which provide upper and lower bounds to the exact binding energy of our equation, is reduced. This emphasizes the need of an exact solution of the two-variable integrodifferential equations in this case. Another advantage of such a solution is that in this way we can in principle compare not only the binding energies, but also the bound-state wave functions, of the two adiabatic approximations, with the exact ones. This will, however, be left for the next paper. In addition, we show that the inclusion of the hypercentral component of the two-body local potential, operating in all orbitals in the IDEA, takes the effect of the higher partial waves largely into account without requiring the solution of a coupled system of integrodifferential equations.

The main purpose of this work is to compare the adiabatic approximations for the wave functions and the binding energies to the results obtained from the exact solution of the two-variable equations for three- and four-nucleon systems, to compare those results with calculations by means of other methods, and to investigate the advantage of incorporating the hypercentral potential in the definition of the Faddeev components of the wave functions.

Another purpose is to demonstrate in particular the excellence of the approximation which includes only two-particle correlations for four-nucleon systems. Finally, we wish to draw attention to the usefulness of Faddeev-type integrodifferential equations in two variables for A nucleon systems with $A \geq 4$.

In the next paper we will study explicitly the configuration space Faddeev-type amplitudes for three- and four-nucleon systems in the spirit of the three-body graphical studies of Payne and co-workers[6–8] and demonstrate the surprising accuracy of the wave functions calculated in the adiabatic approximations. In this paper we briefly sketch the formalism of the integrodifferential equations for A particles in Sec. II, followed by more details for the cases $A=3$ and 4 in Sec. III. The results of our calculations and comparisons with other works are given in Sec. IV. These sections are sup-

plemented by some details of the spin-isospin states and the wave function symmetries in Appendix A, the projection on the $\mathbf{r}_{ij}$ space in Appendix B, the numerical calculations in Appendix C, and finally the adiabatic approximation for the coupled channels in Appendix D. Conclusions and an outlook on future work are given in Sec. V.

II. INTEGRODIFFERENTIAL EQUATION APPROACH FOR A PARTICLES

The A-body wave function of the Schrödinger equation can be written as

$$\Psi(\mathbf{x})=\sum_{i<j\le A}\psi(\mathbf{r}_{ij},\mathbf{x}) , \tag{2.1}$$

where the $\psi(\mathbf{r}_{ij},\mathbf{x})$ represent Faddeev-type components and are defined by

$$(T-E)\psi(\mathbf{r}_{ij},\mathbf{x})=-V(r_{ij})\sum_{k<l\le A}\psi(\mathbf{r}_{kl},\mathbf{x}) . \tag{2.2}$$

Here $\mathbf{x}$ represents all the coordinates and $\mathbf{r}_{ij}=\mathbf{r}_i-\mathbf{r}_j$ if $\mathbf{r}_i$ is the coordinate of particle i. The hyperradius is given by $r^2=2/A\sum r_{ij}^2$. Assuming that the expansion in the so-called potential harmonics (PH) introduced by Fabre de la Ripelle[9] is not only extremely accurate for three-body bound systems, as shown by Erens,[10] but also for the A-body wave function, we can use the ansatz[1,11]

$$\psi(\mathbf{r}_{ij},\mathbf{x})=H_{[L_m]}(\mathbf{x})F(\mathbf{r}_{ij},r) , \tag{2.3}$$

where the harmonic polynomial $H_{[L_m]}(\mathbf{x})$, which defines the ground state of a system of fermions, can be extracted from a sum of harmonic oscillator Slater determinants, where $L_m=\sum_L(2n_L+l_L)$, in terms of the radial (n_L) and orbital (l_L) quantum numbers of the individual harmonic oscillator occupied states.[4,11] This is a consequence of the hypercentral nature of a sum of harmonic oscillator potentials, providing a direct relationship between individual harmonic oscillator states and harmonic polynomials.

Inserting Eq. (2.3) into Eq. (2.2), we obtain

$$(T-E)H_{[L_m]}(\mathbf{x})F(\mathbf{r}_{ij},r)=-V(r_{ij})H_{[L_m]}(\mathbf{x})\sum_{k<l\le A}F(\mathbf{r}_{kl},r) . \tag{2.4}$$

For bosons when all particles, except possibly the pair (ij), are in S states, we have $L_m=0$ and Eq. (2.4) reduces to[14]

$$(T-E)F(\mathbf{r}_{ij},r)=-V(r_{ij})\sum_{k<l\le A}F(\mathbf{r}_{kl},r) , \tag{2.5}$$

which is of the same form as the Faddeev equation for three bodies ($\boldsymbol{\rho}_k$ represents the Jacobi coordinate of the spectator particle),

$$(T-E)F(\mathbf{r}_{ij},\boldsymbol{\rho}_k)=-V(r_{ij})\sum_{l<m\le 3}F(\mathbf{r}_{lm},\boldsymbol{\rho}_n)\quad (l,m)\neq n , \tag{2.6}$$

except that in Eq. (2.5) the coupling between the orbitals $l\neq 0$ of the spectator particle and the interacting pair is neglected, since the spectator particle remains in an S state. For $A=4$ a further cluster decomposition of the wave function then leads to the exact Faddeev-Yakubovsky integrodifferential equations, which take all the higher-order correlations[12] into account. If we project Eq. (2.5) on the $\mathbf{r}_{ij}$ space, it generates the integrodifferential equation applicable to S-state projected potentials, which for $A=3$ reduces to the exact Faddeev three-body equation, since $F(\mathbf{r}_{ij},\boldsymbol{\rho}_k)$ can be written as $F(r_{ij},r)$ in this case.[4] If we have to deal with a local potential which operates in all orbitals, $F(\mathbf{r}_{ij},\boldsymbol{\rho}_k)$ has to be expanded into components associated with an interacting pair in a definite orbital state, while the spectator relative orbitals also have to be taken into account. This results in a relatively large system of coupled integrodifferential equations, especially for realistic forces.

It has been shown previously[3,13] that by extracting the first term $V_0(r)$ of the PH expansion, which takes the most important part of the interaction into account, we obtain the modified equation

$$\left[T+\frac{A(A-1)}{2}V_0(r)-E\right]F(\mathbf{r}_{ij},r)=-[V(r_{ij})-V_0(r)]\sum_{k<l\le A}F(\mathbf{r}_{kl},r) . \tag{2.7}$$

We specifically denote this equation by IDEA,[14,15] while Eq. (2.5) valid for S-state projected potentials only has been denoted by SIDE.[15] In the exact Faddeev equation (2.6) the motions, where the spectator particle is in an $l>0$ orbital, are included, but in our modified equation (2.7) the spectator particle is always in an S state. The most important feature of Eq. (2.7) is that we isolate in the right-hand side that part of the interaction which generates the correlations. The reason for this is that for realistic interactions the potential energy originating from the hypercentral part $V_0(r)$ of the nuclear potential just balances the kinetic energy in such a way that all binding is due to the residual part of the potentials, generating the correlations.

To solve Eq. (2.7), we write the total wave function as

$$\Psi=r^{-(D-1)/2}\sum_{i<j\le A}P(2r_{ij}^2/r^2-1,r) , \tag{2.8}$$

where $D=3(A-1)$. Setting $z=2r_{ij}^2/r^2-1$ and projecting Eq. (2.7) on the $\mathbf{r}_{ij}$ space, we obtain the equation for $P(z,r)$ (Refs. 1 and 14):

$$\left[\frac{\hbar^2}{m}\left[-\frac{\partial^2}{\partial r^2}+\frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2}-\frac{4}{r^2}\frac{1}{W_0(z)}\frac{\partial}{\partial z}\times(1-z^2)W_0(z)\frac{\partial}{\partial z}\right]+\frac{A(A-1)}{2}V_0(r)-E\right]P(z,r)=-[V(r_{ij})-V_0(r)]\Pi(z,r) , \tag{2.9}$$

where $\Pi(z,r)=\sum_{k<l}\langle r_{ij}|F(r_{kl},r)\rangle$ is the projection of

$\sum_{k<l} F(r_{kl},r)$ on the $\mathbf{r}_{ij}$ space and can be written as[11,14]

$$\Pi(z,r)=P(z,r)+\int_{-1}^{+1} f_{(0)}(z,z')P(z',r)dz' , \quad (2.10)$$

where

$$f_K^2=1+\frac{\{2(A-2)P_k(-\frac{1}{2})+[(A-2)(A-3)/2]P_k(-1)\}}{P_k(1)} , \quad (2.12)$$

while the $P_k(z)$ are polynomials with the weight function $W_0(z)$, and $\mathcal{H}_k$ is a normalization constant:

$$h_K=\int_{-1}^{+1}[P_K(z)]^2 W_0(z)dz . \quad (2.13)$$

For A bosons in S states, the weight function is given by

$$W_0(z)=(1-z)^{\alpha}(1+z)^{\beta} , \quad (2.14)$$

where $\alpha=(D-5)/2$ and $\beta=\frac{1}{2}$ are associated with the Jacobi polynomials $P_K^{\alpha,\beta}(z)$ to which $P_K(z)$ in Eqs. (2.11)–(2.13) reduces in this case. When the interacting pair is in an orbital state l, we have $\beta=l+\frac{1}{2}$. The integrodifferential equation of the IDEA (2.9) reduces to the one for S-state projected two-body potentials when we set $V_0(r)=0$.

For fermions defined by a harmonic polynomial of minimal degree L_m, the weight function is of the form

$$W_0(z)=(1-z)^{\alpha}(1+z)^{\beta}\rho(z) , \quad \alpha=L_m+(D-5)/2-n , \quad (2.15)$$

where $\rho(z)$ is a polynomial of degree n which has all its zeros outside the range $-1\le z\le 1$. The method for the calculation of the weight function for fermions is given in Refs. 13–15 where the associated polynomials are also defined.

The calculation of the projection function cannot in general be performed analytically, except for bosons in S states.[11,14,16] It has the general structure

$$f_{(0)}(z,z')=2(A-2)f_{(0)}(z,z',-\tfrac{1}{2})+\frac{(A-2)(A-3)}{2}f_{(0)}(z,z',-1) , \quad (2.16)$$

where $f_{(0)}(z,z',-\frac{1}{2})$ is the projection function for connected pairs and $f_{(0)}(z,z',-1)$ the one for disconnected pairs (which only occur for $A\ge 4$). For bosons in S states, they are given in Ref. 16. In that case we have

$$\int_{-1}^{+1} f_{(0)}(z,z',\cos 2\delta)P(z',r)dz' = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)}\frac{1}{\cos\phi\cos\delta}\left[\frac{1}{\sin\phi\sin\delta}\right]^{2\lambda-1} \times\int_a^b[(u-a)(b-u)]^{\lambda-1}P(2u^2-1,r)u\,du , \quad (2.17)$$

where $\cos\phi=\sqrt{(1+z)/2}$, $\sin\phi=\sqrt{(1-z)/2}$, $\lambda=D/2-2$, and $\delta=2\pi/3$ or $\pi/3$ for connected pairs. The limits a and b are given by $a=\cos(\phi+\delta)$ and $b=\cos(\phi-\delta)$. For disconnected pairs we have $\delta=\pi/2$ and $\cos(\pi/2)=0$. Therefore, we have to take the limit of Eq. (2.15) for $\delta\to\pi/2$, which leads to

$$f_{(0)}(z,z')=W(z')\sum_K\frac{(f_K^2-1)}{h_K}P_K(z)P_K(z') \quad (2.11)$$

and

$$\int_{-1}^{+1} f_{(0)}(z,z',-1)P(z',r)dz' = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda-1)}[(1-z)]^{1/2-\lambda} \times\int_{-1}^{z}[-(z+z')]^{\lambda-2}\sqrt{1+z'}P(z',r)dz' . \quad (2.18)$$

III. INTEGRODIFFERENTIAL EQUATIONS FOR THREE- AND FOUR-NUCLEON SYSTEMS

In this first paper we consider only central spin-dependent nucleon-nucleon forces of either Gaussian or Yukawa type like the Malfliet-Tjon potentials. The extension to tensor forces will be given in a subsequent paper. We consider the ground states of the three- and four-nucleon systems. The solution of the single integrodifferential equation given in Sec. II [Eq. (2.9)] describes a state completely symmetrical in space, but the exchange part of the nuclear potential generates states which mix spin-isospin and space symmetry to form mixed symmetry states. These states must be included in the wave function, leading to a system of coupled integrodifferential equations.

To describe the trinucleon ground state for pairs in an even state, we have to include one completely symmetrical (S) state and one mixed symmetry (S') state. In the case of a tensor force component in the nuclear force, it must be completed by one even and one odd state. However, this will not be considered in this paper.

For four nucleons, once again neglecting the odd part of the nucleon-nucleon force, we again have two components, a fully symmetric and a mixed symmetric one.

We now proceed to give the details of the wave function and the coupled system of two-variable equations for three and four nucleons.

For even states, the even part of the central but spin-dependent nucleon-nucleon potential is given by

$$V^+(r_{ij},\sigma,\tau)=V^{1+}(r_{ij})P_{ij}^{1+}+V^{3+}(r_{ij})P_{ij}^{3+} . \quad (3.1)$$

The projection operators P_{ij}^{3+} and P_{ij}^{1+} act on triplet- and singlet-even states, respectively. For three nucleons the representation of the space-even states is given by (see Appendix A)

$$\langle \mathbf{r}_{ij}|\psi_{ij}^+(\mathbf{x},s,t)\rangle = r^{-5/2}Y_{00}(\omega_{ij})[|A\rangle P_0^S(z,r) + |A'_{ij}\rangle P_0^{S'}(z,r)] , \quad (3.2)$$

in terms of the coordinates z and r, and the fully (S) and mixed (S') symmetric components P^S and $P^{S'}$. The pro-

jection of the total wave function on the $\mathbf{r}_{ij}$ space becomes (see Appendix B)

$$\langle \mathbf{r}_{ij}|\Psi^+(\mathbf{x},s,t)\rangle = r^{-5/2}Y_{00}(\omega_{ij})[|A\rangle\Pi_0^S(z,r) + |A'_{ij}\rangle\Pi_0^{S'}(z,r)] , \quad (3.3)$$

where

$$\Pi_0^S(z,r) = P_0^S(z,r) + \frac{2}{[3(1-z^2)]^{1/2}}\int_{z_-}^{z_+} P_0^S(z',r)dz' \quad (3.4)$$

and

$$\Pi_0^{S'}(z,r) = P_0^{S'}(z,r) - \frac{1}{[3(1-z^2)]^{1/2}}\int_{z_-}^{z_+} P_0^{S'}(z',r)dz' . \quad (3.5)$$

The coupled integrodifferential equations for the IDEA are now given (see Appendix B) by

$$\left[\frac{\hbar^2}{m}\nabla_0^2 - \frac{A(A-1)}{2}V_0(r)+E\right]P_0^S(z,r) = \left[\frac{V^{1+}+V^{3+}}{2} - V_0(r)\right]\Pi_0^S(z,r) + \frac{V^{1+}-V^{3+}}{2}\Pi_0^{S'}(z,r) , \quad (3.6)$$

$$\left[\frac{\hbar^2}{m}\nabla_0^2 - \frac{A(A-1)}{2}V_0(r)+E\right)P_{S'}(z,r) = \frac{V^{1+}-V^{3+}}{2}\Pi_0^S(z,r) + \left[\frac{V^{1+}+V^{3+}}{2} - V_0(r)\right]\Pi_0^{S'}(z,r) , \quad (3.7)$$

where $V^{(\,)}$ stands for $V^{(\,)}(r\sqrt{(1+z)/2})$ and the Laplacian ∇_0^2 is given by

$$\nabla_0^2 = \frac{\partial^2}{\partial r^2} - \frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2} + \frac{4}{r^2}\frac{1}{W_0(z)}\frac{\partial}{\partial z}(1-z^2)W_0(z)\frac{\partial}{\partial z} , \quad (3.8)$$

with $\mathcal{L}_0=(D-3)/2$, while the weight function $W_0(z)$ is given by Eq. (2.14). The hypercentral potential $V_0(r)$ is as mentioned before given by $[D=3(A-1)]$

$$V_0(r) = \frac{\int d\Omega \frac{(V^{1+}+V^{3+})}{2}}{\int d\Omega} = \frac{1}{B[(D-3)/2,\frac{3}{2}]} \times \int_0^1 (1-u^2)^{(D-5)/2}[V^{1+}(ru)+V^{3+}(ru)]u^2\,du , \quad (3.9)$$

where B is the beta function. Setting $V_0(r)$ equal to zero, we revert to the SIDE for three nucleons, which has been derived in Ref. 3.

For four nucleons we have

$$\langle \mathbf{r}_{ij}|\Psi^+(\mathbf{x},s,t)\rangle = r^{-4}Y_{00}(\omega_{ij})[|A\rangle\Pi_0^S(z,r) + |A'_{ij}\rangle P_0^{S'}(z,r)] , \quad (3.10)$$

while the projection of the total wave function Ψ on the $\mathbf{r}_{ij}$ space is given by (see Appendix B)

$$\langle \mathbf{r}_{ij}|\Psi^+(\mathbf{x},s,t)\rangle = r^{-4}Y_{00}(\omega_{ij})[|A\rangle\Pi_0^S(z,r) + |A'_{ij}\rangle\Pi_0^{S'}(z,r)] . \quad (3.11)$$

We obtain, in the same way as for three nucleons,

$$\Pi_0^S(z,r) = P_0^S(z,r) + \int_{-1}^{+1}[4f_{(0)}(z,z',-\tfrac{1}{2}) + f_{(0)}(z,z',-1)]P_0^S(z',r)dz' \quad (3.12)$$

and

$$\Pi_0^{S'}(z,r) = P_0^{S'}(z,r) + \int_{-1}^{+1}[-2f_{(0)}(z,z',-\tfrac{1}{2}) + f_{(0)}(z,z',-1)]P_0^{S'}(z',r)dz' . \quad (3.13)$$

The kernels $f_{(0)}(z,z',-\frac{1}{2})$ and $f_{(0)}(z,z',-1)$ refer to connected and disconnected pairs (ij) and (k,l), respectively, and are given by Eqs. (2.17) and (2.18).

For four nucleons we derive (see Appendix B) the same set of coupled equations [Eqs. (3.6) and (3.7)] as in the three-nucleon case, but now with the new projection functions $\Pi_0^S(z,r)$ and $\Pi_0^{S'}(z,r)$ defined above in Eqs. (3.12) and (3.13).

The normalization of the wave function can be obtained by projecting Ψ on the $\mathbf{r}_{ij}$ space as described in Ref. 3. In the presence of mixed symmetry states the result obtained is given by

$$\langle\Psi|\Psi\rangle = \frac{A(A-1)}{2}[Y_0(D-3)]^{-2}\int_0^\infty dr\int_{-1}^{+1}2^{-D/2}(1-z)^{(D-5)/2}(1+z)^{1/2}[P_0^S(z,r)\Pi_0^S(z,r)+P_0^{S'}(z,r)\Pi_0^{S'}(z,r)]dz , \quad (3.14)$$

for $A=3$ or 4 nucleons.

The coupled integrodifferential equations in two variables have been solved numerically (for the details see Appendix C), and the results are given in Sec. IV. In addition, we have also solved them in the extreme and uncoupled adiabatic approximations (EAA and UAA), which have been applied previously for three- and four-nucleon systems interacting

by means of the spin-independent Malfliet-Tjon V forces.[15] For our coupled system of integrodifferential equations, the adiabatic approximations are generalized in Appendix D. We again start by writing ($P=P^S$ or $P^{S'}$)

$$P(z,r)=P_\lambda(z,r)u_\lambda(r)\ , \tag{3.15}$$

and obtain in the EAA the coupled system

$$\left[\frac{4\hbar^2}{mr^2}\frac{1}{W_0(z)}\frac{d}{dz}(1-z^2)W_0(z)\frac{d}{dz}+U_\lambda(r)\right]P_\lambda^S(z,r)=\left[\frac{V^{1+}+V^{3+}}{2}-V_0(r)\right]\Pi_\lambda^S(z,r)+\frac{V^{1+}-V^{3+}}{2}\Pi_\lambda^{S'}(z,r)\ ,$$
$$\left[\frac{4\hbar^2}{mr^2}\frac{1}{W_0(z)}\frac{d}{dz}(1-z^2)W_0(z)\frac{d}{dz}+U_\lambda(r)\right]P_\lambda^{S'}(z,r)=\frac{V^{1+}-V^{3+}}{2}\Pi_\lambda^S(z,r)+\left[\frac{V^{1+}+V^{3+}}{2}-V_0(r)\right]\Pi_\lambda^{S'}(z,r)\ , \tag{3.16}$$

to determine the eigenpotential $U_\lambda(r)$, while the binding energy is determined from

$$\left[\frac{\hbar^2}{m}\left[\frac{d^2}{dr^2}-\frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2}\right]-\frac{A(A-1)}{2}V_0(r)-U_\lambda(r)+E\right]u_\lambda(r)=0\ . \tag{3.17}$$

These equations are solved numerically in the usual way described in Ref. 14.

IV. CALCULATIONS AND RESULTS

In two previous papers[14,15] we have calculated the binding energy for $A=3$ and 4 nucleons using the MT-V and other spin-independent nucleon-nucleon forces. The two-variable integrodifferential equation was not solved exactly in that case, but only in the EAA and UAA. To estimate the correct binding energy, we used the interpolation formula

$$E_I=E^{\mathrm{UAA}}+0.2(E^{\mathrm{EAA}}-E^{\mathrm{UAA}})\ , \tag{4.1}$$

and compared our results with those obtained by other methods.

In this paper we calculate the binding energies by means of an exact solution of the single two-variable integrodifferential equation [Eq. (2.9)] applicable for this case where no S' state occurs, both for the S-wave projected MT-V potential (SIDE) and the IDEA [inclusion of $V_0(r)$]. In the latter case one should of course compare with calculations where the MT-V has been treated as a true potential, acting in all partial waves. In Table I we compare the results of the EAA, UAA, and the interpolated values E_I with the binding energies calculated by means of the two-variable integrodifferential equation E_{ex} and by other methods. For the four-nucleon system the corresponding data are given in Table II.

It is seen from Table I that the interpolated values E_I of the adiabatic approximations are quite close to the ones found by the exact two-variable solution E_{ex} in the case of the SIDE. The discrepancy can be larger (up to 0.06 MeV) in the case of the IDEA.

From Table II we see that for four nucleons agreement is very good in the case of the SIDE and only slightly less for the IDEA (a discrepancy of up to 0.17 MeV can then occur, but it is in general far less).

These results therefore confirm those previously ob-

TABLE I. Three-nucleon binding energies (in MeV) calculated with Wigner forces for the exact solution of the integrodifferential equation (E_{ex}), the adiabatic approximation (E^{EAA}, E^{UAA}, and E_I), and other methods, i.e., Faddeev (S-projected or fully local potential), hyperspherical harmonic expansion method (HHEM), equivalent two-body method (ETBM), and Green's function Monte Carlo method (GFMC). See. Ref. 15.

	SIDE				Other methods
Potential	E^{EAA}	E^{UAA}	E_I	E_{ex}	Faddeev
MT-V (Ref. 10)	7.88	7.55	7.61	7.59	
MT-V (Ref. 17)	7.83	7.49	7.56	7.54	7.541 (Ref. 6)
MT-V (Ref. 18)	8.34	7.99	8.06	8.04	
S_3 (Ref. 19)	6.90	6.26	6.39	6.41	6.40 (Ref. 20)
S_4 (Ref. 22)	7.18	6.88	6.94	6.93	

	IDEA				Other methods			
	E^{EAA}	E^{UAA}	E_I	E_{ex}	HHEM	ETBM	Faddeev	GFMC
MT-V (Ref. 10)	8.07	7.70	7.78	7.73	7.783 (Ref. 10)	7.778 (Ref. 21)		
MT-V (Ref. 17)	8.03	7.65	7.73	7.68			7.736 (Ref. 17)	
MT-V (Ref. 18)	8.57	8.17	8.25	8.19			8.253 (Ref. 23)	8.26±0.01 (Ref. 18)
S_3 (Ref. 19)	7.24	6.56	6.69	6.67	6.695 (Ref. 10)	6.677 (Ref. 21)	6.696 (Ref. 17)	
S_4 (Ref. 22)	7.37	7.01	7.08	7.08	7.05 (Ref. 10)	7.04 (Ref. 21)		

TABLE II. As in Table I, but for the four-nucleon binding energies.

Potential	SIDE E^{EAA}	SIDE E^{UAA}	SIDE E_I	SIDE E_{ex}	Other methods Faddeev-Yakubovsky
MT-V (Ref. 10)	29.98	28.46	28.57	28.56	
MT-V (Ref. 17)	28.91	28.38	28.48	28.47	
MT-V (Ref. 18)	30.19	29.63	29.74	29.74	
S_3 (Ref. 19)	26.18	25.26	25.44	25.38	25.5 (Ref. 20)
S_4 (Ref. 22)	28.09	27.66	27.75	27.74	

	IDEA E^{EAA}	IDEA E^{UAA}	IDEA E_I	IDEA E_{ex}	Other methods HHEM	ETBM	ATMS	GFMC
MT-V (Ref. 10)	29.97	29.28	29.42	29.46				
MT-V (Ref. 17)	29.91	29.20	29.34	29.37				
MT-V (Ref. 18)	31.22	30.48	30.63	30.68			31.36 (Ref. 23)	31.3±0.2 (Ref. 24)
S_3 (Ref. 19)	28.09	26.63	26.92	27.09	26.0 (Ref. 25)	26.47 (Ref. 27)		
S_4 (Ref. 22)	29.33	28.55	28.71	28.80	27.9 (Ref. 25)	28.18 (Ref. 27)		

tained in the adiabatic approximation only and all the conclusions we have drawn from them.[14,16]

We now consider the spin-dependent MT-I/III force and solve the two-variable integrodifferential equation both in the SIDE and IDEA, and compare the results with those obtained in the EAA, UAA, and with the interpolated binding energy E_I. The results for three and four nucleons are given in Tables III and IV, respectively. Once again we find a surprisingly small discrepancy of only about 0.03 MeV between E_I and E_{ex}, demonstrating the accuracy of the interpolation formula for E_I. For four nucleons the difference is only about 0.28 MeV. which is also an excellent result.

The excellent agreement between E_I and E_{ex} provides convincing support for a major point made in our previous work,[14,15] namely, that the redefinition of the Faddeev components by inclusion of the hypercentral component of the two-body local potential into the integrodifferential equation takes the effect of the higher partial waves largely into account, obviating to a large extent the need for solving a system of coupled integrodifferential equations, when the two-body potential is a true local potential acting in all partial waves. The results of Table II confirm another important but related point; namely, at least for spin-independent local potentials, the effect of the omission of particle correlations of order higher than 2, implied in the use of our two-variable integrodifferential equation, is small. This becomes clear if we compare the binding energy of 30.68 MeV obtained by the exact solution of the two-variable equation for the MT-V potential, with the parameters of Zabolitzky,[18] to the results obtained by means of the essentially exact GFMC method[24] and to the ATMS method[23] of 31.36 MeV. The difference of 0.68 MeV amounts to only 2%, which is surprisingly small and shows how little the higher-order correlations actually contribute to the binding energy in this case.

For the MT-I/III force which generates a mixed symmetry (S') component in the three- and four-nucleon wave functions, we recover for the SIDE the binding energy of the triton of 8.536 MeV obtained by Payne *et al.*[6] Unfortunately, there appear to be no calculations available for the MT-I/III treated as a local potential (and not as an S-wave projected potential) for comparison purposes. The IDEA, however, does produce a significant improvement of 0.3 MeV over the SIDE, which is considerably more than the improvement of about 0.15 MeV found from Table I for the MT-V potential. For ^{4}He, though, the increase in the binding energy for the MT-I/III potential, by going from the SIDE to the IDEA, is about 1.3 MeV, which is somewhat more than the about 0.9 MeV increase achieved for the MT-V. In the case of the MT-I/III potential, we have again found no four-nucleon results in the literature for the fully local potential to compare ours with. For S-wave projected potentials the binding energy of ^{4}He has been calculated by

TABLE III. Three-nucleon binding energies (in MeV) for the MT-I/III and S_3 potentials.

Potential	SIDE E^{EAA}	SIDE E^{UAA}	SIDE E_I	SIDE E_{ex}	Other Faddeev	Other GFMC
MT-I/III	8.99	8.46	8.57	8.54	8.536 (Ref. 6)	
S_1						

Potential	IDEA E^{AA}	IDEA E^{UAA}	IDEA E_I	IDEA E_{ex}	Other Faddeev	Other GFMC
MT-I/III	9.40	8.76	8.89	8.86		
S_3				8.75	8.76 (Ref. 29)	8.73±0.10 Ref. 32)

TABLE IV. As in Table III, but for the four-nucleon binding energies.

			SIDE		
Potential	E^{EAA}	E^{UAA}	E_I	E_{ex}	Others
MT-I/III	30.45	29.91	30.02	29.74	30.36 (Ref. 26) 29.6 (Ref. 27)
S_3				27.93	
			IDEA		
Potential	E^{EAA}	E^{UAA}	E_I	E_{ex}	Others
MT-I/III	31.78	30.99	31.14	31.02	
S_3				30.37	

Sofianos *et al.*[26] for the MT-I/III in the Unitary Pole Approximation (UPA) and by Tjon[27] using the Hilbert-Schmidt expansion to solve the Faddeev-Yakubovsky equation. The underbinding of 0.62 MeV, which we obtain in the SIDE compared to Ref. 27, is clearly due to the omission of higher-order correlations in our calculations. Finally, we should point out that in the case of the interpolated binding energies, E_I of the triton and ^{4}He are in nearly perfect agreement with the exact binding energies obtained by the numerical solution of the coupled two-variable integrodifferential equations.

For comparison purposes, we also consider the S_3 force regarded as a spin-dependent force.[19] For three nucleons, the binding energy is also given in Table III for the SIDE and IDEA. The latter is compared to the value obtained by Friar[28] in the case of S_3 force acting in all partial waves (fully local potential), by solving the system of 34 coupled two-variable Faddeev equations, and to the essentially exact Green's function Monte Carlo calculations by Carlson.[32] The IDEA with only two coupled integrodifferential equations for the mixed symmetry states S and S', but including $V_0(r)$, practically reproduces the exact results of Friar and Carlson. To be precise our binding energy in this case is 8.75 MeV, while Friar obtained a value of 8.76 MeV and Carlson a value of 8.73 ± 0.10 MeV. The IDEA is even better in this case than for the MT-V potential where there is a difference of 0.06 MeV between the results of the IDEA and the exact solution of the coupled Faddeev equations by Friar.[17] This may be due to the fact that the S_3 potential, which consists of a sum of Gaussians, finite at the origin, generates a solution which converges exponentially in terms of the number K of harmonics occurring in the expansion of the wave function in contrast to a potential like the MT-V. The latter exhibits a r^{-1} singularity at the origin, which leads to a lower convergence of the order of K^{-4} (see Ref. 32). We should always keep in mind that our IDEA has been derived by summing up all potential harmonics occurring in the expansion of the wave functions, and that the faster the convergence of the expansion, the smaller is the contribution of the nonpotential harmonics, describing the N-body correlations with $N>2$, neglected in the IDEA, which only contains two-body correlations.

We note that all our binding energies in Tables I and II are smaller than the corresponding binding energies obtained by means of converged Faddeev calculations. Green's function Monte Carlo, Faddeev-Yakubovsky, or ATMS methods, which are essentially exact or very accurate. This is not surprising since although the set of HH basis functions occurring in the expansion employed for the derivation of our two-variable integrodifferential equations starting from Eq. (2.2) is complete and orthogonal for the partition (i.e., the set of Jacobi coordinates[14]) under consideration, by projecting on the subset of potential harmonics, associated with one particle pair only, we neglected N-body correlations with $N>2$. We therefore omit that part of the Hilbert space spanned by the basis which is associated with many-particle correlations and is orthogonal to the potential basis. This is taken into account by means of additional coupled equations, generating an increase in binding energy, when fully local potentials (not S-wave projected ones[14]) are treated exactly. Only when very accurate solutions are not available, as for the S_3 and S_4 potentials if $A=4$, do we obtain a larger binding energy than the ones published in the literature. Otherwise, our binding energies as compared to essentially exact ones always leave some room for the contribution coming from N-particle correlations with $N>2$.

V. CONCLUSIONS AND OUTLOOK

The most important conclusions to be drawn are in the first place that the interpolated binding energies obtained from the extreme and uncoupled adiabatic approximations are in very good agreement with those calculated by exactly solving the two-variable integrodifferential equation both in the SIDE (S-wave projected potential) and the IDEA (fully local potential) for the MT-V potential. This implies that all conclusions drawn from the values of E_I in Refs. 14 and 15 are valid.

In particular, the binding energies obtained in the IDEA are a significant improvement over those calculated in the SIDE and reflect the fact that the IDEA takes a large part of the increase in binding energy into account, produced by the higher partial waves in the system of coupled integrodifferential equations, by means of a single integrodifferential equation. The IDEA for the MT-V potential indeed produces an underbinding of less than 0.7 MeV compared to the essentially exact GFMC result for ^{4}He.

In this paper we have also treated the MT-I/III force,

requiring the solution of a system of two coupled integrodifferential equations in two variables for the fully and mixed symmetric S and S' components, both for the triton and ^{4}He. The results obtained are qualitatively the same as for the MT-V potential. The interpolated binding energies (from the adiabatic approximation) are in nearly perfect agreement with the exact results for the two-variable equations.

In the SIDE our coupled equations are exact for the three-body case and constitute only a different coordinate space formulation, as compared to the one usually employed by Payne *et al.*,[6] of the three-nucleon Faddeev equation in coordinate space, and as expected we recover the result of Ref. 6. To our knowledge there are no calculations available in the literature which treat the MT-I/III as a fully local potential either for the three- or four-nucleon systems, and we can therefore make no comparisons with our results for the IDEA. For the spin-dependent S_3 force, however, the IDEA with two coupled equations for the mixed symmetry states S and S' reproduces within 0.01 MeV, the exact result of Friar[28] using the S_3 force as a fully local potential in the set of Faddeev equations coupling not only the mixed symmetry states S and S', but also different partial waves. For the SIDE and ^{4}He calculations by Sofianos *et al.*,[26] using the MT-I/III potential as an S-wave projected potential, are available. However, they treat the MT-I/III in the UPA and therefore may have underestimated the exact four-nucleon binding energy. Comparison of these SIDE binding energies from Table II show that omission of higher-order correlations in the SIDE could produce an underbinding of the order of 0.7 MeV. The inclusion of the hypercentral potential in the IDEA (assuming a fully local MT-I/III potential) results in a significant increase of 1.3 MeV in the binding energy. It might be that even greater improvements are possible for realistic nucleon-nucleon forces by means of the IDEA.

This paper is only the first one in a series of papers on the triton and ^{4}He ground states. In the second one we will present graphical studies of the three- and four-nucleon ground-state wave functions with the Malfliet-Tjon forces for the SIDE and IDEA, and compare the exact wave functions with those obtained in the adiabatic approximations. In the third paper of the series realistic nucleon-nucleon forces will be treated.

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APPENDIX A: SPIN-ISOSPIN AND WAVE FUNCTION SYMMETRIES

Three-body system

We may construct the spin-isospin states for three nucleons, $|\xi;\sigma\tau;[f]_r\rangle$, of specific symmetry $[f]_r$, from the direct product of the [21]-spin and [21]-isospin states. For the spin case, the particles (ij) are coupled to the particle k to give the total spin $\sigma=\frac{1}{2}$ either when they are in the triplet (symmetric s) state or in the singlet (antisymmetric a) state. These states are denoted by

$$|\sigma\rangle^n\equiv|\sigma,[21]^n\rangle,\quad n=s \text{ or } a\ , \tag{A1}$$

and similarly for the isospin. Then the following symmetries can be constructed:

$$|A\rangle\equiv|A;\sigma\tau;[111]\rangle \equiv\frac{1}{\sqrt{2}}(|\sigma\rangle^a|\tau\rangle^s-|\sigma\rangle^s|\tau\rangle^a)\ , \quad \text{completely antisymmetric ;} \tag{A2}$$

$$|A'_{ij}\rangle=|A'_{ij};\sigma\tau;[21]^a\rangle \equiv\frac{1}{\sqrt{2}}(|\sigma\rangle^a|\tau\rangle^s+|\sigma\rangle^s|\tau\rangle^a)\ , \quad \text{antisymmetric with respect to } (ij) ; \tag{A3}$$

$$|S'_{ij}\rangle\equiv|S'_{ij};\sigma\tau;[21]^s\rangle \equiv\frac{1}{\sqrt{2}}(|\sigma\rangle^s|\tau\rangle^s-|\sigma\rangle^a|\tau\rangle^a)\ , \quad \text{symmetric with respect to } (ij) ; \tag{A4}$$

$$|S\rangle\equiv|S;\sigma\tau;[3]\rangle \equiv\frac{1}{\sqrt{2}}(|\sigma\rangle^s|\tau\rangle^s+|\sigma\rangle^a|\tau\rangle^a)\ , \quad \text{completely symmetric .} \tag{A5}$$

Four-body system

The spin-isospin states $|\xi;ST;[f]_r\rangle$ for the case $S=T=0$ are given by the direct product of the two irreducible representations

$$[22]\times[22]=[4]+[22]+[1111]\ . \tag{A6}$$

With the definition

$$|S\rangle^n\equiv|S,[22]^n\rangle,\quad n=s \text{ or } a\ , \tag{A7}$$

for the spin and a corresponding one for the isospin, we may construct, similarly to the three-nucleon case, the following states:

$$|A\rangle\equiv|A;ST;[1111]\rangle \equiv\frac{1}{\sqrt{2}}(|S\rangle^a|T\rangle^s-|S\rangle^s|T\rangle^a)\ , \quad \text{completely antisymmetric ;} \tag{A8}$$

$$|A'_{ij}\rangle\equiv|A'_{ij};ST;[22]^a\rangle \equiv\frac{1}{\sqrt{2}}(|S\rangle^a|T\rangle^s+|S\rangle^s|T\rangle^a)\ , \quad \text{antisymmetric with respect to } (ij) ; \tag{A9}$$

$$|S'_{ij}\rangle \equiv |S'_{ij};ST;[22]^s\rangle$$
$$\equiv \frac{1}{\sqrt{2}}(|S\rangle^s|T\rangle^s - |S\rangle^a|T\rangle^a) ,$$
symmetric with respect to (ij) ; (A10)
$$|S\rangle \equiv |S;ST;[4]\rangle$$
$$\equiv \frac{1}{\sqrt{2}}(|S\rangle^s|T\rangle^s + |S\rangle^a|T\rangle^a) ,$$
completely symmetric . (A11)

Wave functions for the three- and four-nucleon systems

For pure central interactions the three-body Faddeev components can be written as

$$\psi_{ij}(\mathbf{x};\sigma,\tau) = |A\rangle\psi^S_{ij}(\mathbf{x}) + |A'_{ij}\rangle\psi^{S'}_{ij}(\mathbf{x}) + |S'_{ij}\rangle\psi^{A'}_{ij}(\mathbf{x}) + |S\rangle\psi^A_{ij}(\mathbf{x}) , \quad \text{(A12)}$$

where the spatial partial waves have the parity opposite to the one of the associated spin-isospin states in an exchange of the nucleons i and j. Considering only the space-even states with respect to the pair (ij), we have

$$\psi_{ij}(\mathbf{x};\sigma,\tau) = |A\rangle\psi^S_{ij}(\mathbf{x}) + |A'_{ij}\rangle\psi^{S'}_{ij}(\mathbf{x}) . \quad \text{(A13)}$$

However, the states $|A'_{ki}\rangle$ and $|A'_{jk}\rangle$ are also required. They can be easily expressed in terms of the (ij) states via the usual recoupling procedure. The result is

$$|A'_{ki}\rangle = -\frac{1}{2}|A'_{ij}\rangle + \frac{\sqrt{3}}{2}|S'_{ij}\rangle ,$$
$$|A'_{jk}\rangle = -\frac{1}{2}|A'_{ij}\rangle - \frac{\sqrt{3}}{2}|S'_{ij}\rangle . \quad \text{(A14)}$$

The four-body Faddeev components are treated in the same way. Keeping only the even states, we have

$$\psi_{ij}(\mathbf{x};S,T) = |A\rangle\psi^S_{ij}(\mathbf{x}) + |A'_{ij}\rangle\psi^{S'}_{ij}(\mathbf{x}) . \quad \text{(A15)}$$

The additional spin-isospin states for the (ki), (jk), (li), (jl), and (kl) configurations are given by

$$|A'_{kl}\rangle = |A'_{ij}\rangle ,$$
$$|A'_{jl}\rangle = |A'_{ki}\rangle = -\frac{1}{2}|A'_{ij}\rangle + \frac{\sqrt{3}}{2}|S'_{ij}\rangle , \quad \text{(A16)}$$
$$|A'_{li}\rangle = |A'_{jk}\rangle = -\frac{1}{2}|A'_{ij}\rangle - \frac{\sqrt{3}}{2}|S'_{ij}\rangle .$$

APPENDIX B: PROJECTION ON THE $\mathbf{r}_{ij}$ SPACE

Three-body system

Assuming a nucleon-nucleon potential of the form given in Eq. (3.1) and considering only the space-even states as in Eq. (A13), we project the three-body Faddeev components for the configurations (ij,k), (ki,j), and (jk,i) on the $\mathbf{r}_{ij}$ space:

$$\langle \mathbf{r}_{ij}|\psi^+_{ij}(\mathbf{x},s,t)\rangle = \frac{Y_{00}}{r^{-5/2}}[|A\rangle P^S_0(z,r) + |A'_{ij}\rangle P^{S'}_0(z,r)] ,$$
$$\langle \mathbf{r}_{ij}|\psi^+_{ki}(\mathbf{x},s,t)\rangle = \frac{Y_{00}}{r^{-5/2}}\left[|A\rangle I^S_0(z,r) - \tfrac{1}{2}|A'_{ij}\rangle I^{S'}_0(z,r) + \frac{\sqrt{3}}{2}|S'_{ij}\rangle I^{S'}_0(z,r)\right] , \quad \text{(B1)}$$
$$\langle \mathbf{r}_{ij}|\psi^+_{jk}(\mathbf{x},s,t)\rangle = \frac{Y_{00}}{r^{-5/2}}\left[|A\rangle I^S_0(z,r) - \tfrac{1}{2}|A'_{ij}\rangle I^{S'}_0(z,r) - \frac{\sqrt{3}}{2}|S'_{ij}\rangle I^{S'}_0(z,r)\right] ,$$

with

$$I^n_0(z,r) = \frac{1}{[3(1-z^2)]^{1/2}}\int_{z_-}^{z_+} P^n_0(z',r)dz' , \quad n = S \text{ or } S' , \quad \text{(B2)}$$
$$z_\pm = \tfrac{1}{2}\{-z \pm [3(1-z^2)]^{1/2}\} .$$

Therefore, by projecting the Faddeev equation on the spin-isospin states $\langle A|$ and $\langle A'_{ij}|$, we get the result for the SIDE:

$$\left[\frac{\hbar^2}{m}\nabla^2_0 + E\right]P^S_0(z,r) = \frac{V^{1+}+V^{3+}}{2}\Pi^S_0(z,r) + \frac{V^{1+}-V^{3+}}{2}\Pi^{S'}_0(z,r) , \quad \text{(B3)}$$
$$\left[\frac{\hbar^2}{m}\nabla^2_0 + E\right]P^{S'}_0(z,r) = \frac{V^{1+}-V^{3+}}{2}\Pi^S_0(z,r) + \frac{V^1+V^{3+}}{2}\Pi^{S'}_0(z,r) , \quad \text{(B4)}$$

where we have made use of the matrix elements

$$\langle A|P^{1+}_{ij}|A\rangle = \langle A'_{ij}|P^{1+}_{ij}|A'_{ij}\rangle = \langle A|P^{3+}_{ij}|A\rangle = \langle A'_{ij}|P^{3+}_{ij}|A'_{ij}\rangle = \langle A|P^{1+}_{ij}|A'_{ij}\rangle = -\langle A|P^{3+}_{ij}|A'_{ij}\rangle = \tfrac{1}{2} . \quad \text{(B5)}$$

The functions $\Pi^S_0(z,r)$ and $\Pi^{S'}_0(z,r)$ are defined in Eqs. (3.4) and (3.5), respectively. Introducing the hypercentral potential $V_0(r)$ in the same way as in the boson case (see Sec. II), the coupled equations for the IDEA are finally obtained [Eqs. (3.6) and (3.7)].

Four-body system

As in the three-body case we consider only the fully and mixed symmetric spatial states; i.e., we start with Eq. (A15). Projection on the $\mathbf{r}_{ij}$ space yields[14]

$$\langle \mathbf{r}_{ij}|\psi_{ij}^{+}(\mathbf{x},s,t)\rangle = \frac{Y_{00}}{r^{-4}}[|A\rangle P_0^S(z,r)+|A'_{ij}\rangle P_0^{S'}(z,r)] , \tag{B6}$$

$$\langle \mathbf{r}_{ij}|\psi_{nm}^{+}(\mathbf{x},s,t)\rangle = \frac{Y_{00}}{r^{-4}}[|A\rangle I_0^S(z,r,\varphi_N) + |A'_{nm}\rangle I_0^{S'}(z,r,\varphi_N)] ,$$

where the kernels $I_0(z,r,\varphi_N)$ are given by

$$I_0^n(z,r,\varphi_N)=\int_{-1}^{+1} f_{(0)}(z,z',\cos 2\varphi_N) P_0^n(z',r)\,dz' , \tag{B7}$$

and φ_N takes the values $2\pi/3$ and $\pi/2$ for connected pairs like (ij,ki) and disconnected pairs like (ij,kl), respectively. Inserting the results of Eq. (A16) into Eq. (B6), we obtain for the total four-body wave function, projected on the $\mathbf{r}_{ij}$ space, the identity

$$\langle \mathbf{r}_{ij}|\psi_{ij}^{+}(\mathbf{x},s,t)\rangle = \frac{Y_{00}}{r^{-4}}\{|A\rangle[P_0^S(z,r)+4I_c^S(z,r)+I_d^S(z,r)]+|A'_{ij}\rangle[P_0^{S'}(z,r)-2I_c^{S'}(z,r)+I_d^{S'}(z,r)]\} . \tag{B8}$$

The indices c and d indicate connected and disconnected pairs, respectively. Using the fact that Eq. (B5) remains valid in the four-body case, we finally arrive at the same set of coupled equations as for the three-body system, i.e., Eq. (B4), the only difference being that now the functions $\Pi_0^S(z,r)$ and $\Pi_0^{S'}(z,r)$ are given by Eqs. (3.12) and (3.13).

APPENDIX C: EXACT SOLUTION OF THE TWO-VARIABLE INTEGRODIFFERENTIAL EQUATIONS

In order to solve the two-variable integrodifferential equations (2.9), (3.6), and (3.7) exactly without introducing the adiabatic approximation, we have used an expansion of the Faddeev-type components in terms of Hermite splines. The resulting equations have then been solved by the method of orthogonal collocation.[30] However, since we have mainly followed the treatment of Payne[31] in this respect, we do not give any details here. There remains only one point to be discussed.

Regularity of the solution requires the following asymptotic behavior at the boundaries ($E=\hbar^2\kappa^2/m$):

$$\begin{aligned} &P_0(z,r)\xrightarrow[z\to -1]{}\text{const}\times f(r) ,\\ &P_0(z,r)\xrightarrow[z\to +1]{}\text{const}\times f(r) , \end{aligned} \tag{C1}$$

$$\begin{aligned} &P_0(z,r)\xrightarrow[r\to 0]{} r^{(\mathcal{L}_0+1)} f(z) ,\\ &P_0(z,r)\xrightarrow[r\to \infty]{} e^{-\kappa r} f(z) . \end{aligned} \tag{C2}$$

The fact that the irregular solution with respect to the coordinate z tends to infinity at the boundaries while the regular one is giving a finite, but nonzero value [see Eq. (C1)], suggests that $P_0(z,r)$ be replaced by the function $p_0(z,r)$:

$$P_0(z,r)=\frac{e^{-\kappa r}p_0(z,r)}{(1-z)^{\alpha}(1+z)^{\beta}} . \tag{C3}$$

The exponential factor is also extracted from the function $P_0(z,r)$, because $p_0(z,r)$ is expected to behave more smoothly for large values of the hyperradius r. The new function $p_0(z,r)$ fulfills the boundary conditions

$$\begin{aligned} &p_0(z,r)\xrightarrow[z\to -1]{}(1+z)^{\beta} f(r) ,\\ &p_0(z,r)\xrightarrow[z\to +1]{}(1-z)^{\alpha} f(r) , \end{aligned} \tag{C4}$$

$$\begin{aligned} &p_0(z,r)\xrightarrow[r\to 0]{} r^{(\mathcal{L}_0+1)} f(z) ,\\ &p_0(z,r)\xrightarrow[r\to \infty]{}\text{const}\times f(z) , \end{aligned} \tag{C5}$$

and is a solution of the equation (units $\hbar=m=1$)

$$\begin{aligned} &\left[\frac{\partial^2}{\partial r^2}-2\kappa\frac{\partial}{\partial r}+\frac{4(\alpha+\beta)-\mathcal{L}_0(\mathcal{L}_0+1)}{r^2}\right.\\ &\quad+\frac{4}{r^2}\left[(1-z^2)\frac{\partial^2}{\partial z^2}+[\alpha-\beta+(\alpha+\beta-2)z]\frac{\partial}{\partial z}\right]\\ &\quad\left.-\frac{A(A-1)}{2}V_0(r)\right]p_0(z,r)\\ &\quad=\{V[r\sqrt{(1+z)/2}]-V_0(r)\}\pi_0(z,r) , \end{aligned} \tag{C6}$$

where $\pi_0(z,r)$ is defined by

$$\Pi_0(z,r)=\frac{e^{-\kappa r}\pi_0(z,r)}{(1-z)^{\alpha}(1+z)^{\beta}} . \tag{C7}$$

The same statements are valid in the case of coupled equations for the mixed symmetry states.

As mentioned above, the orthogonal collocation technique has been used for the computation of the solution of Eq. (C6). The intervals for the coordinates z and r have been divided into I_z and I_r subintervals, respectively. In the three-body case one needs at least $I_z=12$ and $I_r=9$ partitions to achieve an accuracy of 0.01 MeV, both for the MT-V and the MT-I/III potentials, while the values $I_z=18$ and $I_r=13$ are required for an accuracy of 0.001 MeV. This means that the numerical accuracy of the integrodifferential equation written in r-z coordinates is comparable to the one given in r-θ coordinates, which is used by Payne *et al.*[6] To be precise, they need $I_\theta=17$ and $I_r=15$ subintervals for the θ and r coordinates to achieve an accuracy of 0.001 MeV[31]. In the four-nucleon case values of $I_z=15$ and $I_r=12$ are required for an accuracy of 0.01 MeV.

APPENDIX D: ADIABATIC APPROXIMATION FOR THE COUPLED CHANNELS

Introducing the definitions

$$D_r \equiv \frac{\partial^2}{\partial r^2} - \frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2} ,$$
$$D_z \equiv \frac{1}{W_0(z)} \frac{\partial}{\partial z}(1-z^2)W_0(z)\frac{\partial}{\partial z} , \tag{D1}$$

and

$$V_{11}=V_{22}=\frac{V^{1}+V^{3+}}{2}, \quad V_{12}=V_{21}=\frac{V^{1+}-V^{3+}}{2} , \tag{D2}$$

we can write the SIDE for the coupled channels in the matrix form ($\hbar=m=1$)

$$\begin{bmatrix} D_r+\frac{4}{r^2}D_z+E & 0 \\ 0 & D_r+\frac{4}{r^2}D_z+E \end{bmatrix} \begin{bmatrix} P_0^S \\ P_0^{S'} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Pi_0^S \\ \Pi_0^{S'} \end{bmatrix} . \tag{D3}$$

The basic assumption of the adiabatic approximation is that the orbital motion can be separated from the radial motion, which implies that the functions $P_0^n(z,r)$, $n=S$ or S', be replaced by the ansatz

$$P_0^n(z,r)=P_{0\lambda}^n(z,r)u_\lambda(r) . \tag{D4}$$

Therefore, the newly defined functions $P_{0\lambda}^n(z,r)$ should vary slowly with respect to the variable r, in comparison to the variable z. Writing Eq. (D3) in a concise form,

$$\left[1(D_r+E)+\frac{4}{r^2}1D_z\right]\mathbf{P}=\mathbb{V}\mathbf{\Pi} , \tag{D5}$$

and defining the eigenpotential $U_\lambda(r)$ via

$$\left[\frac{4}{r^2}1D_z+1U_\lambda(r)\right]\mathbf{P}_\lambda=\mathbb{V}\mathbf{\Pi}_\lambda , \tag{D6}$$

we finally arrive at the following equation, which determines the binding energy:

$$(D_r+E-U_\lambda(r))u_\lambda(r)=0 . \tag{D7}$$

Explicitely, i.e., not using the matrix notation, the SIDE for the coupled channels is given by

$$\left[\frac{4}{r^2}\frac{1}{W_0(z)}\frac{d}{dz}(1-z^2)W_0(z)\frac{d}{dz}+U_\lambda(r)\right]P_{0\lambda}^S(z,r) = \frac{V^{1+}+V^{3+}}{2}\Pi_{0\lambda}^S(z,r)+\frac{V^{1+}-V^{3+}}{2}\Pi_{0\lambda}^{S'}(z,r) , \tag{D8}$$

$$\left[\frac{4}{r^2}\frac{1}{W_0(z)}\frac{d}{dz}(1-z^2)W_0(z)\frac{d}{dz}+U_\lambda(r)\right]P_{0\lambda}^{S'}(z,r) = \frac{V^{1+}-V^{3+}}{2}\Pi_{0\lambda}^S(z,r)+\frac{V^{1+}+V^{3+}}{2}\Pi_{0\lambda}^{S'}(z,r) ,$$

$$\left[\frac{d^2}{dr^2}-\frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2}-U_\lambda(r)+E\right]u_\lambda(r)=0 . \tag{D9}$$

Introducing the hypercentral potential $V_0(r)$ in the usual way, Eqs. (D8) and (D9) are easily generalized to the case of the IDEA to obtain Eqs. (3.16) and (3.17).

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