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rms radius of the deuteron

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An explanation is given for the nearly linear relationship between the deuteron radius and the triplet scattering length as predicted by potential models of the *NN* interaction. This is accomplished by showing that the quantity $\langle r^2 \rangle / a_t^2$ can be expanded to high accuracy in terms of the small parameter $x = r_t / a_t$. We show that shape dependence only enters at order x^3 and higher. Possible reasons for the disagreement with the experimental result are discussed.

I. INTRODUCTION

The most loosely bound stable nucleus, the deuteron, is traditionally viewed as a neutron-proton bound state with its properties largely governed[1] by the one-pion exchange potential (OPEP) tail of the *NN* potential. For realistic *NN* potentials the ^3S scattering length, a_t, and the calculated deuteron rms radius, $\langle r^2 \rangle^{1/2}$, fall within a narrow range of values, typically varying in the third significant figure. Klarsfeld *et al.*[2] have made the interesting observations that a plot of $\langle r^2 \rangle^{1/2}$ vs a_t yields, in this narrow range, points that tend to fall on a straight line and that the experimental point is clearly above this line. Recently, van Dijk[3] has examined this problem with various potential models and found that potentials with strong nonlocality may correct this deficiency. In view of these findings we address ourselves here to the following question: To what extent is the mean square radius of the deuteron a shape-independent quantity in potential theory?

It is natural to look for an expansion of the ratio $\langle r^2 \rangle / a_t^2$ in terms of low-energy properties of the potential. However, the choice of expansion parameter is not so trivial since for some choices of this parameter the expansions appear not to converge. For example, if K is the deuteron binding energy and b is a potential width parameter, then we have found that expansions of $\langle r^2 \rangle / a_t^2$ in powers of K/b do not converge. However, we show in Sec. II that this ratio can be expressed as an apparently convergent power series in terms of the small parameter $x = r_t / a_t$, where r_t is the triplet effective range. The first two terms of this series, of order x^0 and x^2, are *universal*, and give the rms radius of the deuteron (for a central potential) to 99% accuracy. This is the shape-independent part, irrespective of the nonlocality or the shape of the potential. However, coefficients of order x^3 and higher do depend sensitively on the nature of the potential. In fact, by considering various models, to be discussed below, we demonstrate that local potentials and nonlocal separable potentials give opposite signs for these coefficients, confirming van Dijk's observation.[3]

In Sec. III we derive an approximate modification of the expansion in the presence of a tensor potential. This modification is given in terms of η, the asymptotic D/S ratio. The resulting series is compared to actual numerical results for some realistic potentials.

Section IV considers the effect of isobar channels on the $\langle r^2 \rangle^{1/2} / a_t$ plot. Our results indicate that inclusion of explicit isobar channels does not help in getting agreement with the experimental value of $\langle r^2 \rangle^{1/2} / a_t$.

II. CENTRAL POTENTIALS

Consider first only a central potential acting in the S state. Denote the normalized bound-state wave function by $u(r)$ where

$$\int_0^\infty u^2(r)dr = 1 \; .$$

Asymptotically $u(r) \sim A e^{-\kappa r}$, where κ is related to the deuteron binding energy B by

$$B = \hbar^2 \kappa^2 / 2\mu$$

and μ is the reduced neutron-proton mass. The asymptotic constant A may be determined from the definition of the effective range

$$\tfrac{1}{2} r_t = \int_0^\infty \left[e^{-2\kappa r} - \frac{1}{A^2} u^2(r) \right] dr \; , \tag{1}$$

from which follows the relation

$$A = [2\kappa / (1 - \kappa r_t)]^{1/2} \; . \tag{2}$$

The asymptotic form of the wave function $u(r)$ is therefore given by

$$u(r) \sim [2\kappa/(1-\kappa r_t)]^{1/2} e^{-\kappa r} . \quad (3)$$

If we compute the mean-square radius of the deuteron

$$\langle r^2 \rangle = \tfrac{1}{4} \int_0^\infty r^2 u^2(r) dr \quad (4)$$

by using for $u(r)$ its asymptotic form given by Eq. (3), we then obtain

$$\langle r^2 \rangle \simeq \frac{1}{8\kappa^2}(1-\kappa r_t)^{-1} . \quad (5)$$

Since our objective is to obtain an approximate expression for $\langle r^2 \rangle$ from effective-range theory, it is consistent to use the approximate relation

$$\kappa \simeq \frac{1}{a_t} + \frac{1}{2} r_t \kappa^2 , \quad (6)$$

whose solution is

$$\kappa = \frac{1}{r_t}[1-(1-2r_t/a_t)^{1/2}] . \quad (7)$$

Substituting this expression for κ into Eq. (5) one obtains the expansion

$$\langle r^2 \rangle / a_t^2 \simeq \tfrac{1}{8} + \tfrac{1}{32} x^2 + \tfrac{1}{16} x^3 + \cdots , \quad (8)$$

where

$$x = r_t / a_t \quad (9)$$

is a small ($\simeq 0.3$) shape-independent parameter. We should point out that the expression, Eq. (1), for r_t is really $r_t(-B,-B)$ in the notation of Wilson[4] whereas normally one means by r_t

$$r_t \equiv r_t(0,0) = \frac{d^2}{dk^2}[k \cot\delta(k)]_{k=0} . \quad (10)$$

The difference between these two quantities is of order κ^2 and does not affect the first two terms in Eq. (8). We proceed now to show, by considering various exactly solvable central potentials, that the first two terms in the expansion, Eq. (8), to order x^2, are exact. Terms of order x^3 and higher are shape dependent but small. The following exactly solvable potentials are considered.

(a) The separable nonlocal effective-range interaction

$$V(p,p') = -\lambda_1 v(p) v(p') , \quad (11)$$

where

$$v(p) = \gamma \frac{p}{(p^2+\gamma^2)^{1/2}}$$

and λ_1, γ are constants. Our notation here follows that of van Dijk.[3]

(b) The Yamaguchi separable nonlocal potential, of the same form as Eq. (11), but with $v(p)$ given by

$$v(p) = \frac{p\beta^2}{(p^2+\beta^2)} . \quad (12)$$

Note that whereas the expansion for $k \cot\delta(k)$ terminates exactly with the k^2 term in the case of potential (a), potential (b) gives an additional k^4 term.

(c) The local Bargmann potential[5] with Jost function[6]

$$F(k) = \frac{k-i\kappa}{k+ib}, \quad b > 0 .$$

There is actually an infinite family of such Bargmann potentials[6] characterized by a parameter c, $0 < c < \infty$. However, only the case $c=2$ is physically acceptable since all other values of c include a long-range repulsion in the potential. For $c=2$ the potential is

$$V(r) = -8b^2 s \frac{e^{-2br}}{(1+se^{-2br})^2} , \quad (13)$$

where

$$s = \frac{b+\kappa}{b-\kappa} .$$

This potential, with the appropriate choice of parameters, is phase equivalent to the separable form (a). In particular, phase equivalence is established by setting $b=\gamma$ and adjusting λ_1 so that potential (a) gives the binding energy $\hbar^2\kappa^2/2\mu$.

(d) The square well without a hard core;

$$V(r) = -V_0, \quad 0 \le r \le b ;$$

$$V(r) = 0, \quad r > b .$$

(e) The square well with a hard core of radius r_c;

$$V(r) = \infty, \quad r \le r_c ;$$

$$V(r) = -V_0, \quad r_c < r \le (b+r_c) ;$$

$$V(r) = 0, \quad r > (b+r_c) .$$

(f) The delta-shell potential

$$V(r) = -\lambda\delta(r-a) .$$

For all these potentials, the series expansion

$$\langle r^2 \rangle / a_t^2 = \sum_{n=0}^{\infty} \alpha_n x^n , \quad (14)$$

with x as defined by Eq. (9) and r_t as defined by Eq. (10), can be obtained exactly. We find in all cases that $\alpha_0 = \frac{1}{8}$, $\alpha_1 = 0$, and $\alpha_2 = \frac{1}{32}$ as in Eq. (8). Table I displays the coefficients α_3 and α_4. We note that for the square well $\alpha_3 = 0.021\,66$ which is slightly less than the δ-shell value of 0.025. The addition of a repulsive core to the square well increases the value of α_3; i.e., for g ranging from 0 to 1, α_3 varies from 0.021 66 to 0.033 85. Perhaps more interesting is the fact that α_3 and α_4 are both positive for the local potential models (c)–(f) while they are negative for the two nonlocal potentials (a) and (b) considered here. Recall again that potentials (a) and (c) are exactly phase equivalent. The nontrivial algebraic manipulations required to obtain these expansions were carried out using MAPLE. The convergence of the series expansions will be discussed shortly.

TABLE I. The expansion coefficients α_3 and α_4 of the series in Eq. (14) for the potentials (a)–(f), where $N_1=(30-2\pi^2)+g(48+3\pi^2)+18g^2\pi^2+12g^3\pi^2$ and $N_2=\pi^2(13\pi^2-48)+g(-4224+2016\pi^2-70\pi^4)+g^2(-6144+5280\pi^2-48\pi^4)+g^3(3072+656\pi^2)\pi^2+1040g^4\pi^4+416g^5\pi^4$.

Potential	α_3	α_4
(a) Separable	$-5/128$	$-3/64$
(b) Separable	$-5/432$	$-233/10\,368$
(c) Bargmann	$\frac{1297}{110\,592}$	$\frac{27\,061}{884\,736}$
(d) Square well	$\frac{5}{8\pi^2}-\frac{1}{24}$	$\frac{13}{384}-\frac{1}{8\pi^2}$
(e) Square well with core (Here $g=r_c/b$)	$\frac{N_1}{48(1+2g)^3\pi^2}$	$\frac{N_2}{384(1+2g)^5\pi^4}$
(f) Delta shell	$\frac{1}{40}$	$\frac{33}{1280}$

III. APPROXIMATE INCLUSION OF THE TENSOR POTENTIAL

Our next task is to incorporate the effect of the tensor force into the expansion, Eq. (14). In this case, the S- and D-wave radial wave functions of the deuteron are denoted by $u(r)$ and $w(r)$, respectively, with the normalization

$$\int_0^\infty (u^2+w^2)dr=1 \ .$$

For large r these wave functions have the asymptotic form[1]

$$u(r)\sim Ae^{-\kappa r} \ ,$$
$$w(r)\sim A\eta e^{-\kappa r}\left[1+\frac{3}{\kappa r}+\frac{3}{\kappa^2 r^2}\right] \ . \tag{15}$$

The most recent experimental value of η, the asymptotic D/S ratio, gives $\eta=0.0256(4)$ (Ref. 7). Now Eq. (2) becomes modified to read

$$A^2(1+\eta^2)=\frac{2\kappa}{(1-\kappa r_t)} \ , \tag{16}$$

and the mean-square radius is computed from

$$\langle r^2\rangle=\tfrac{1}{4}\int_0^\infty (u^2+w^2)r^2dr \ . \tag{17}$$

The asymptotic form for $w(r)$ cannot be used directly in the above integral because of the divergence at $r=0$. How accurate is it to simply drop the last term proportional to $(\kappa r)^{-2}$ in Eq. (15) for $w(r)$? Consider for example, the quadrupole moment

$$Q=\frac{\sqrt{2}}{10}\int_0^\infty w(r)\left[u(r)-\frac{1}{\sqrt{8}}w(r)\right]r^2dr \ . \tag{18}$$

Neglecting the part proportional to w^2, and using the truncated asymptotic form for $w(r)$, one obtains the result

$$Q\approx\frac{1}{\sqrt{50}}\frac{A^2\eta}{\kappa^3}\simeq\frac{2\eta}{\sqrt{50}(1-\kappa r_t)\kappa^2} \ . \tag{19}$$

The approximation, Eq. (19), underestimates the quadrupole moment by less than 20%. Since the effect of introducing the D state in the deuteron is to alter the rms radius typically by only about 2%, it is sufficient to use the truncated form

$$w(r)\sim A\eta e^{-\kappa r}\left[1+\frac{3}{\kappa r}\right]$$

to evaluate $\langle r^2\rangle$. We then obtain, instead of Eq. (5), the modified form

$$\langle r^2\rangle=\frac{1}{8\kappa^2}(1-\kappa r_t)^{-1}(1+24\eta^2) \ . \tag{20}$$

This simple form is expected[8] to underestimate $\langle r^2\rangle$. Following the earlier procedure, we obtain (see notes added in proof)

$$\langle r^2\rangle/a_t^2\simeq(1+24\eta^2)(\tfrac{1}{8}+\tfrac{1}{32}x^2+\cdots) \ , \tag{21}$$

where $x=r_t/a_t$ as before and the dots denote the shape-dependent terms of order x^3 and higher.

In Table II we list some commonly used realistic potentials along with their corresponding x, η, and numerically calculated $\langle r^2\rangle$ according to Eq. (17). Also listed are the potential models (a)–(f) described earlier. One notes from this table that the ratio $\langle r^2\rangle^{1/2}/a_t$ is almost constant for realistic potentials, although individual variations in $\langle r^2\rangle^{1/2}$ and a_t are much greater.[2] Further, the approximate universal form given by Eq. (21), ignoring the shape-dependent terms of order x^3 and higher, reproduces the exact numerical values to better than 99%. Finally, the convergence of the series, Eq. (14), can be examined for the various solvable models (a)–(f). Numerically the convergence appears to be excellent, although we do not know if the series, Eq. (14), is truly convergent for $x\simeq0.3$ or only an asymptotic expansion. For example, the square well with $x=0.32$ gives a series which goes as

$$\langle r^2\rangle/a_t^2=0.125+0.0032+0.000\,71+0.000\,29+\cdots \ .$$

Finally, we note that only for the nonlocal separable potential models (a) and (b) are the higher-order coefficients

TABLE II. Comparison of $\langle r^2\rangle^{1/2}/a_t$ for various potential models, computed exactly and approximately using Eq. (21). (a) Some entries use $r_t = r_t(-B,-B)$. This should not affect the seventh column to the accuracy given. (b) Only terms up to and including x^2 are retained. The model calculations (a)–(f) use $\kappa=0.2316$, $M=940$ MeV, and $\hbar c=197.3$ MeV fm.

Potential	a_t (fm)	$\langle r^2\rangle^{1/2}$ (fm) Eq. (17)	η	$x=r_t/a_t$ (a)	$\langle r^2\rangle^{1/2}/a_t$ Exact	$\langle r^2\rangle^{1/2}/a_t$ Eq. (21) (b)	Ref.
RSC	5.390	1.9569	0.0262	0.3195	0.3631	0.3610	12
TRS	5.435	1.9754	0.0262	0.3286	0.3635	0.3612	12
PARIS	5.427	1.9716	0.0261	0.3249	0.3633	0.3611	12
V_{14}	5.450	1.9814	0.0266	0.3302	0.3636	0.3614	9
R-Bonn	5.423	1.9691	0.0260	0.3244	0.3631	0.3611	10
Q-Bonn	5.424	1.9684	0.0262	0.3245	0.3629	0.3611	10
Separable (a)	5.426	1.9257	0	0.3250	0.3549	0.3582	
Separable (b)	5.426	1.9370	0	0.3278	0.3570	0.3583	
Bargmann (c)	5.426	1.9529	0	0.3250	0.3599	0.3582	
Square well (e) ($g=0.4210$)	5.426	1.9562	0	0.3305	0.3605	0.3583	

α_n, $n\geq 3$, negative. Negative shape-dependent terms would push the $\langle r^2\rangle^{1/2}$ vs a_t slope towards the "experimental" value.

IV. EFFECT OF ISOBAR CONFIGURATIONS

In the presence of isobar configurations[11] one has additional parts to the deuteron wave function. Considering only the lowest isobar resonance, the $\Delta(1232)$, one obtains the following $\Delta\Delta$ components in the deuteron wave function: $^3S_1$, $^3D_1$, $^7D_1$, and $^7G_1$. The normalization condition is now given by

$$\int_0^\infty \left[u^2+w^2+\sum_{i=1}^{4}(u_i^{\Delta\Delta})^2\right]dr=1\ ,$$

where the $u_i^{\Delta\Delta}$ denote the various radial parts of the $\Delta\Delta$ components in the wave function. In order to study the effect of these isobar channels on $\langle r^2\rangle^{1/2}$ and a_t, a coupled-channel calculation is necessary. An impulse approximation calculation is not sufficient since this would neglect the dynamical influence of the $\Delta\Delta$ on the *NN* part of the wave function, which is important for a precise determination of $\langle r^2\rangle^{1/2}$ and a_t. Unfortunately there are only a few potential models constructed for coupled-channel calculations with N and Δ degrees of freedom. We consider here the Argonne potential V_{28} (Ref. 9) which gives a $\Delta\Delta$ percentage in the deuteron of 0.52%. The calculation of the low-energy parameters leads to the following results:

$$\langle r^2\rangle^{1/2}=1.9863\ \text{fm}\ ,$$

$$a_t=5.460\ \text{fm}\ ,$$

$$\langle r^2\rangle^{1/2}/a_t=0.3638\ .$$

Thus the value of $\langle r^2\rangle^{1/2}/a_t$ is very similar to that given by the other realistic potentials listed in Table II. It appears unlikely, therefore, that the inclusion of isobars can lead to a better agreement with experiment.

V. CONCLUSIONS

We have shown that $\langle r^2\rangle/a_t^2$ is given to a high degree of accuracy by

$$\langle r^2\rangle/a_t^2=(1+24\eta^2)(\tfrac{1}{8}+\tfrac{1}{32}x^2)+O(x^3)\ ,$$

where $x=r_t/a_t\sim 0.3$. Terms of order x^3 and higher depend on the details of the potential model. In view of this result the task of obtaining agreement with experiment appears challenging,[13] especially since our results in Sec. IV indicate that the inclusion of isobar components appears not to help. Since only the two separable nonlocal potentials yielded negative shape-dependent terms one could speculate, as does van Dijk,[3] that some such explicit form of nonlocality is required.

Note added in proof. The authors have recently received a preprint by D.W.L. Sprung, Hua Wu, and J. Martorell which gives a more detailed derivation of our result. In their work the *D*-wave correction is only about one-half of ours, i.e., $1+11\eta^2$ instead of our $1+24\eta^2$. Thus there is a larger degree of uncertainty about this particular correction. We observe, however, that if in Table II one writes $(\text{Exact})^2=(1+\lambda\cdot\eta^2)(\frac{1}{8}+\frac{1}{32}x^2)$, then λ turns out to be about 40. Using the value of Sprung *et al.* of $11\eta^2$ would change our values in column (b) of Table II from 0.361 to 0.359–0.360.

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