Analytical model for the triton asymptotic D-state parameters

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We suggest an analytic model for the triton asymptotic D to S normalization ratio η^t and the closely related D-state parameter D_2^t. This model explains correlations among D_2^t/η^d and E_t and among η^t/η^d and E_t in three-nucleon calculations, where η^d is the deuteron asymptotic D to S normalization ratio and E_t is the triton binding energy. The model yields numerical values of η^t/η^d and D_2^t/η^d in close agreement with results of dynamical three-nucleon calculations. We conclude that η^d, E_t, and other low energy on-shell nucleon-nucleon observables determine the triton asymptotic D-state parameters within an estimated error of 15%.

I. INTRODUCTION

Recently, we have emphasized[1] the role of the deuteron asymptotic D to S normalization ratio η^d in making a theoretical estimate for the triton asymptotic D to S normalization ratio η^t. There has been considerable[2] interest for theoretical and experimental determination of the asymptotic D to S ratio of small nuclei ever since Amado and co-workers[3] have suggested that η should be given the "experimental" status of a single quantity to measure the D state of small nuclei. In our recent study[1] we suggested a strong correlation between η^t/η^d and the triton binding energy E_t in theoretical trinucleon calculations and emphasized its importance in making a theoretical estimate of η^t/η^d. Lacking[2] a precise experimental value of η^d, the above correlation only yields the ratio η^t/η^d ($=1.68\pm0.04$). Londergan *et al.*[2] estimated the error in η^d to be of the order of 10% which is much larger than the estimate of error in Ref. 4: $\eta^d=0.0271\pm0.004$. In view of the above uncertainty in the experimental value of η^d, any attempt to estimate η^t from dynamical three-nucleon calculations as in Ref. 5 will involve an error at least equal to that of η^d (10%). In Ref. 1 we also justified the strong correlation between η^t/η^d and E_t in a simple model which, in addition, explained the trend of the dependence of η^t/η^d on E_t.

Motivated by the success of the model of Ref. 1 in predicting the qualitative properties of η^t/η^d in three-nucleon calculations, we present in this paper an improved analytic model for triton asymptotic D-state properties. Under several simplifying assumptions the present model reduces to and has all the qualitative features of the model of Ref. 1. The present model uses Yamaguchi form factors for the deuteron wave function and for the triton S state to be compared to zero range (minimal) wave functions used in Ref. 1. Inclusion of these form factors in the present model makes it theoretically more acceptable than that of Ref. 1 and numerical calculation based on the present model yields triton asymptotic D-state observables in agreement within 10–20 % of those obtained in realistic three-nucleon calculations.

Apart from the asymptotic D to S ratio η there is another closely related model independent parameter for measuring the D state of light nuclei. This is the distorted wave D-state parameter or, simply, the D-state parameter D_2 which, like the asymptotic D to S ratio η, can be extracted from experiments and can also be estimated theoretically in a model-independent way. This is why there has been a great deal of recent activity in estimating the triton D-state parameter D_2^t both theoretically[7–9] and experimentally.[10–12] In addition to calculate η^t in our model we also undertake the task of the theoretical evaluation of D_2^t.

The main objective of our work is, however, not to present precise results for triton D-state asymptotic parameters, rigorous dynamical calculations for which are now available, but to identify the minimum ingredients needed for obtaining precise results for these asymptotic parameters. This will allow us to extend our analytical model to the study of the D-state asymptotic parameters of other nuclei for which exact dynamical calculation is not an easy task. From the work of Ref. 1 and this present investigation we verify that in order to reproduce the correct triton asymptotic normalization parameters η^t and D_2^t, the minimum ingredients required of a model are the correct low energy deuteron properties including η^d, and the triton binding energy E_t. The study of our analytic model for the triton and a comparison of our results with that obtained in exact dynamical calculations will give us an idea about the reliability of our approach, which will prove to be very useful in problems where exact results are not easily obtainable.

The plan of the paper is as follows. In Sec. II we present our definitions and notations. In Sec. III we present our analytic model. Section IV contains a numerical investigation using our model and comparison with exact calculation. Finally, in Sec. V we present a discussion of our results.

II. DEFINITIONS AND NOTATIONS

At this stage it is convenient to present our notations and definitions[13] which we shall use in the following. Let us consider a two-body bound state Ψ_t (without the isospin formalism) of a neutron 1 and a deuteron (23) forming a triton; $V \equiv V_2 + V_3$ is the neutron-deuteron interaction with V_i the interaction between nucleon j and k, $i \neq j \neq k \neq i$. The vertex function $f_l(q_1)$ of the $t \leftrightarrow dn$ vertex is defined (neglecting a three-nucleon potential) as[7]

$$f_l(q_1) = \langle \phi_d^{(23)}, q_1 l | \Psi_t \rangle = -2m_R \frac{1}{q_1^2+\mu^2} \langle \phi_d^{(23)}, q_1 l | V_2 + V_3 | \Psi_t \rangle \ , \quad (2.1)$$

where m_R is the reduced mass of the system, $q_1(l)$ is the relative momentum (orbital angular momentum) between the neutron and the deuteron, $\phi_d^{(23)}$ is the deuteron wave function of the nucleons 2 and 3, and $\mu = \sqrt{4(E_t - E_d)/3}$, E_d being deuteron binding energy. We are using $\hbar = m = 1$, where m is the nucleon mass. For simplicity we omit the other relevant quantum numbers.

The asymptotic form of the wave function in configuration space is given by

$$\mathcal{U}_l(r) \equiv \langle \phi_d ; rl | \Psi_t \rangle \underset{r\to\infty}{\to} -m_R \sqrt{2\pi} \frac{e^{-\mu r}}{r} \lim_{q\to i\mu} \langle \phi_d ; ql | V_2 + V_3 | \Psi_t \rangle \mathcal{U}_l^A(r) \quad (2.2)$$

with

$$\mathcal{U}_l^A(r) = (-i)^l \sqrt{3\mu/2}\, C_l \frac{e^{-\mu r}}{r} \ , \quad (2.3)$$

where C_l is the asymptotic normalization parameter for the bound state. The coordinate and momentum space wave functions are related by

$$\mathcal{U}_l(r) = i^l \sqrt{2/\pi} \int_0^\infty dq\, q^2 j_l(qr) f_l(q) \quad (2.4)$$

with j_l as the usual bessel function. The asymptotic normalization parameter C_l of Eq. (2.3) is explicitly defined by

$$C_l = i^{l+1} \sqrt{4\mu\pi/3} \lim_{q\to i\mu} (q - i\mu) f_l(q) \quad (2.5)$$

$$= -m_R i^l \sqrt{4\pi/3\mu} \lim_{q\to i\mu} \langle \phi_d ; ql | V_2 + V_3 | \Psi_t \rangle \ . \quad (2.6)$$

Finally, the on-shell t matrix $t_0(k^2)$ has the following behavior at the bound state pole:

$$t_0(k^2) \equiv \frac{e^{i\delta}\sin\delta}{k} \underset{k\to i\mu}{\to} -\frac{3\mu C_0^2}{k^2+\mu^2} \ . \quad (2.7)$$

In Appendix A we show the connection between the present definitions and those used in Refs. 1 and 7. With this brief summary of our notations and definitions we present our analytic model for the triton asymptotic D-state parameters in Sec. III.

III. ANALYTIC MODEL FOR η^t/η^d AND D_2^t/η^d

For the trinucleon system the reduced mass $m_R = \frac{2}{3}$. The parameter D_2^t is defined by[7,13]

$$D_2^t = -\lim_{k\to 0} \left[\frac{f_2(k)}{k^2 f_0(k)} \right] \ , \quad (3.1)$$

where

$$f_l(q) = -\frac{4}{3} \frac{1}{q^2+\mu^2} \langle q, \phi_d^{(23)}, J_t = \tfrac{1}{2}, l, \hat{S} | V_2 + V_3 | \Psi_t \rangle \ , \quad (3.2)$$

in terms of which the parameter C is defined by[7,13]

$$C_l = -\tfrac{2}{3} \sqrt{4\pi/3\mu}\, i^l \lim_{q\to i\mu} \langle q, \phi_d^{(23)}, J_t = \tfrac{1}{2}, l, \hat{S} | V_2 + V_3 | \Psi_t \rangle \ . \quad (3.3)$$

The triton asymptotic D to S ratio is defined by

$$\eta^t \equiv C_D^t / C_S^t \ ,$$

where $C_D^t \equiv C_2$ and $C_S^t \equiv C_0$ refer to triton asymptotic normalization for the D and S states, respectively. In Eqs. (3.2) and (3.3), $\hat{S}$ is obtained by coupling the total spin of deuteron (23) and the spin of the neutron 1. We have two possibilities: $l=0, \hat{S}=\frac{1}{2}$ and $l=2, \hat{S}=\frac{3}{2}$.

The matrix elements appearing on the right-hand side of Eqs. (3.2) and (3.3) can be written in terms of the Faddeev components $\Psi_i (\Psi_t = \sum_{i=1}^3 \Psi_i)$ as

$$m_l(q_1) \equiv \langle q_1, \phi_d^{(23)}, J_t = \tfrac{1}{2}, l, \hat{S} | V_2 + V_3 | \Psi_t \rangle = -\langle q_1, \phi_d^{(23)}, J_t = \tfrac{1}{2}, l, \hat{S} | E_t + H_0 | \Psi_2 + \Psi_3 \rangle \ , \quad (3.4)$$

where H_0 is the kinetic energy operator of the three-nucleon system.

The matrix element (3.4) is essential for the evaluation of D_2^t and η^t. We shall present an approximate analytical model for calculating this matrix element. As in Ref. 1 we neglect the spin singlet nucleon clusters (13) and (12) in the Faddeev component Ψ_2 and Ψ_3, respectively, and be content with the spin triplet nucleon-nucleon pair states (13) and (12). Assuming that nucleon 1 and 3 are neutrons and that only the S-wave nucleon-nucleon interaction is important, only Ψ_3 can contain two nucleons (12) in the spin triplet state. Thus, in our analytic model we shall take only the contribution of Ψ_3 in Eq. (3.4), which is rewritten as

$$m_l(q_1) = -\langle q_1, \phi_d^{(23)}, J_t = \tfrac{1}{2}, l, \hat{S} | E_t + H_0 | \Psi_3 \rangle \ . \quad (3.5)$$

Because of the approximations Eq. (3.5) should be taken as a model rather than a result of some approximations to a complete dynamical theory.

For the evaluation of (3.5) we shall employ two types of analytic functions for ϕ_d: the deuteron wave function for the tensor Yamaguchi separable interaction and the minimal (zero range) deuteron wave function used in Ref. 1. These wave functions can be written in form[14]

$$\phi_d(\mathbf{p}) = \frac{1}{E_d + p^2} \left[\frac{4E_d^{1/2}}{\pi} \right]^{1/2} N_d \sum_{L=0,2} g_L(p) \mathcal{Y}_{L1}^{1M}(\hat{p}) \ , \quad (3.6)$$

where $\mathcal{Y}_{L1}^{1M}$ is the spin angular momentum function defined by[15]

$$\mathcal{Y}_{LS}^{1M}(\hat{p}) = \sum_{M_S, M_L} C_{MM_L M_S}^{JLS} Y_{LM_L}(\hat{p}) |SM_S\rangle , \tag{3.7}$$

where the Clebsh-Gordan coefficient C is as defined in Ref. 15.

For the Yamaguchi interaction in Eq. (3.6)

$$g_0(p) = (\alpha_0^2 + p^2)^{-1}, \quad g_2(p) = tp^2(\alpha_2^2 + p^2)^{-2} ,$$

where the α's are range parameters, t is a constant, and the normalization N_d is given by

$$N_d^2 = \frac{\pi}{4E_d^{1/2}} \int_0^\infty \frac{dp\, p^2}{(E_d + p^2)^2} [g_0^2(p) + g_2^2(p)] , \tag{3.8}$$

which yields

$$N_d^2 = \left[\frac{1}{\alpha_0(\alpha_0 + \sqrt{E_d})^3} + \frac{t^2}{8} \frac{\sqrt{E_d}}{\alpha_2} \frac{5\sqrt{E_d} + \alpha_2}{(\sqrt{E_d} + \alpha_2)^5} \right]^{-1} . \tag{3.9}$$

The S-state asymptotic normalization parameter C_S^d and the asymptotic D to S ratio η^d in this case are given, respectively, by

$$\begin{aligned} C_S^d &= N_d(\alpha_0^2 - E_d)^{-1} , \\ \eta^d &= -tE_d(\alpha_0^2 - E_d)(\alpha_2^2 - E_d)^{-1} , \end{aligned} \tag{3.10}$$

in terms of which the wave function (3.6) can be rewritten as

$$\phi_d(\hat{p}) = \frac{1}{E_d + p^2} \left[\frac{4E_d^{1/2}}{\pi} \right]^{1/2} \times C_S^d [\bar{g}_0(p) \mathcal{Y}_{01}^{1M}(\hat{p}) - \eta^d \bar{g}_2(p) \mathcal{Y}_{21}^{1M}(\hat{p})] , \tag{3.11}$$

where for the Yamaguchi tensor interaction

$$\begin{aligned} \bar{g}_0(p) &= (\alpha_0^2 - E_d)(\alpha_0^2 + p^2)^{-1} , \\ \bar{g}_2(p) &= \frac{(\alpha_2^2 - E_d)^2}{(\alpha_2^2 + p^2)^2} \frac{p^2}{E_d} . \end{aligned} \tag{3.12}$$

For the (minimal) zero range deuteron wave function Eq. (3.11) is valid but now with $(\alpha_0, \alpha_2 \to \infty)$

$$\bar{g}_0(p) = 1, \quad \bar{g}_2(p) = p^2/E_d . \tag{3.13}$$

It is worthwhile to note that the zero range wave function given by Eqs. (3.11) and (3.13) is not normalizable as in Eq. (3.8). We shall use the deuteron wave functions given by Eqs. (3.11)–(3.13) for our approximate evaluation of Eq. (3.5).

Next we shall use an approximate analytic form for the wave function Ψ_3 needed for the evaluation of Eq. (3.5). In the momentum space as in Ref. 1, the wave function is written in a form reminiscent of the form which naturally appears when one employs a separable interaction between the nucleons. Assuming the presence of only S-wave nucleon-nucleon interactions one has

$$\langle \mathbf{p}_3, \mathbf{q}_3 | \Psi_3 \rangle = \left[\frac{2}{\pi} E_d \right]^{1/2} \frac{C_S^d}{E_t + (3/4)q_3^2 + p_3^2} \chi(q_3) Y_{00}(\hat{q}_3) Y_{00}(\hat{p}_3) , \tag{3.14}$$

where $\mathbf{p}_3$ is the relative momentum between nucleon 1 and 2, and $\mathbf{q}_3$ is the relative momentum between neutron 3 and deuteron (12). The function $\chi(q_3)$ is called the spectator function and is essentially the bound state wave function of a deuteron and a neutron forming the triton. In the zero range model it is taken as

$$\chi(q_3) = \sqrt{3/2} \frac{C_0}{\mu^2 + q_3^2} \sqrt{4\mu/\pi} . \tag{3.15}$$

The factor of $\sqrt{3/2}$ in Eq. (3.15) comes from the difference between the present definition of C_0 and that used in Ref. 1 (see Appendix A).

Of course, one can now easily construct a more realistic wave function Ψ_3. First, we allow the deuteron to exist both in S and D states and employ the more realistic deuteron wave function (3.11) in place of the S-wave zero range deuteron used in the construction of Eq. (3.14). Next, we employ a realistic form for the spectator function χ. In particular, we employ the following S-wave bound state with the Yamaguchi form factor

$$\chi(q_3) = \sqrt{3/2} \frac{C_0}{\mu^2 + q_3^2} \sqrt{4\mu/\pi} \bar{g}_t(q_3) \tag{3.16}$$

with

$$\int_0^\infty dq\, q^2 |\chi(q)|^2 = 1 .$$

In Eq. (3.16) we use an S-wave Yamaguchi interaction between the neutron and the deuteron with the form factor $g_t(q) = (\alpha_t^2 + q^2)^{-1}$, where α_t is a range parameter. Now using Eqs. (3.9)–(3.12) it is easy to realize that in Eq. (3.16)

$$\begin{aligned} C_0 &= \sqrt{2/3} [\alpha_t(\alpha_t + \mu)^3]^{1/2} (\alpha_t^2 - \mu^2)^{-1} , \\ \bar{g}_t(q) &= (\alpha_t^2 - \mu^2)(\alpha_t^2 + q^2)^{-1} . \end{aligned} \tag{3.17}$$

The zero range wave function is recovered, in the limit $\bar{g}_t(q) = 1$, in Eq. (3.16). If we now allow the Faddeev component Ψ_3 to include the wave functions (3.11) and (3.16), then a more realistic form of Ψ_3 of (3.14) becomes

$$\langle \mathbf{p}_3, \mathbf{q}_3 | \Psi_3 \rangle = \frac{2}{\pi} (3\mu E_d^{1/2})^{1/2} \frac{C_0}{\mu^2 + q_3^2} \bar{g}_t(q_3) Y_{00}(\hat{\mathbf{q}}_3) \frac{C_S^d}{E_t + p_3^2 + (3/4)q_3^2} \times \sum_{m, m_s} C_{M_t m m_s}^{1/2 1 1/2} [\bar{g}_0(p_3) \mathcal{Y}_{01}^{1m}(\hat{\mathbf{p}}_3) | \tfrac{1}{2} m_s \rangle \tag{3.18}$$

where $\bar{g}_0$, $\bar{g}_2$, and $\bar{g}_t$ are defined in Eqs. (3.12) and (3.17). In Eq. (3.18) the spin angular momentum functions are explicitly shown.

We shall use the deuteron wave function ϕ_d of Eq. (3.11) and the Faddeev component of the triton wave function Ψ_3 of Eq. (3.18) in order to evaluate the matrix element of Eq. (3.5). This matrix element will be utilized via Eqs. (3.1)–(3.4) for our evaluation of D_2^t and η^t. The Faddeev component of the triton wave function given by Eq. (3.18) is very flexible and includes several possibilities. For example, $\bar{g}_0(p_3)=1$ and $\bar{g}_2(p_3)=p_3^2/E_d$ together with $\bar{g}_t(p_3)=1$ yield a zero range deuteron bound to a nucleon via a zero range wave function, $\bar{g}_0(p_3)$ and $\bar{g}_2(p_3)$ of the Yamaguchi tensor interaction given by Eq. (3.11) together with $\bar{g}_t(p_3)=1$ yield a tensor Yamaguchi deuteron bound to a nucleon via a zero range wave function, and finally, $\bar{g}_0(p_3)$ and $\bar{g}_2(p_3)$ of Eq. (3.11) together with $\bar{g}_t(p_3)$ of Eq. (3.17) yield a tensor Yamaguchi deuteron bound to a nucleon via a S-wave Yamaguchi interaction. Taking appropriate choices of the form factors one can have all these possibilities. Of course, when we evaluate the matrix element (3.5) it is understood that the same type of deuteron wave function is to be employed both in $\phi_d^{(23)}$ and Ψ_3.

Using ϕ_d of Eq. (3.11) and Ψ_3 of Eq. (3.18), the matrix element of Eq. (3.5) denoted as $m_l(q_1)$ can be written for $l=2$ as

$$m_2(q_1)=-\frac{4}{\pi}\sqrt{3\mu E_d/\pi}(C_S^d)^2 C_0\int_0^\infty dq_3 q_3^2\frac{\mathcal{T}(q_1,q_3)}{q_3^2+\mu^2}\ , \tag{3.19}$$

where

$$\mathcal{T}(q_1,q_3)=\tfrac{1}{2}\sum \mathcal{C}^{1/2\,2\,3/2}_{m m_{l_1} m_{S_1}}\mathcal{C}^{3/2\,1/2\,1}_{m_{S_1} m_1 m_{23}}\mathcal{C}^{1/2\,0\,1/2}_{m 0 m}\mathcal{C}^{1/2\,1/2\,1}_{m m_3 m_{12}}$$
$$\times\int d\Omega_{q_3} d\Omega_{q_1}\langle \mathcal{S}_1 m_1|[\bar{g}_0(p_1)\mathcal{Y}_{01}^{1m_{23}^*}(\hat{\mathbf{p}}_1)-\eta^d\bar{g}_2(p_1)\mathcal{Y}_{21}^{1m_{23}^*}(\hat{\mathbf{p}}_1)]$$
$$\times[\bar{g}_0(p_3)\mathcal{Y}_{01}^{1m_{12}}(\hat{\mathbf{p}}_3)-\eta^d\bar{g}_2(p_3)\mathcal{Y}_{21}^{1m_{12}}(\hat{\mathbf{p}}_3)]|\mathcal{S}_3 m_3\rangle Y^*_{2m_{l_1}}(\hat{q}_1)Y_{00}(\hat{\mathbf{q}}_3)[E_d+(q_1^2/4)+q_3^2+\mathbf{q}_1\cdot\mathbf{q}_3]^{-1}\ . \tag{3.20}$$

The momenta $\mathbf{q}_i$, spin $\mathcal{S}_i=\frac{1}{2}$, projection m_i, relative orbital angular momentum l_i, and projection m_{li} all refer to the nucleon. The two wave functions in the square brackets of Eq. (3.20) refer to two deuterons formed out of nucleons (23) and (12), respectively. The deuteron (23) is the deuteron which explicitly appears in Eq. (3.5) and the deuteron (12) is contained in the Faddeev component Ψ_3. The first two Clebsh-Gordan coefficients in Eq. (3.20) construct the total spin ($=\frac{1}{2}$) of the triton in the final state with $l_1=2$ and the last two Clebsh-Gordan coefficients construct the total spin ($=\frac{1}{2}$) of the triton in the initial state with $l_3=0$. The sum in Eq. (3.20) is over all permitted spin and angular momentum projections m's. The momentum $\mathbf{p}_1$ ($\mathbf{p}_3$) is the relative momentum of nucleon 23 (12) forming the deuteron in the angular momentum state $L_1 M_{L_1}$ ($L_3 M_{L_3}$) and is given by

$$\mathbf{p}_1=\mathbf{q}_3+\frac{\mathbf{q}_1}{2},\quad \mathbf{p}_3=-\mathbf{q}_1-\frac{\mathbf{q}_3}{2}\ .$$

Using the definition (3.7) for the spin angular function, Eq. (3.20) can be rewritten as

$$\mathcal{T}(q_1,q_3)=\tfrac{1}{2}\sum \mathcal{C}^{1/2\,2\,3/2}_{m m_{l_1} m_{S_1}}\mathcal{C}^{3/2\,1/2\,1}_{m_{S_1} m_1 m_{23}}\mathcal{C}^{1/2\,0\,1/2}_{m 0 m}$$
$$\times\mathcal{C}^{1/2\,1/2\,1}_{m m_3 m_{12}}\mathcal{C}^{1 L_1 1}_{m_{23} m_{L_1} M_{23}}\mathcal{C}^{1\,1/2\,1/2}_{M_{23} m_3 m_2}\mathcal{C}^{1 L_3 1}_{m_{12} m_{L_3} M_{12}}\mathcal{C}^{1\,1/2\,1/2}_{M_{12} m_2 m_1}$$
$$\times(-\eta^d)^{(L_1+L_3)/2}\int (d\Omega q_1 d\Omega q_3/\sqrt{4\pi})\bar{g}_{L_1}(p_1)\bar{g}_{L_3}(p_3)\frac{Y^*_{L_1 m_{L_1}}(\hat{\mathbf{p}}_1)Y^*_{2m_{l_1}}(\hat{\mathbf{q}}_1)Y_{L_3 m_{L_3}}(\hat{\mathbf{p}}_3)}{E_d+(q_1^2/4)+q_1^2+\mathbf{q}_1\cdot\mathbf{q}_3}\ . \tag{3.21}$$

The sum in Eq. (3.21) now extends over all permitted spin and angular momentum projections and L_1 and L_3. We calculate the leading term of Eq. (3.21) proportional to η^d, where we take either $L_1=0$, $L_3=2$ or $L_1=2$, $L_3=0$. The detail of this calculation is given in Appendix B and we state the result here. Equation (3.21) when substituted into Eq. (3.19) yields

$$m_2(q_1)=\frac{4}{\pi}\sqrt{3\mu E_d/\pi}(C_S^d)^2 C_0\eta^d q_1^2 h(q_1)\ , \tag{3.22}$$

where

$$h(q_1)=\int_0^\infty \frac{dq_3 q_3^2}{\mu^2+q_3^2}\bar{g}_t(q_3)\left[\tfrac{1}{2}\mathcal{J}_{l=0}^{L_1=0,L_3=2}(q_1,q_3)-\tfrac{1}{16}\mathcal{J}_{l=0}^{L_1=2,L_3=0}(q_1,q_3)+\frac{1}{2}\frac{q_3}{q_1}\mathcal{J}_{l=1}^{L_1=0,L_3=2}(q_1,q_3)\right.$$
$$\left.-\frac{1}{4}\frac{q_3}{q_1}\mathcal{J}_{l=1}^{L_1=2,L_3=0}(q_1,q_3)+\frac{1}{8}\frac{q_3^2}{q_1^2}\mathcal{J}_{l=2}^{L_1=0,L_3=2}(q_1,q_3)-\frac{1}{4}\frac{q_3^2}{q_1^2}\mathcal{J}_{l=2}^{L_1=2,L_3=0}(q_1,q_3)\right], \tag{3.23}$$

and where

$$\mathcal{J}_l^{L_1,L_3}(q_1,q_3)=\int_{-1}^{+1}dx\frac{P_l(x)\bar{g}_{L_1}[|(\mathbf{q}_1/2)+\mathbf{q}_3|]\bar{g}_{L_3}[|\mathbf{q}_1+(\mathbf{q}_3/2)|]}{[E_d+(q_1^2/4)+q_3^2+\mathbf{q}_1\cdot\mathbf{q}_3]|(\mathbf{q}_1/2)+\mathbf{q}_3|^{L_1}|\mathbf{q}_1+(\mathbf{q}_3/2)|^{L_3}} . \tag{3.24}$$

Equations (3.22)–(3.24) constitute the set of equations we shall use to study η^t and D_2^t. From Eqs. (3.3), (3.5), and (3.22) we have

$$\frac{\eta^t}{\eta^d}=\frac{16}{3\pi}(C_S^d)^2\sqrt{E_d}\mu^2 h(i\mu) , \tag{3.25}$$

which is our analytic expression for η^t/η^d and which we shall use in Sec. IV for numerical investigation. Note that each side of Eq. (3.25) refers to a different part of the triton wave function; we included in Eq. (3.25) an extra minus sign, due to the Pauli principle.

In order to find our analytic expression for D_2^t/η^d it is useful to express $f_0(0)$ of Eqs. (3.1) and (3.2) in terms of C_0. The following approximate relation between $f_0(0)$ and C_0 is easily derived from Eqs. (3.2), (3.3), and (3.5) if we take the matrix element of Eq. (3.5) as a slowly varying function of momentum:

$$C_0=\sqrt{(\pi/3)}\mu^{3/2}f_0(0) . \tag{3.26}$$

This relation has been tested by Gibson and Lehman[7] in a separable potential model and has been found to hold within an estimated error of 10–15 %. [They had the relation $C_0=\pi\mu^{3/2}f_0(0)$, because they used a different normalization for f_0, Eq. (A3).] Next in Eq. (3.26) we introduce the momentum dependence as given by Eq. (3.17):

$$f_0(0)=\sqrt{3/\pi}\frac{C_0}{\mu^{3/2}}\frac{\alpha_t^2-\mu^2}{\alpha_t^2} . \tag{3.27}$$

From Eqs. (3.1), (3.2), (3.5), (3.22), and (3.27) we have

$$\frac{D_2^t}{\eta^d}=\frac{16}{3\pi}(C_S^d)^2\sqrt{E_d}\frac{\alpha_t^2}{\alpha_t^2-\mu^2}\lim_{q_1\to 0}h(q_1) , \tag{3.28}$$

which is our analytic expression for D_2^t/η^d and will be used in Sec. IV for numerical investigation.

Equations (3.25) and (3.28) are the principal results of this paper. They show two important features. If the triton (and also the deuteron) binding energies are maintained constant, D_2^t and η^t are linearly proportional to η^d (assuming that C_S^d is also held constant). Moreover, they exhibit the correct variation of η^t/η^d and D_2^t/η^d with E_t expected of a realistic trinucleon calculation; namely, the ratio η^t/η^d (D_2^t/η^d) increases (decreases) as E_t increases in a dynamical trinucleon calculation. We shall verify this second point in numerical calculations using Eqs. (3.25) and (3.28) in Sec. IV.

IV. NUMERICAL RESULTS

We have studied in great detail the η^t/η^d-E_t correlation in Ref. 1 using results of trinucleon calculation. In particular we demonstrated the following. Firstly, we verified directly that if E_t is held constant and η^d is varied in a dynamical calculation the resulting η^t is linearly proportional to η^d. This yields a $\eta^t-\eta^d$ correlation at a fixed E_t passing through $\eta^t=\eta^d=0$, which is consistent with Eq. (3.25). Secondly, the above correlation was indirectly verified by studying the η^t/η^d versus E_t correlation which has a much lesser dispersion than the η^t versus E_t correlation when one allows a wide range of variation of η^d than that suggested by most of the theoretical potentials. (The wider range of variation of η^d is suggested by experimental studies[2].) In this way we predicted η^t/η^d to be consistent with the experimental value of E_t. The experimental value of η^d has to be provided for one to find η^t from this correlation. Ishikawa and Sasakawa studied the η^t versus E_t correlation in realistic trinucleon calculations and predicted the value of η^t from this correlation. Their plot has a much lower dispersion than our study of η^t-E_t correlation of Ref. 1, essentially because their theoretical models had a very narrow variation of η^d($=0.0263\pm0.0003$), whereas we permitted a much larger variation of η^d as suggested recently.[2] The prediction of η^t as has been done by Ishikawa and Sasakawa is not independent of the value of η^d of their model and will suffer modification if the experimental η^d falls outside the narrow range predicted by their model.

With this summary of the numerical investigations of the η^t/η^d versus E_t correlation of Ref. 1, we turn to the study of correlations involving D_2^t. Extensive dynamical calculations of D_2^t have been performed recently[5,7–9] using both separable nucleon- nucleon potential models[7] and realistic and field-theoretic nucleon-nucleon potential models.[5,8,9] Realistic nucleon-nucleon potential models include Paris, Argonne, Urbana, and also three-nucleon interactions. We shall use the results of these calculations in order to substantiate the conclusions of Sec. III. We have seen in Sec. III the very important role played by η^d in any calculation of D_2^t. Any attempt at a theoretical evaluation of D_2^t must take into consideration the correct value of η^d. The parameter D_2^t, being an asymptotic observable for the D state, is very sensitive to its two-nucleon counterpart η^d (we recall at this stage that in the deuteron, η^d is closely related to its D-state parameter); a specification of E_t alone (at a fixed E_d) is not enough to determine D_2^t in a three-nucleon model calculation.

To see the importance of η^d in a model three-nucleon calculation of D_2^t we plot in Fig. 1 the values of $D_2^t/\eta^d_{\text{expt.}}$ as a function of triton binding energy E_t, where $\eta^d_{\text{expt.}}$ ($=0.0271$) is the experimental η^d of Ericson and Rosa-Clot.[4] This plot is equivalent to that of D_2^t versus E_t as $\eta^d_{\text{expt.}}$ is just a constant. In Fig. 1 we show results of Gibson and Lehman,[7] Ishikawa and Sasakawa,[5] and Schiavilla, Pandharipande, and Wiringa.[8] We also show the experimental results of Refs. 10 and 11. Results of Refs. 5 and 8 involve realistic potential models with and without three-nucleon forces, and those of Ref. 7 involve separable potential models. As the various models have different values of η^d, the width of the band for $D_2^t/\eta^d_{\text{expt.}}$ versus E_t is large, which makes a model independent estimate for D_2^t difficult.

Next we study in Fig. 2 the correlation between D_2^t/η^d and E_t. In Fig. 2 we show results corresponding to all calculations of Fig. 1. We verify that the parameter D_2^t/η^d does not contain much information about the off-shell behavior of nucleon-nucleon interaction except that contained in the value of E_t. All types of potential models yield essentially identical results for D_2^t/η^d once they produce identical E_t. At the triton binding energy $E_t = 8.48$ MeV, our study yields the expected value of D_2^t/η^d ($=7.9\pm0.4$ fm^2).

Next we would like to present numerical calculations

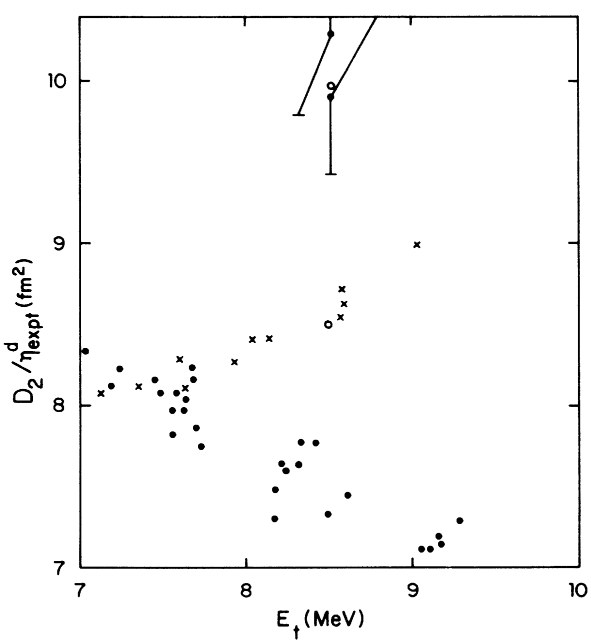

FIG. 1. $D_2^t/\eta^d_{\text{expt}} - E_t$ plot. The crosses are separable potential model calculations of Gibson and Lehman (Ref. 7), solid circles are the realistic calculations of Ishikawa and Sasakawa (Ref. 5), open circles are the realistic calculations of Schiavilla *et al.* (Ref. 8), and the points with error bars are experimental results (Refs. 10 and 11).

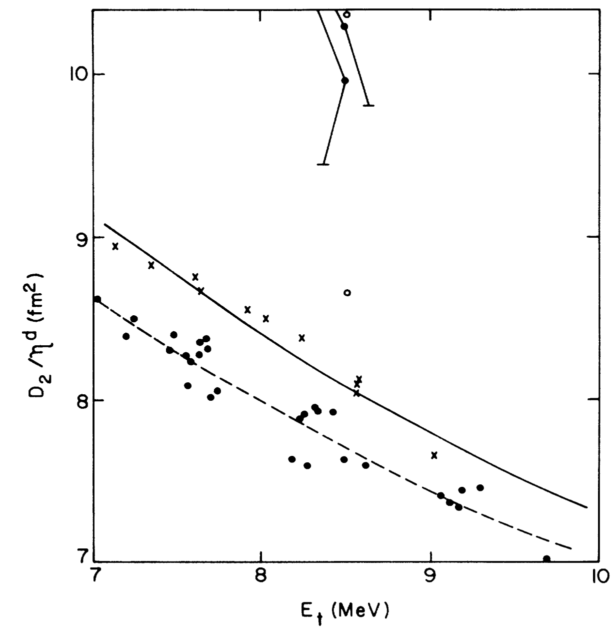

FIG. 2. $D_2^t/\eta^d - E_t$ plot. Same as in Fig. 1. Solid line represents the present calculation with $C_0^2 = 2.2$ and the dashed line with $C_0^2 = 3.3$.

of η^t and D_2^t based on Eqs. (3.25) and (3.28), respectively. We present three types of calculations in Table I marked A, B, and C. The calculation A takes a zero range deuteron bound to a nucleon via a zero range wave function. The calculation B takes a tensor Yamaguchi deuteron bound to a nucleon via a zero range wave function. Finally, calculation C uses a tensor Yamaguchi deuteron bound to a nucleon via an S-wave Yamaguchi interaction.

The tensor Yamaguchi interaction we use is defined by

$$\langle pJLM|V|p'JL'M\rangle = -\lambda_t g_L(p) g_{L'}(p') , \qquad (4.1)$$

with

$$g_0(p) = (\alpha_0^2 + p^2)^{-1}, \quad g_2(p) = tp^2(\alpha_2^2 + p^2)^{-1} ,$$

$$\alpha_0 = 1.2560 \text{ fm}^{-1} , \quad \alpha_2 = 1.7545 \text{ fm}^{-1} ,$$

$$t = -3.0398, \text{ and } \lambda = 2.2228 \text{ fm}^{-3} ,$$

TABLE I. Result of our calculation of η^t/η^d and D_2^t/η^d our analytic model Eq. (3.25) and (3.28). Model A: no form factor at $d \to NN$ and $t \to dN$ vertices. Model B: tensor Yamaguchi form factor at the $d \to NN$ vertex and no form factor at the $t \to dN$ vertex. Model C: tensor Yamaguchi form factor at the $d \to NN$ vertex and S-wave Yamaguchi form factor at the $t \to dN$ vertex.

Model	η^d	E_t	η^t/η^d	D_2/η^d (fm^2)
A	0.027	8.48	4.26	20.4
B	0.027	8.48	2.22	9.48
	0.025	8.48	1.45	7.82
C	0.027	8.48	1.47	7.70
	0.029	8.48	1.46	7.45
Expt.	0.0271	8.48	1.68±0.04	7.9±0.4

which yields a deuteron with $\eta^d=0.027$, $E_d=2.225$ MeV, quadrupole moment $Q^d=0.2859$ fm^2, and triplet scattering length $^3a_1=5.424$ fm. The S-wave Yamaguchi interaction which binds the deuteron to a nucleon is given by

$$\langle \mathbf{p}|V|\mathbf{p}'\rangle = -\lambda_t g_t(p) g_t(p') \ , \tag{4.2}$$

with

$$g_t(p)=(\alpha_t^2+p^2)^{-1}, \quad \alpha_t=0.9806 \text{ fm}^{-1},$$
$$\lambda_t=0.1522 \text{ fm}^{-3} \ .$$

This leads to a triton of correct binding energy $\mu=0.4485$ fm^{-1} and correct S-wave asymptotic normalization $C_0^2=3.3$. With these definitions of $g_0(p)$, $g_2(p)$, and $g_t(p)$ it is straightforward to construct $\bar{g}_0(p)$, $\bar{g}_2(p)$, and $\bar{g}_t(p)$ which enter in Eqs. (3.25) and (3.28) using Eqs. (3.12) and (3.17).

The result of our calculation is shown in Table I. As expected, model A produces the poorest result as all the vertices used there are unrealistic. Both η^t/η^d and D_2^t/η^d are larger than their "exact" values by a factor of ~ 2.5. Model B uses a more realistic deuteron vertex and this improves the results very much. Finally, model C uses Yamaguchi type form factors both at the deuteron vertex and at the triton vertex and yields for $E_t=8.48$ MeV results correct to within about 15% of the exact values of η^t/η^d and D_2^t/η^d. It is remarkable that the present simple analytic model is able to reproduce results of dynamical trinucleon calculations. The difference of our model and the exact calculations can be reduced if we take the normalization of the effective two-body wave function (3.16), lesser than 1, which is reasonable since the probability to find a $n-d$ state inside the triton is less than 1.

For the sake of completeness we have shown in Fig. 2 the result of our calculation for D_2^t/η^d at various E_t for $\eta^d=0.027$. These calculations have been done assuming $C_0^2=2.2$, this modification was necessary in order to bring our model in close agreement with the exact calculations; this procedure has the effect of reducing the normalization of the effective two-body wave function (3.16). Finally, in Fig. 3 we plot η^t/η^d versus E_t where we exhibit results of realistic calculations of various authors and also the results of our analytic model C with $\eta_t=0.027$. Our analytic model again produces results which are in good agreement with the exact calculations.

The results of Table I demonstrate that at a fixed E_t, η^t/η^d is reasonably independent of η^d. This proves that at a fixed E_t, η^t is proportional to η^d. In addition, our simple analytic model is able to reproduce correctly the E_t dependence of η^t/η^d. The same also holds for the observable D_2^t/η^d.

The present study and that of Ref. 1 yield the following estimates: $D_2^t/\eta^d=7.9\pm0.4$ fm^2 and $\eta^t/\eta^d=1.68\pm0.04$. As the precise experimental value of η^d is not known,[2] it will be premature to predict an experimental D_2^t and η^t. Assuming the experimental η^d of Ericson and Rosa-Clot ($=0.0271\pm0.0004$) we have $D_2^t=0.21\pm0.01$ fm^2 and

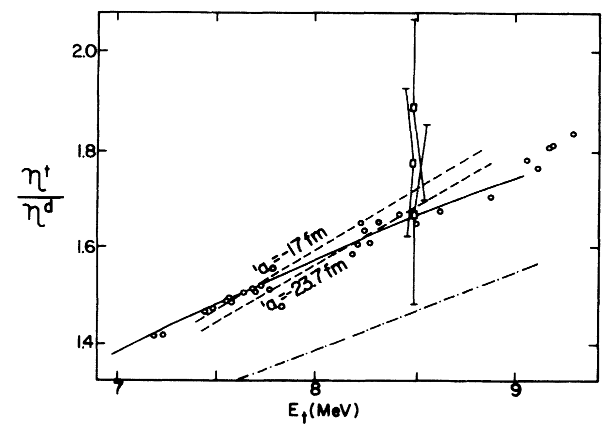

FIG. 3. η^t/η^d-E_t correlation. The dashed lines are generated by varying both $^1r_0$ and η^d in the separable model calculation of Ref. 1 for $^1a_0=-17$ and 23.7 fm. The open circles are from Ref. 5 and the points with error bars are experimental results from Ref. 6. The solid line is our present calculation with $C_0^2=2.2$ and the dashed-cotted line $C_0^2=3.3$.

$\eta^t=0.0455\pm0.0015$.

The study of Ref. 1 yielded $\eta^t/\eta^d=1.68\pm0.04$. Now with our estimate, $D_2/\eta^d=7.9\pm0.4$ fm^2, we are prepared to verify the approximate relation

$$R\equiv D_2/(\eta^t/\mu^2)\cong 1 \ . \tag{4.3}$$

Using the above estimate of D_2^t and η^t we have $R=0.95\pm0.05$ in close agreement with Eq. (4.3).

V. DISCUSSION

In this paper we have suggested and studied a new correlation between D_2^t/η^d and E_t in three-nucleon calculations. This correlation is important in estimating the D-state parameter of the triton. In view of our discussion in this paper and in Ref. 1, one must bear in mind that a good theoretical estimate of D_2^t and η^t cannot be separated from a precise knowledge of η^d. Based on an analaytic model for asymptotic D-state parameters of the trinucleon, we predicted the essential energy dependence of D_2^t/η^d and η^t/η^d as triton energy E_t is varied. Using our studies of correlations of D_2^t/η^d (and η^t/η^d) with E_t we made a theoretical estimate of these parameters: $\eta^t/\eta^d=1.68\pm0.04$ (Ref. 1) and $D_2^t/\eta^d=7.9\pm0.4$ fm^2. These values are associated with a wide class of possible off-shell variations of nucleon-nucleon interactions in triton calculations. Our study emphasizes the importance of the asymptotic part of the trinucleon wave function in the calculation of D_2^t, as well as η^t, as we used essentially the correct asymptotic part of this wave function in our consideration.[1] This, in turn, explains the sensitivity of D_2^t and η^t on E_t. The fluctuation observed in the values of D_2^t/η^d and η^t/η^d in dynamical calculations at a fixed E_t reflects the changes in the off- and on-shell behaviors of the nucleon-nucleon interaction not constrained by E_t

and sets a limit on the universality of these correlations. In order to obtain new information about the nucleon-nucleon interaction from a study of D_2^t/η^d (η^t/η^d), beyond those contained in the value of E_t, the experimental error in this observable should be lowered to less than 5% (3%), which is the scale for breakdown of the strong model independence of these correlations. Our estimate for D_2^t/η^d is in contradiction with those of Refs. 8, 10, and 11.

It has long been expected[4] that the study of two- and three-nucleon systems should yield information about the tensor part of the nucleon-nucleon interaction. Apart from the deuteron parameters η^d and C_S^d, the other interesting observables to study are the deuteron quadrupole moment Q^d, and the parameters η^t and D_2^t. It is interesting to note that all these three observables are correlated with the deuteron asymptotic normalizations through the relation[1,4]

$$\frac{\mathcal{O}}{\eta^d} \sim (C_s^d)^2 f \ , \tag{5.1}$$

where $\mathcal{O}$ stands for Q^d, η^t, or D_2^t. The function f depends on E_d and E_t in the cases of η^t and D_2^t and on E_d only in the case of Q^d, while other low-energy on-shell nucleon-nucleon observables are held fixed. Correlation (5.1) has been studied by Ericson and Rosa-Clot in the case of Q^d in Ref. 4 where they emphasized the linear correlation between $Q^d/(C_S^d)^2$ and η^d. Equation (3.28) of this paper and Eq. (10) of Ref. 1 suggest correlation (5.1) in the cases of D_2^t and η^t, respectively. Special cases of correlation (5.1) were studied in the case of η^t in Refs. 4 and 5. In Ref. 4 Ericson and Rosa-Clot studied the η^d dependence of η^t neglecting its dependence on E_t and C_S^d. In Ref. 5 Ishikawa and Sasakawa studied the E_t dependence of η^t neglecting its dependence on η^d and C_S^d. However, in Refs. 4 and 5 no explanation was given for the existence of correlation (5.1) in the three-nucleon system. If correlation (5.1) were exact, no information about the nucleon-nucleon interaction could be obtained from the study of η^t and D_2^t, which is not implicit in the values of E_d, E_t, C_S^d, and η^d. However, correlation (5.1) is approximate and some information about the nucleon-nucleon interaction may be obtained from the study of Q^d, η^d, η^t, and D_2^t from the breakdown of correlation (5.1) as mentioned previously.

The present investigation and that of Ref. 1 generalize a conjecture made in a different context.[16] It was conjectured in Ref. 16 that the low energy trinucleon central observables—those observables which do not require a noncentral nucleon-nucleon interaction for calculation—do not feel the detail of the nucleon-nucleon or three-nucleon potential models used for their study except for the information contained in the value of E_t. In the case of three-nucleon observables sensitive to noncentral nucleon-nucleon interaction, such as η^t or D_2^t, the information about the nucleon-nucleon or the three-nucleon potential models is contained in C_S^d, E_t, and η^d, apart from the other two nucleon observables. The above conjecture of Ref. 16 about the model independence of central trinucleon observables was verified in Ref. 17 in relation to second order polarization-transfer parameters in nucleon-deuteron scattering. In view of our study once E_t, C_S^d, and η^d are specified the study of the parameters n^t or D_2^t will not contain much information about the detail of the tensor nucleon-nucleon interaction.

Because of the drastic approximation, the present study should be considered as a model and not an approximate theory. It works surprisingly well in explaining many qualitative features of the triton D state. Using very simple expressions for the deuteron and the triton wave functions, the model predicts the essential behavior of D_2^t and η^t expected of a dynamical three-nucleon calculation and also the approximate values of D_2^t/η^d and η^t/η^d to within an estimated error of 15% (see Table I). The main ingredients of the model are the correct low-energy deuteron properties (such as binding energy, scattering length, effective range, and the deuteron asymptotic parameters) and the correct triton binding energy. The model predicts a linear correlation between η^t/η^d (D_2^t/η^d) and E_t, which is verified[1] in a separable potential model dynamical three-nucleon calculation. In dynamical three-nucleon calculations involving realistic nucleon-nucleon and three-nucleon potentials a correlation between η^t (D_2^t) and E_t seems to be enough[5] because the η^d of the working nucleon-nucleon potential models do not have the large spread ($\sim 10\%$) predicted by Londergan *et al.*[2] If η^d is allowed to vary significantly in realistic nucleon-nucleon potential models the necessity of η^t/η^d (D_2^t/η^d) versus E_t correlations is expected to become obvious. This can be verified in the future by constructing realistic nucleon-nucleon potentials with one- and two-pion exchange tails, which can accomodate a largely different η^d and by performing dynamical three-nucleon calculations with these nucleon-nucleon potentials with and without a three-nucleon potential. We expect the essential conclusions of this paper to remain unmodified after such realistic calculations.

ACKNOWLEDGMENTS

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APPENDIX A

In this Appendix we show how to relate our present definitions and those used in Refs. 1 and 7. Equations (2) and (17) of Ref. 7 are used to express C_0 in terms of the deuteron-triton overlap and they read as

$$_A\langle \phi_d ; \mathbf{q}\tfrac{1}{2}m_n | \Psi_t \rangle = \tfrac{3}{2} f_0^{\text{G.L.}}(q)\, \mathcal{C}^{1/2\,1/2\,1}_{m_t m_n m_d} \sqrt{4\pi}\, Y_{00}(\hat{\mathbf{q}}) \tag{A1}$$

and

$$C_0 = 2\pi i \sqrt{\mu} \lim_{q \to i\mu} (q - i\mu) f_0^{\text{G.L.}}(q) \ ,$$

where m_n, m_d, and m_t are the spin projections, $Y_{00}(\hat{\mathbf{q}})$ is the spherical harmonic, and $f_0^{\text{G.L.}}$ are the momentum overlap of Ref. 7; here we choose for simplicity to show only the S state. The label A denotes an antisymmetrized state.

The definition of antisymmetrization of Eqs. (B3) and (B4) of Ref. 18 inserted in Eq. (A1) gives

$$\langle \phi_d; q\tfrac{1}{2}m_n | \Psi_t \rangle = \sqrt{3\pi} f_0^{\mathrm{G.L.}}(q) \mathcal{C}^{1/21/21}_{m_t m_n m_d} Y_{00}(\hat{\mathbf{q}}) , \quad \text{(A2)}$$

and comparing it with Eq. (2.1), we find

$$f_0(q) = \sqrt{3\pi} f_0^{\mathrm{G.L.}}(q) . \quad \text{(A3)}$$

The asymptotic normalization parameter C_0 can be expressed by the neutron-deuteron overlap function through Eqs. (A1) and (A2):

$$C_0 = i\sqrt{2/3}\sqrt{2\mu\pi} \lim_{q \to i\mu} (q - i\mu) \langle \phi_d; ql = 0 | \Psi_t \rangle . \quad \text{(A4)}$$

Equation (A4) is the $l=0$ case of Eq. (2.5), and the residue of the elastic $n-d$ amplitude at the triton pole is given by $-3\mu C_0^2$ as in Eq. (2.7). Note that in the zero range limit ($\mu \to 0$) the elastic $n-d$ amplitude must have the form

$$t_0(k^2) = \frac{1}{-\mu - ik} \underset{k \to i\mu}{\longrightarrow} -\frac{2\mu}{k^2+\mu^2} . \quad \text{(A5)}$$

Comparing the residue of Eq. (A5) and $-3\mu C_0^2$ results in $C_0 \underset{\mu \to 0}{\longrightarrow} \sqrt{2/3}$. This fact motivated the definition used in Ref. 1, where the factor $\sqrt{2/3}$ has been dropped from Eqs. (2.5) and (A4). The definition of C_l in the present work must be multiplied by the factor $\sqrt{3/2}$ to agree with that of Ref. 1.

APPENDIX B

Here we detail the angular decomposition of Eq. (3.21). For this purpose the decomposition of $Y_{LM}(\mathbf{a}+\mathbf{b})$ (Ref. 19) is useful:

$$Y_{LM}(\mathbf{a}+\mathbf{b}) = \frac{\sqrt{4\pi}}{|\mathbf{a}+\mathbf{b}|^L} \sum_{l=0}^{L} \left[\frac{(2L)!}{(2l)![2(L-l)]!} \right]^{1/2} a^l b^{L-l} \sum_m \mathcal{C}^{LlL-l}_{MmM-m} Y_{lm}(\hat{\mathbf{a}}) Y_{L-l,M-m}(\hat{\mathbf{b}}) . \quad \text{(B1)}$$

We define the angular projection of the kernel as

$$\mathcal{J}^{L_1,L_3}(\mathbf{q}_1,\mathbf{q}_3) = \sum_{l=0}^{\infty} \frac{(2l+1)}{2} P_l(\cos\theta) \mathcal{J}_l^{L_1,L_3}(q_1,q_3) , \quad \text{(B2)}$$

where

$$\mathcal{J}_l^{L_1,L_3}(q_1,q_3) = \int_{-1}^{+1} dx \frac{P_l(x) g_{L_1}[|(\mathbf{q}_1/2)+\mathbf{q}_3|] g_{L_3}[|\mathbf{q}_1+(\mathbf{q}_3/2)|]}{[E_d + (q_1^2/4) + q_3^2 + \mathbf{q}_1 \cdot \mathbf{q}_3] |(\mathbf{q}_1/2)+\mathbf{q}_3|^{L_1} |\mathbf{q}_1 + (\mathbf{q}_3/2)|^{L_3}} .$$

The angular integration in Eq. (3.21) is performed after the introduction of Eqs. (B1) and (B2):

$$\begin{aligned}
\mathcal{T}(q_1,q_3) = \tfrac{1}{2} \sum_{\substack{0 \le g \le L_1 \\ 0 \le g \le L_3}} & (-1)^{L_1+m_l} \mathcal{C}^{1/223/2}_{mm_{l_1}m_{S_1}} \mathcal{C}^{3/21/21}_{m_{S_1}m_1m_{23}} \mathcal{C}^{1/21/21}_{mm_3m_{12}} \mathcal{C}^{1L_11}_{m_{23}m_{L_1}M_{23}} \mathcal{C}^{11/21/2}_{M_{23}m_3m_2} \mathcal{C}^{1L_31}_{m_{12}m_{L_3}M_{12}} \mathcal{C}^{11/21/2}_{M_{12}m_2m_1} \\
& \times \mathcal{C}^{L_1gL_1-g}_{m_{L_1}m_gm_{L_1}-m_g} \mathcal{C}^{L_3\bar{g}L_3-\bar{g}}_{m_{L_3}m_gm_{L_3}-m_g} \begin{pmatrix} 2 & g & \xi \\ m_{l_1} & m_g & m_\xi \end{pmatrix} \begin{pmatrix} \bar{g} & l & \xi \\ m_{\bar{g}} & -m_l & m_\xi \end{pmatrix} \begin{pmatrix} L_3-\bar{g} & l & \bar{\xi} \\ m_{L_3}-m_{\bar{g}} & m_l & m_{\bar{\xi}} \end{pmatrix} \\
& \times \begin{pmatrix} 0 & L_1-g & \bar{\xi} \\ 0 & m_{L_1-m_g} & m_{\bar{\xi}} \end{pmatrix} \begin{pmatrix} 2 & g & \xi \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{g} & l & \xi \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & L_1-g & \xi \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_3-\bar{g} & l & \xi \\ 0 & 0 & 0 \end{pmatrix} \\
& \times \sqrt{5(2L_1+1)(2L_3+1)}(2\xi+1)(2\bar{\xi}+1)(2l+1) \left[\frac{(2L_1)!(2L_3)!}{(2g)!(2L_1-2g)!(2\bar{g})!(2L_3-2\bar{g})!} \right]^{1/2} \\
& \times \frac{q_1^{g+\bar{g}} q_3^{L_1-g+L_3-\bar{g}}}{2^{g+L_3-\bar{g}+1}} \mathcal{J}_l^{L_1,L_3}(q_1,q_3)(-\eta^d)^{(L_1+L_3/2)} .
\end{aligned} \quad \text{(B3)}$$

The sum in Eq. (B3) is over all repeated indices. We reduce Eq. (B3) to a product of invariants with the help of the graphical summation method:[20]

$$\mathcal{T}(q_1,q_3)=\sum_{\substack{0\le g\le L_1\\0\le\bar{g}\le L_3}}(-1)^{(1/2)+Z+\xi+\bar{g}}\begin{bmatrix}2 & g & \xi\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}\bar{g} & l & \xi\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & L_1-g & \bar{\xi}\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}L_3-\bar{g} & l & \bar{\xi}\\0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\bar{g} & L_3 & L_3-\bar{g}\\\bar{\xi} & l & \xi\end{Bmatrix}$$

$$\times\begin{Bmatrix}\frac{3}{2} & 2 & \frac{1}{2}\\0 & \frac{1}{2} & 2\end{Bmatrix}\begin{Bmatrix}Z & \frac{1}{2} & L_3\\1 & 1 & \frac{1}{2}\end{Bmatrix}\begin{Bmatrix}Z & \frac{1}{2} & 1\\\frac{1}{2} & \frac{1}{2} & 1\end{Bmatrix}\begin{Bmatrix}\frac{3}{2} & 2 & \frac{1}{2}\\L_3 & Z & L_1\end{Bmatrix}\begin{Bmatrix}Z & \frac{3}{2} & L_1\\1 & 1 & \frac{1}{2}\end{Bmatrix}\begin{Bmatrix}0 & L_1-g & \bar{\xi}\\2 & g & \xi\\2 & L_1 & L_3\end{Bmatrix}$$

$$\times 180\sqrt{5}(2Z+1)(2\xi+1)(2l+1)(2\bar{\xi}+1)(2L_1+1)(2L_3+1)\left[\frac{(2L_1)!(2L_3)!}{(2g)!(2L_1-2g)!(2\bar{g})!(2L_3-2\bar{g})!}\right]^{1/2}$$

$$\times\frac{q_1^{g+\bar{g}}q_3^{L_3-g+L_1-\bar{g}}}{2^{g+L_3-\bar{g}+1}}\mathcal{J}_l^{L_1,L_3}(q_1,q_3)(-\eta^d)^{(L_1+L_3/2)}\ . \qquad \text{(B4)}$$

Approximating $\mathcal{T}(q_1,q_3)$ by the terms linear in η^d, which is given by the insertion of the values of (0,2) and (2,0) for (L_1,L_3) and introducing Eq. (B4) in Eq. (3.19), the final results of Eqs. (3.22) and (3.23) are reached.

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