Triton asymptotic normalization constants by the hyperspherical harmonics expansion method

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We have calculated the asymptotic normalization constants (ANC) of the triton by comparing the asymptotic form of the triton wave function (obtained by the hyperspherical harmonics expansion method) with an appropriate tensor product of the deuteron wave function and the asymptotically free-neutron wave function. The convergence behavior of the binding energy (BE) and ANC have been studied for a simple S-projected potential ($S3$) and two exponential potentials. It is seen that convergence of the ANC needs many more partial waves than that of the BE, especially for potentials with a long tail.

Asymptotic normalization constants (ANC) are important properties of the trinucleon system and can be extracted from experimental results.[1-3] As the name implies, these constants depend on the asymptotic nature of the trinucleon wave function. The experimental value of C_0^2 (S-wave ANC) varies widely with the method of measurement and ranges from 2.6 (Ref. 2) to 3.3 (Ref. 3) with an average error of ± 0.3. The measured value of the ratio C_2/C_0 (where C_2 is the D-wave ANC) is 0.048 ± 0.007.[4] Thus, the experimental values of C_0 and C_2 have rather large error bars. Thus, a theoretical calculation of these quantities can shed light on the asymptotic behavior of the wave function. Several calculations have been reported[5-11] based on the Faddeev calculation of the trinucleon bound-state wave function. These results agree reasonably with the experimental numbers, showing that the Faddeev wave functions for the chosen realistic interactions have the correct asymptotic behavior. An alternative and fairly common method for the treatment of the trinucleon bound states is the hyperspherical harmonics expansion (HHE) method,[12] in which the wave function is expanded in a complete basis of the hyperspherical harmonics (HH) functions spanning the hyperangular space. Binding energy (BE), charge radius, charge form factor, etc., calculated by the HHE method[12-18] agree fairly well with those calculated by the Faddeev method. However, no calculation of the ANC by this method has so far been reported. The fact that convergence in the BE is obtained fairly easily by retaining a tractable number of partial waves in the optimal subset approximation,[12] does not guarantee that the appropriate asymptotic behavior is reached with the same number of partial waves. This is because the BE is determined mainly by the minimum of the effective potential well and not by its asymptotic part. Hence, to investigate the nature of the asymptotic behavior of the wave function obtained by the HHE method, it is interesting to calculate the ANC by this method. In this work we present such a calculation and comment on the asymptotic nature of the wave function.

We obtain the asymptotic normalization constants C_0 and C_2 (corresponding to S and D partial waves of the triton wave function) by comparing the triton asymptotic wave function obtained by the HHE method with the direct product of asymptotically free-neutron and -deuteron wave functions, i.e., in the limit $y\rightarrow\infty$ (Fig. 1 displays the Jacobi coordinates $\mathbf{x}$ and $\mathbf{y}$ for the triton, particle 1 is a proton, and the pair (12) forms the deuteron with the relative coordinate $\mathbf{x}$).

We define a function

$$f_l(y)=\langle [\Phi_d(\mathbf{x})\otimes {}^2l_j(\hat{y})]^{1/2}|\psi_{^3H}(\mathbf{x},\mathbf{y})\rangle , \tag{1}$$

where $\Phi_d(\mathbf{x})$ is the deuteron wave function including the isospin part and ${}^2l_j(\hat{y})$ represents the third-particle (spectator) spin-angular-isospin function (in the spectroscopic notation) and $\psi_{^3H}(\mathbf{x},\mathbf{y})$ is the fully antisymmetric triton wave function. The integration in Eq. (1) is over $\mathbf{x}$ and over the orientations of $\mathbf{y}$. The asymptotic behaviors of $f_l(y)$ are given by[9]

$$f_0(y)\xrightarrow[y\rightarrow\infty]{} C_0 N_{ZR}\frac{e^{-\beta y}}{y} , \tag{2}$$

$$f_2(y)\xrightarrow[y\rightarrow\infty]{} C_2 N_{ZR}\frac{e^{-\beta y}}{y}\left[1+\frac{3}{\beta y}+\frac{3}{\beta^2 y^2}\right] , \tag{3}$$

where

$$N_{ZR}=\sqrt{2\beta}$$

and

$$\beta=[(4M/3\hbar^2)(B_T-B_D)]^{1/2} .$$

B_T and B_D are the triton and deuteron binding energies. The quantities C_0 and C_2 are the S- and D-wave ANC's, respectively.

The HH expansion of the triton wave function $\psi_{^3H}(\mathbf{x},\mathbf{y})$ in the optimal subset approximation, defined in Ref. 12, is given by

TABLE I. Convergence behavior for the $S3$ potential.

$K_{\max}^{(S)}$	$K_{\max}^{(S')}$	B_T (MeV)	P_S	$P_{S'}$	C_0	β (fm^{-1}) (extracted from the wave function)
4		5.088	100.0		0.250	0.3616
6		6.074	100.0		0.368	0.3941
8		6.331	100.0		0.433	0.4020
10		6.453	100.0		0.473	0.4056
12		6.504	100.0		0.506	0.4073
12	2	8.921	97.1	2.9	0.814	0.4757
12	4	9.097	96.9	3.1	1.086	0.4801
12	6	9.175	95.1	4.9	1.366	0.4818
12	8	9.218	96.1	3.9	1.640	0.4826
12	10	9.233	96.1	3.9	1.820	0.4826
12	12	9.237	95.9	4.1	1.860	0.4826

$$\psi_{^3H}(\mathbf{x},\mathbf{y})=\Gamma_{\frac{1}{2}\frac{1}{2}}(A)\sum_K P_{2K}^{(0)}(\Omega)r^{-5/2}U_{2K}^{(S)}(r)+\sum_K \frac{1}{\sqrt{2}}[\Gamma_{\frac{1}{2}\frac{1}{2}}(M-)P_{2K}^{(+)}(\Omega)-\Gamma_{\frac{1}{2}\frac{1}{2}}(M+)P_{2K}^{(-)}(\Omega)]r^{-5/2}U_{2K}^{(S')}(r)$$

$$+\sum_K \frac{1}{\sqrt{2}}[{}^{(+)}D_{2K+2}(\Omega)|(0\tfrac{1}{2})\tfrac{1}{2},-\tfrac{1}{2}\rangle^T+{}^{(-)}D_{2K+2}(\Omega)|(1\tfrac{1}{2})\tfrac{1}{2},-\tfrac{1}{2}\rangle^T]r^{-5/2}U_{2K+2}^{(D)}(r)\ , \quad (4)$$

where $\Gamma_{TS}(R)$ is a three-body isospin-spin function of the total isospin, spin, and symmetry components T, S, and R, respectively,[12] and $|(t\frac{1}{2})T,M_T\rangle^T$ is the three-body isospin wave function, t being the isospin of the (12) pair. The hyperangular functions $P_{2K}^{(\epsilon)}(\Omega)$ with $\epsilon=+$, $-$, or 0 and ${}^{(\epsilon)}D_{2K+2}(\Omega)$ with $\epsilon=+$ or $-$ having specified symmetry under particle exchange are given in Ref. 12 and will not be quoted here for reasons of brevity. The symbol Ω represents the collection of five hyperangles constituted by the polar angles (θ_x,ϕ_x) and (θ_y,ϕ_y) of $\mathbf{x}$ and $\mathbf{y}$, respectively, and a hyperangle ϕ defined through $\phi=\tan^{-1}(x/y)$. The hyperradius r is given by $r=(x^2+y^2)^{1/2}$. The hyperradial partial waves for the S, S', and D states of the triton are $U_{2K}^{(S)}(r)$, $U_{2K}^{(S')}(r)$, and $U_{2K+2}^{(D)}(r)$, respectively.

With expression (4) for $\psi_{^3H}(\mathbf{x},\mathbf{y})$, it is straightforward to calculate $f_l(y)$ from Eq. (1) and it is given by

$$f_0(y)=\int_0^\infty \left[\frac{\omega_0(x)}{\sqrt{2}} \left\{ \sum_K \left[\frac{1}{\sqrt{2}} U_{2K}^{(S')}(r)\,{}^{(+)}N_{2K}\,{}^{(+)}F_{2K}^{0,0}(\pi/2) - U_{2K}^{(S)}(r)\,{}^{(0)}N_{2K}\,{}^{(0)}F_{2K}^{0,0}(\pi/2) \right] {}^{(2)}P_{2K}^{0,0}(\phi) \right\} \right.$$

$$\left. + \frac{\omega_2(x)}{\sqrt{2}} \left[\sum_K U_{2K+2}^{(D)}(r)\,{}^{(+)}N_{2K+2}\,{}^{(+)}F_{2K+2}^{0,2}(\pi/2)\,{}^{(2)}P_{2K+2}^{0,2}(\phi) \right] \right] r^{-5/2} x\, dx\ , \quad (5)$$

TABLE II. Convergence behavior for the EXP I and EXP II potentials with respect to the D-state partial wave contribution ($K_{\max}^{(S)}=12$, $K_{\max}^{(S')}=12$).

Potential	$K_{\max}^{(D)}$	B_T (MeV)	P_S	$P_{S'}$	P_D	C_0	β (fm^{-1}) (extracted from the wave function)	$10^2\times C_2$
EXP I	4	6.648	93.9	2.1	4.0	0.637	0.4113	0.18
	6	7.115	92.3	3.3	4.4	0.899	0.4250	0.36
	8	7.339	91.1	4.2	4.7	1.295	0.4313	0.77
	10	7.461	90.2	5.0	4.8	1.861	0.4346	0.76
	12	7.524	89.7	5.4	4.9	2.448	0.4362	0.97
EXP II	4	7.069	94.7	2.3	3.0	0.827	0.4235	0.25
	6	7.375	93.6	3.1	3.3	1.117	0.4323	0.44
	8	7.514	92.8	3.7	3.5	1.500	0.4361	0.85
	10	7.587	92.3	4.2	3.5	1.950	0.4380	0.78
	12	7.622	92.0	4.5	3.5	2.364	0.4389	0.92

$$f_2(y)=\int_0^\infty \Bigg\{ \frac{\omega_0(x)}{\sqrt{2}} \sum_K [U^{(D)}_{2K+2}(r)^{(+)}N_{2K+2}{}^{(+)}F^{2,0}_{2K+2}(\pi/2)^{(2)}P^{2,0}_{2K+2}(\phi)]$$
$$+\frac{\omega_2(x)}{2} \sum_K [-\sqrt{2}U^{(S)}_{2K}(r)^{(0)}N_{2K}{}^{(0)}F^{2,2}_{2K}(\pi/2)^{(2)}P^{2,2}_{2K}(\phi)+U^{(S')}_{2K}(r)^{(+)}N_{2K}{}^{(+)}F^{2,2}_{2K}(\pi/2)^{(2)}P^{2,2}_{2K}(\phi)$$
$$-U^{(D)}_{2K+2}(r)^{(+)}N_{2K+2}{}^{(+)}F^{2,2}_{2K+2}(\pi/2)^{(2)}P^{2,2}_{2K+2}(\phi)] \Bigg\} r^{-5/2}x\,dx \ , \qquad (6)$$

where the hyperspherical function ${}^{(2)}P_L^{l_1,l_2}(\phi)$, normalization constants ${}^{(\epsilon)}N_L$, and symmetrizing factors ${}^{(\epsilon)}F_L^{l_1,l_2}$ are given in Ref. (12) and $\omega_l(x)$ is the radial deuteron S and D partial waves for $l=0$ and 2, respectively.

We solve the Schrödinger equation for the triton by the HHE method[12,18] when the nucleons interact via a suitable two-body interaction. We have chosen the Afnan-Tang $S3$ potential[19] as well as the EXP I and EXP II potentials of Ref. 20 for the two-body interaction. Note that, since $S3$ has no tensor part, there are no D-state components in the ground state of either the deuteron or the triton. However, for the EXP I and EXP II potentials of Ref. 20 all the components of the ground-state wave function exist. The deuteron binding energies (B_D) for the $S3$, EXP I, and EXP II potentials are 2.224, 2.2253, and 2.2079 MeV, respectively. Calculations have been done on an HP 1000/A700 computer. The convergence in the BE to better than 0.05% for the $S3$ potential is obtained with $K^{(S)}_{\max}=12$, $K^{(S')}_{\max}=12$, where $K^{(R)}_{\max}$ is the maximum number of K values for a given symmetry component R. However, the convergence in the BE is only by about 0.9% and 0.5% with respect to the inclusion of the last two D partial waves for the EXP I and EXP II potentials, respectively (Tables I and II).

To calculate C_0 and C_2, we compare the asymptotic forms of Eqs. (5) and (6) with Eqs. (2) and (3), respectively. This is done graphically by plotting $\log_e[yf_0(y)]$ and

$$\log_e\left[yf_2(y)\left[1+\frac{3}{\beta y}+\frac{3}{\beta^2y^2}\right]^{-1}\right]$$

against y, when a straight line is obtained for $y>50$ fm (Figs. 2 and 3). The straightness of the lines in Figs. 2 and 3 convincingly demonstrate that the asymptotic nature represented by Eqs. (2) and (3) is reached by Eqs. (5) and (6) for $y>50$ fm. The intercepts of the straight lines give the values of C_0 and C_2. We have calculated C_0 and C_2 by a least-squares fit of calculated $f_l(y)$ for $y>50$ fm.

The convergence behavior of various calculated quantities has been shown in Table I for the $S3$ potential and in Table II for the EXP I and EXP II potentials. In Table I, the triton BE (B_T), S, and S' probabilities ($P_S, P_{S'}$), calculated C_0, and β extracted from the tail of the triton wave function (slope of the straight line in Fig. 2) have been presented for different values of $K^{(S)}_{\max}$ and $K^{(S')}_{\max}$. It is seen that C_0 increases by about 2.2%, while B_T increases by less than 0.05% for the addition of the last two S' partial waves. The value of β extracted from the wave function shows an excellent convergence (better than 0.01%). By contrast, the convergence behavior for the

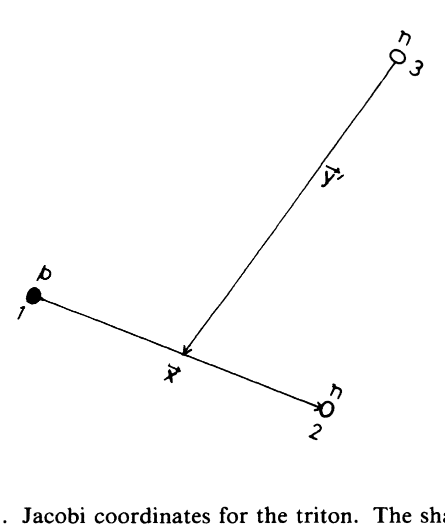

FIG. 1. Jacobi coordinates for the triton. The shaded particle is a proton [$\mathbf{y}=(2/\sqrt{3})\mathbf{y}'$].

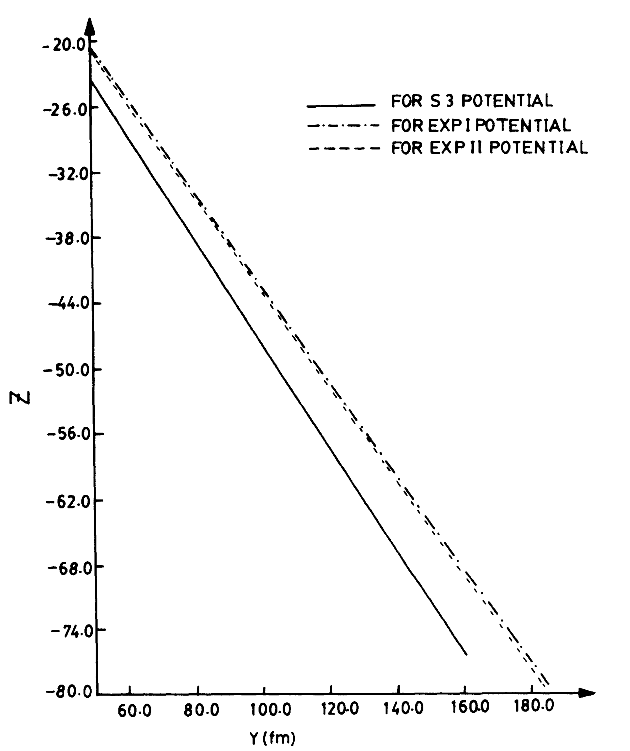

FIG. 2. Plot of $Z=\log_e[yf_0(y)]$ against y for the calculation of C_0.

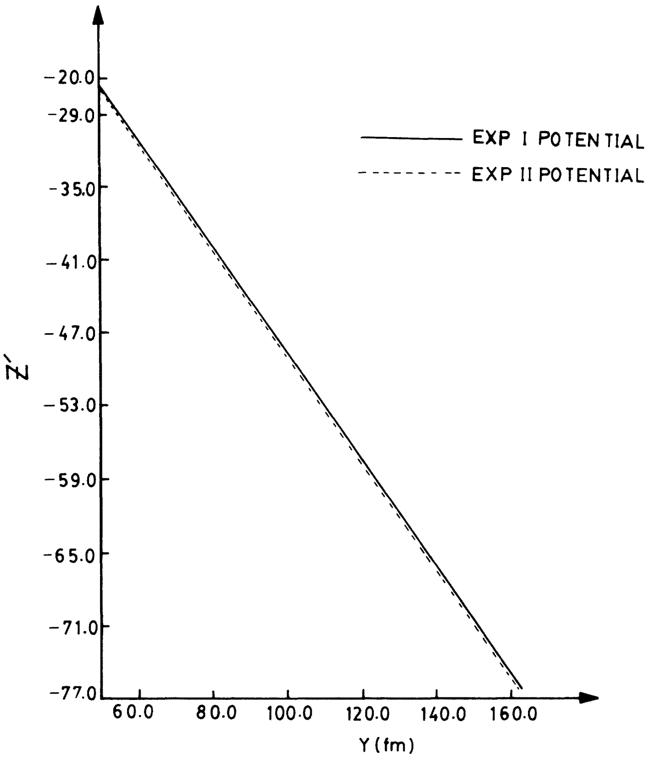

FIG. 3. Plot of

$$Z'=\log_e\left[yf_2(y)\left[1+\frac{3}{\beta y}+\frac{3}{\beta^2y^2}\right]^{-1}\right]$$

against y for the calculation of C_2.

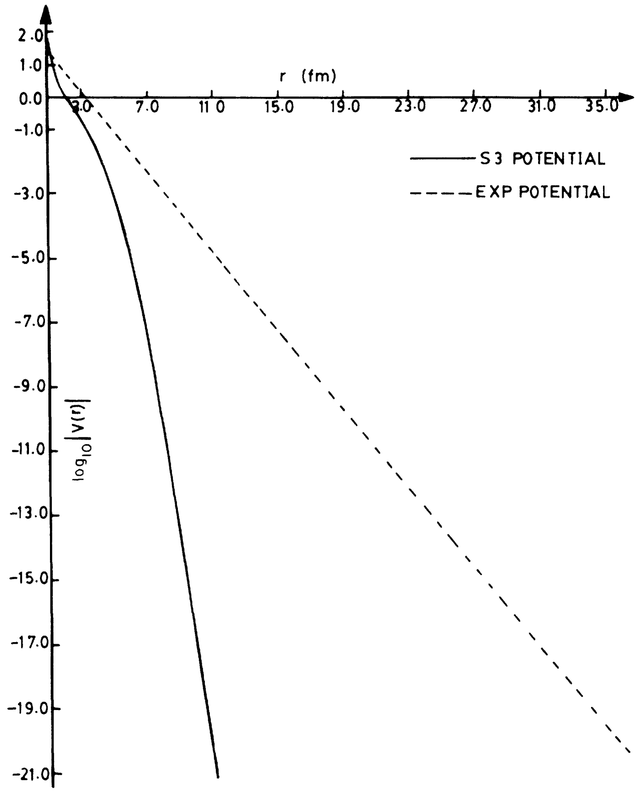

FIG. 4. Plot of $\log_{10}|V(r)|$ vs r for the $S3$ potential (continuous line) and exponential potentials (dotted line). Note that the EXP I and EXP II curves are practically overlapping.

exponential potentials is markedly worse. In Table II, we present the calculated B_T, P_S, $P_{S'}$, and P_D (respectively, S-, S'-, and D-state probabilities), C_0, β (extracted from the wave function), and C_2 values for various values of $K_{\max}^{(D)}$ with $K_{\max}^{(S)}=K_{\max}^{(S')}=12$. It is seen that the convergence of B_T, C_0, and β are worse than those for the $S3$ potential by at least an order of magnitude. While the convergence in C_0 is to about 2.2% for the $S3$ potential, it is only to about 31 and 21 %, respectively, for the EXP I and EXP II potentials. More remarkable is the lack of convergence in the β value which increases by about 0.36 and 0.2 % for the addition of the last two D-state partial waves for the EXP I and EXP II potentials, respectively. The calculated value of C_2 is at least an order of magnitude smaller than the experimental number (which, however, has a large error bar), and its convergence behavior with respect to $K_{\max}^{(D)}$ is not clear. This may be due to smallness of the quantity and associated large relative numerical error. The smallness of C_2 may be due to (1) a weak tensor force in exponential potentials, and (2) incomplete convergence of the asymptotic form of the wave function (affecting both C_0 and C_2 for EXP I and EXP II) as compared to those of the $S3$ potential. This can be understood from the fact that the exponential potentials have a long tail relative to the $S3$ potential. This is clearly demonstrated in a plot of $\log_{10}|V(r)|$ against r (Fig. 4). Thus, the exponential potentials are felt at a much larger two-nucleon separation than the $S3$ potential, invoking higher partial waves to play an important role.

That the BE converges much faster than the asymptotic form of the wave function can be understood from the fact that the major contribution to the BE comes from near the minimum of the effective hyperradial potential well where the wave function also has a maximum. From Table I, we see that the convergent value of C_0 for the $S3$ potential may be expected to be in the range 1.9–1.95, which is close to the values obtained for the S-projected potentials in the Faddeev calculation.[6,8,9] However, no definite values of C_0 and C_2 can be inferred from the present calculation for the exponential potentials, due to the lack of convergence.

We conclude that even though the HHE method is particularly convenient for the calculation of certain bound-state properties (which are not very sensitive to the asymptotic nature of the wave function), its asymptotic form is not as reliable as the BE for the same number of partial waves. This means that one has to be cautious in using the HHE expansion in three-body scattering problems. For a reliable calculation, a larger number of partial waves (than necessary for the BE) must be employed. This is particularly emphasized for potentials with a relatively long tail.

From these we have the following conclusions: (1) The BE converges faster (at least by an order of magnitude) than the asymptotic form of the wave function for the same number of partial waves. (2) Convergence of all calculated quantities is much worse for the exponential potentials.

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