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Transverse form factors in the Riemann rotational model

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The Riemann rotational model is an extension of the Bohr-Mottelson collective model which allows for linear nuclear currents, indexed by the rigidity r, between the limits of rigid rotation, $r=1$, and irrotational flow, $r=0$. The Riemann moment of inertia and transverse form factors are shown to be weighted linear combinations of the corresponding rigid and irrotational values. The resulting simple analytic formulas for the transverse factors are useful for the analysis and interpretation of data from state of the art electron scattering experiments.

The ultimate character of nuclear rotational motion is one of the unsolved basic science problems in nuclear structure physics. To solve this problem, a direct determination of the nuclear current is required. Transverse form factors probe the current and, as measured in state of the art electron scattering experiments, provide a practical tool for our purpose.[1,2]

The Bohr-Mottelson model is a theoretical structure for analyzing the macroscopic geometrical properties of rotating nuclei. However, the model is limited to only two choices for the current: rigid rotation (RR) and irrotational flow (IF). A satisfactory model for analyzing current measurements must allow for the possibility of currents intermediate between RR and IF.

The Riemann rotational model is a simple extension of the Bohr-Mottelson model in which the velocity field is assumed to be a linear function of position. The Riemann model, in its classical form, provides a theory for rotating stars.[3] For nuclei, a linear velocity field was proposed first by Cusson.[4] The general collective motion group GCM(3) provides a unified algebraic framework for both classical and quantum Riemann models. The classical Riemann rotor is a Hamiltonian dynamical system on a co-adjoint orbit of GCM(3).[5] The quantum Riemann models are irreducible unitary representations of GCM(3).[6,7] The connection between the classical and quantum models is achieved by geometric quantization.[8]

A Riemann rotor is characterized by a parameter f, defined as the ratio of the uniform vorticity ζ to the angular velocity ω.[3,5] The uniform vorticity is the curl of the body-fixed velocity field. The curl of the laboratory frame velocity vector field $\mathbf{U}(\mathbf{r})$, projected onto the body-fixed frame, is $\mathbf{curl}\ \mathbf{U}(\mathbf{r})=\boldsymbol{\zeta}+2\boldsymbol{\omega}$. It is assumed that both the uniform vorticity and angular velocity are aligned with a principal axis, say the x axis, whence $\mathbf{curl}\ \mathbf{U}(\mathbf{r})=(f+2)\boldsymbol{\omega}$. When $f=0$, the body is rotating rigidly; when $f=-2$, the flow is irrotational. Alternatively, define the rigidity $r=1+f/2$ for which the limits are $r=1$ (RR) and $r=0$ (IF).

The Kelvin circulation is the line integral of the velocity field around the ellipse bounding the y-z principal plane, which, by Stoke's theorem, is equivalent to a surface integral,

$$\mathcal{L}=(M/5\pi)\oint \mathbf{U}\cdot dl=(M/5\pi)\int \mathbf{curl}\ \mathbf{U}\cdot \mathbf{n}dS$$
$$=(M/5)a_2a_3(f+2)\omega\,.$$

The ratio of the Kelvin circulation $\mathcal{L}$ to the angular momentum L is given by

$$\frac{\mathcal{L}}{L}=\frac{\chi r}{1+\chi^2(r-1)}\,, \tag{1}$$

where $\chi\equiv(1-\mathcal{J}_{\mathrm{IF}}/\mathcal{J}_{\mathrm{RR}})^{1/2}$. $\mathcal{J}_{\mathrm{RR}}$ and $\mathcal{J}_{\mathrm{IF}}$ denote the RR and IF moments of inertia.

The moment of inertia $\mathcal{J}_r$ of a Riemann rotor is a convex combination of the rigid and irrotational values,[5]

$$\mathcal{J}_r=r\mathcal{J}_{\mathrm{RR}}+(1-r)\mathcal{J}_{\mathrm{IF}}\,. \tag{2}$$

The velocity vector field $\mathbf{U}(\mathbf{r})$ is also a convex combination of rigid and irrotational contributions,

$$\mathbf{U}(\mathbf{r})=r\mathbf{U}_{\mathrm{RR}}+(1-r)\mathbf{U}_{\mathrm{IF}}\,, \tag{3}$$

$$\mathbf{U}_{\mathrm{RR}}=\boldsymbol{\omega}\times\mathbf{r}=\omega\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\mathbf{r}\,, \tag{4a}$$

$$\mathbf{U}_{\mathrm{IF}}=-(\mathcal{J}_{\mathrm{IF}}/\mathcal{J}_{\mathrm{RR}})^{1/2}\omega\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\mathbf{r}\,. \tag{4b}$$

Expressed as spherical tensors, the rigid and irrotational fields are associated with rank one and two tensors, respectively,

$$U(\mathbf{r})^{(1)}_{\mu}=r(\hat{V}^{(1)}\times r^{(1)})^{(1)}_{\mu}+(1-r)(\hat{V}^{(2)}\times r^{(1)})^{(1)}_{\mu}\,, \tag{5}$$

where $\mathbf{U}(\mathbf{r})=\sum(-1)^{\mu}U(\mathbf{r})^{(1)}_{-\mu}\xi_{\mu}$; the spherical basis is $\xi_0=\mathbf{e}_z$, $\xi_{\pm1}=\mp(\mathbf{e}_x\pm i\mathbf{e}_y)/\sqrt{2}$, $\hat{V}^{(1)}_{\mu}=-i\sqrt{2}\omega_{\mu}$, $\hat{V}^{(2)}_{\mu}=i\sqrt{10/3}(\mathcal{J}_{\mathrm{IF}}/\mathcal{J}_{\mathrm{RR}})^{1/2}\mu\omega_{\mu}$, $\mu=\pm1$, and $\hat{V}^{(2)}_{\mu}=0$, otherwise, $\omega_{\mu}=-\mu\omega/\sqrt{2}$, $r^{(1)}_{\mu}=\sqrt{4\pi/3}\mathbf{r}Y^{(1)}_{\mu}$.

The transverse electric multipole, in the body-fixed frame, is given in the Born approximation by

$$\hat{T}^{E\lambda}_{\mu}(q)=\sum_l f_l(\lambda)\int d^3r\, j_l(qr)\mathbf{Y}^{\mu}_{\lambda l}(\Omega)\cdot\hat{\mathbf{J}}(\mathbf{r})\,, \tag{6}$$

where

$$f_l(\lambda)\equiv(i^{\lambda+1}/\hat{\lambda})(\sqrt{\lambda+1}\delta_{l,\lambda-1}-\sqrt{\lambda}\delta_{l,\lambda+1})\,,$$

and the current is the product of the proton charge density and the velocity field, $\hat{\mathbf{J}}(\mathbf{r})=\rho^{P}(\mathbf{r})\mathbf{U}(\mathbf{r})$.

Since the electric multipoles are linear in the current, $\hat{T}^{E\lambda}_{\mu}(q)$ is a convex combination of rigid and irrotational terms. Both terms may be evaluated in a common formalism. First, note the identity involving the vector spherical harmonics,

$$\mathbf{Y}^{\mu}_{\lambda l}(\Omega)\cdot(\hat{V}^{(k)}\times r^{(1)})^{(1)} = \sum_{L}\sqrt{3}\hat{l}\hat{\lambda}\hat{L}\begin{pmatrix} l & 1 & L\\ 0 & 0 & 0\end{pmatrix} W(\lambda l k 1;1L)\sum_{\nu M}(-1)^{\mu}\begin{pmatrix} k & \lambda & L\\ \nu & -\mu & -M\end{pmatrix} r Y_{L-M}(\Omega)\hat{V}^{(k)}_{\nu}\,. \tag{7}$$

Next, expand the charge density in spherical harmonics, $\rho^{P}(\mathbf{r}) = \sum_L \rho^{P}_{L}(r) Y_{L0}(\Omega)$. Only even harmonics contribute for axially symmetric nuclei. Then, using the identity (7), we have rigid rotor ($k=1$) and irrotational ($k=2$) contributions to the multipoles,

$$\hat{T}^{E\lambda}_{\mu}(q;k) = \sum_{l,L} f_l(\lambda)\int_0^{\infty} r^3 dr\, j_l(qr)\rho^{P}_{L}(r)\sqrt{3}\hat{l}\hat{\lambda}\hat{L}\begin{pmatrix} l & 1 & L\\ 0 & 0 & 0\end{pmatrix} W(\lambda l k 1;1L)(-1)^{\mu}\begin{pmatrix} k & \lambda & L\\ \mu & -\mu & 0\end{pmatrix}\hat{V}^{(k)}_{\mu}\,. \tag{8}$$

Specializing to $E2$ multipoles, the rigid rotor sum is restricted to $L=2$, due to angular momentum coupling and axial symmetry. For the irrotational term, $L=0$ is the leading term [of order $O(\beta)$] in an expansion in the deformation β. Thus, the integrals for the rigid and irrotational cases are given to leading order in β by

$$\hat{T}^{E2}_{\mu}(q;k=1) = -(Ze)\sqrt{3/\pi}\,\mathcal{J}_{\mathrm{RR}}\mathfrak{F}^{E2}_{\mathrm{RR}}(q)\mu\omega_{\mu}\,, \tag{9a}$$

$$\hat{T}^{E2}_{\mu}(q;k=2) = -(Ze)\sqrt{3/\pi}\,\mathcal{J}_{\mathrm{IF}}\mathfrak{F}^{E2}_{\mathrm{IF}}(q)\mu\omega_{\mu}\,, \tag{9b}$$

where the form factors are

$$\mathfrak{F}^{E2}_{\mathrm{RR}}(q) = \frac{\sqrt{30}}{40}\frac{Q_0}{R_0\mathcal{J}_{\mathrm{RR}}}[j_1(qR_0) - \tfrac{2}{3}j_3(qR_0)]\,, \tag{10a}$$

$$\mathfrak{F}^{E2}_{\mathrm{IF}}(q) = \frac{\sqrt{30}}{40}\frac{Q_0}{R_0\mathcal{J}_{\mathrm{IF}}}[j_1(qR_0) + j_3(qR_0)]\,, \tag{10b}$$

and the intrinsic quadrupole moment $Q_0 = 3(5\pi)^{-1/2}R_0^2\beta$, and $R_0 = 1.12A^{1/3}$. Since both the rigid and irrotational terms factor similarly, the $E2$ multipole simplifies to

$$\hat{T}^{E2}_{\mu}(q) = -(Ze)\sqrt{3/\pi}\,\mathfrak{F}^{E2}_{r}(q)\mu\mathcal{J}_r\omega_{\mu}\,, \tag{11}$$

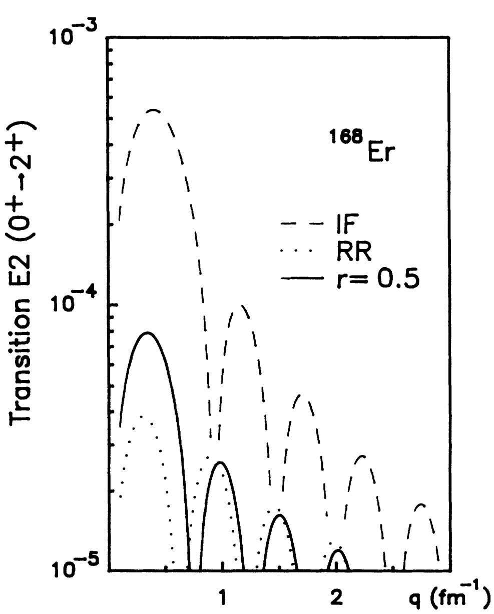

FIG. 1. The transition $E2$ form factor for ^{168}Er, $0^{+}\rightarrow 2^{+}$. Intermediate between the rigid rotor (RR) and irrotational flow (IF) factors is the Riemann curve for $r=0.5$.

where the form factor for a Riemann rotor is

$$\mathfrak{F}^{E2}_{r} = [r\mathcal{J}_{\mathrm{RR}}\mathfrak{F}^{E2}_{\mathrm{RR}} + (1-r)\mathcal{J}_{\mathrm{IF}}\mathfrak{F}^{E2}_{\mathrm{IF}}]/\mathcal{J}_r\,. \tag{12}$$

The $E2$ multipole may be transformed now into an operator in the Hilbert space of nuclear wave functions by replacing $\mathcal{J}_r\omega_{\mu}$ by the angular momentum operator in the body-fixed frame $\hat{L}_{\mu}$. Rotating into the lab frame and inserting the resulting $E2$ multipole operator between Bohr-Mottelson rotational wave functions yields the transition multipole[2]

$$F^{E2}(q)_{I_i\rightarrow I_f} = \frac{Ze}{\sqrt{4\pi}}[I_f(I_f+1) - I_i(I_i+1)] \times (I_i 020\,|\,I_f 0)\mathfrak{F}^{E2}_{r}\,. \tag{13}$$

A simple analytic formula has been derived for the transverse $E2$ form factor $\mathfrak{F}^{E2}_{r}$, Eq. (12), which expresses the Riemann form factor as a weighted combination of the rigid and irrotational form factors. In order to apply this result, the rigidity must be determined. Assuming the energy spectrum is entirely kinetic, the experimental moment of inertia determines the rigidity r, Eq. (2). The form factor, multiplied by $\sqrt{4\pi}/Ze$, for the $0^{+}\rightarrow 2^{+}$ transition in ^{168}Er is plotted in Fig. 1 for the rigid rotor, irrotational flow, and Riemann fluid $r=0.5$ cases. As expected, the Riemann form factor falls between the curves for RR and IF.

The magnetic form factor in the Riemann model may be derived easily using the same techniques described here for the electric multipoles.

It is hoped that this simple extension of the Bohr-Mottelson model will prove valuable in the analysis of transition multipole data. By fitting the Riemann model to the experimental multipoles, a direct determination of the rigidity r, and hence the Kelvin circulation $\mathcal{L}$, can be achieved.

Although the macroscopic Riemann model is useful for interpreting transverse multipole measurements, a complete understanding of nuclear rotational motion must be founded on microscopic theory, e.g., PHF, cranking,[9] or the shell model.[10] In the framework of the Riemann model, the problem is then to predict the rigidity r from microscopic theory.

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[1]T. W. Donnelly and J. D. Walecka, Annu. Rev. Nucl. Sci. **25**, 329 (1975).
[2]E. Moya de Guerra, Phys. Rep. **138**, 293 (1986).
[3]S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium* (Yale Univ. Press, New Haven, 1969).
[4]R. Y. Cusson, Nucl. Phys. **A114**, 289 (1968).
[5]G. Rosensteel, Ann. Phys. (N.Y.) **186**, 230 (1988).
[6]G. Rosensteel and D. J. Rowe, Ann. Phys. (N.Y.) **96**, 1 (1976).
[7]O. L. Weaver, L. C. Biedenharn, and R. Y. Cusson, Ann. Phys. (N.Y.) **102**, 493 (1976).
[8]G. Rosensteel and E. Ihrig, Ann. Phys. (N.Y.) **121**, 113 (1979).
[9]D. Berdichevsky, P. Sarriguren, E. Moya de Guerra, M. Nishimura, and D. W. L. Sprung, Phys. Rev. C **38**, 338 (1988).
[10]M. G. Vassanji and D. J. Rowe, Phys. Lett. B **190**, 7 (1987).