Cluster-orbital shell model with continuum discretization and its application to the He isotopes

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A model is formulated for a system of several valence nucleons coupled to a core. Single-particle orbits in this model can be determined consistently with underlying potentials between the core and the valence nucleon. A continuum spectrum for the single-particle state is taken into account through its discretization. A very simple system, the He isotopes, is used to show the feasibility of the model.

I. INTRODUCTION

The recent development in the sophisticated experimental technique of using radioactive nuclear beams has led to the possibility of nuclear spectroscopy of nuclei far from stability.[1] One example of the most interesting results is the determination of the matter radii of light nuclei deduced from the interaction cross sections of high-energy heavy-ion collisions.[2] We find some noteworthy features in Fig. 1 which summarizes the matter radii of p-shell isotopes obtained by Tanihata *et al.*[2] For example, the fact that the radii of ^{6}He and ^{8}He are almost the same and much larger than that of ^{4}He suggests that the extra neutrons in ^{6}He and ^{8}He fill an extended $p_{3/2}$ orbit. This is also consistent with the result on the Li isotopes, because the excess neutrons should fill the $p_{3/2}$ orbit up to ^{9}Li and the last two neutrons of ^{11}Li occupy a further extended $p_{1/2}$ orbit.[3] It seems that even in very neutron-rich nuclei the single-nucleon spectrum at least approximately follows the conventional order of the shell model established in stable nuclei, but the spatial extension of the orbits is quite different between the stable and neutron-rich nuclei. The difference in the spatial extension of the orbits is certainly correlated with the very weak binding of the last neutrons, as typically shown in the anomalously large radii of the neutron dripline nuclei, ^{11}Li, ^{14}Be, and ^{17}B.

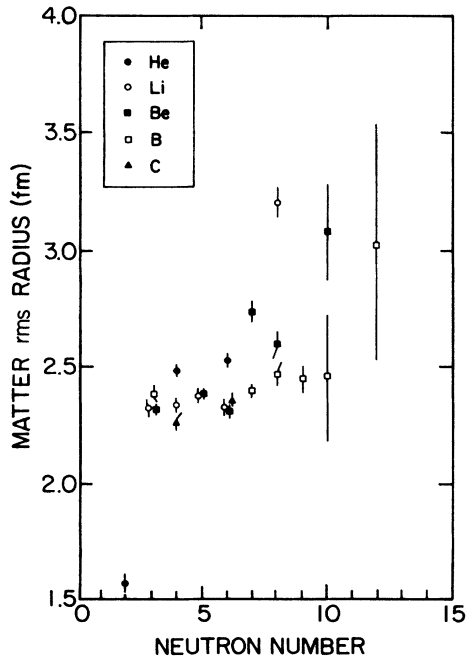

FIG. 1. Matter root-mean-square radii of the p-shell isotopes taken from Ref. 2.

An attempt[4] at understanding the structure of light exotic nuclei like neutron-rich nuclei has been undetaken in a shell model approach without employing an inert core. The nucleons are distributed mostly among major shell configurations and effective interactions are determined from an empirical procedure using both energies and electromagnetic moments of isospin $T \leq 2$ nuclei as input. The binding energies obtained, however, deviate from the empirical values for large neutron/proton ratio and it is conjectured that an effective interaction appropriate for a unified description of normal and exotic nuclei cannot be determined uniquely from data on normal nuclei only. In contrast to experiment, the calculation predicts a gradual increase of the matter radii of the Li isotopes with increasing mass number and reproduces no anomalous rise of the radius at ^{11}Li.

The purpose of this paper is to explore in more detail the cluster-orbital shell model outlined in Ref. 5, keeping in mind its application to the structure study of exotic nuclei. Although we investigate the properties of the He isotopes with this model, the emphasis of the paper is to establish a method for studying the structure of the exotic nuclei in which very extended single-nucleon orbits are inevitably included. The question of what the single-

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particle wave functions look like is very important, particularly in the structure study of the exotic nuclei. Suppose we try to know the single-particle wave function of the last nucleon in ^{6}He, ^{8}He, and ^{6}Li from the spectra of ^{5}He and ^{5}Li. As neither ^{5}He nor ^{5}Li has any bound state but a low-lying $p_{3/2}$ resonance, we guess that the sought single-particle wave function has relevance to the resonance wave function. Accordingly, we in general have to deal with a continuum spectrum in constructing the single-particle orbits of the exotic nuclei. The cluster-orbital shell model is formulated particularly to suit this purpose.

We start from the assumption that nuclei consist of a stable core and a group of valence nucleons weakly coupled to the core. The motion of the valence nucleons is described in terms of the radius vectors measured from the center of mass of the core. By the introduction of this set of coordinates of independent particle nature, our formalism can still make use of existing shell model codes and moreover becomes completely free from spurious center-of-mass motion no matter how highly the valence nucleons are excited. In Sec. II we define the cluster-orbital shell model and discuss the core-valence nucleon potential that serves to generate the single-particle wave functions of the valence nucleons. Section III is concerned with two methods of continuum discretization to provide a square integrable, orthogonal basis for the valence nucleons in the continuum. One is called the momentum-bin discretization method, which borrows the notion of eigendifferentials. Another is based on the direct diagonalization of an appropriate single-nucleon Hamiltonian in a certain square integrable basis. Our formalism is applied in Sec. IV to calculate the energies, matter radii, and densities of the He isotopes. Section V gives a summary.

II. FORMULATION OF THE CLUSTER-ORBITAL SHELL MODEL

A. The cluster-orbital shell model

In neutron-rich nuclei like the He isotopes, we assume that weakly bound nucleons move around a stable core. Let n and f be the number of the valence nucleons and the mass number of the core, respectively, and $\mathbf{r}_1, \ldots, \mathbf{r}_n$, $\mathbf{r}_{n+1}, \ldots, \mathbf{r}_{n+f}$ be the position vectors of the nucleons. A system of coordinates used in the cluster-orbital shell model[5] is

$$\boldsymbol{\xi}_i = \left[\frac{f}{f+1}\right]^{1/2} \mathbf{x}_i = \left[\frac{f}{f+1}\right]^{1/2} \left[\mathbf{r}_i - \left[\frac{1}{f}\right]^{1/2} \mathbf{R}_c\right] \quad (i=1, 2, \ldots, n) , \tag{1}$$

$$\mathbf{R} = \left[\frac{1}{n+f}\right]^{1/2} (\mathbf{r}_1 + \cdots + \mathbf{r}_{n+f}) = \left[\frac{1}{n+f}\right]^{1/2} (\mathbf{r}_1 + \cdots + \mathbf{r}_n) + \left[\frac{f}{n+f}\right]^{1/2} \mathbf{R}_c , \tag{2}$$

where $\mathbf{R}_c$ is the normalized center-of-mass coordinate of the core defined by

$$\mathbf{R}_c = \left[\frac{1}{f}\right]^{1/2} (\mathbf{r}_{n+1} + \cdots + \mathbf{r}_{n+f}) . \tag{3}$$

The coordinate $\mathbf{x}_i$ is the radius vector from the center of mass of the core to the ith valence nucleon, $\boldsymbol{\xi}_i$ its corresponding normalized coordinate, and $\mathbf{R}$ the normalized total center-of-mass coordinate. Let ρ_c and ρ denote a collection of all the internal spatial coordinates of the core and the total system including the core and the valence nucleons, respectively. Since the transformation of Eqs. (1) and (2) is not orthogonal, we need to express the volume element $d\rho$ in terms of the new coordinates. By the definition we have

$$d\rho\, d\mathbf{R} = d\mathbf{r}_1 d\mathbf{r}_2 \cdots d\mathbf{r}_{n+f} = d\mathbf{r}_1 \cdots d\mathbf{r}_n d\rho_c d\mathbf{R}_c . \tag{4}$$

The transformation of Eqs. (1) and (2) enables us to express

$$d\boldsymbol{\xi}_1 \cdots d\boldsymbol{\xi}_n d\mathbf{R} = \left[\frac{f}{f+1}\right]^{3n/2} \times \left[\frac{n+f}{f}\right]^{3/2} d\mathbf{r}_1 \cdots d\mathbf{r}_n d\mathbf{R}_c . \tag{5}$$

Comparing Eqs. (4) and (5) leads to

$$d\rho = \left[\frac{f+1}{f}\right]^{3n/2} \left[\frac{f}{n+f}\right]^{3/2} d\boldsymbol{\xi}_1 \cdots d\boldsymbol{\xi}_n d\rho_c = \left[\frac{f}{n+f}\right]^{3/2} d\mathbf{x}_1 \cdots d\mathbf{x}_n d\rho_c . \tag{6}$$

The next step is to express the total Hamiltonian in terms of the coordinates, $\mathbf{x}_i$ and ρ_c. The kinetic energy, K, with the center-of-mass kinetic energy subtracted is given by

$$K = K_c + \sum_{i=1}^{n} \frac{1}{2\mu} \mathbf{p}_i^2 + \sum_{j>i=1}^{n} \frac{1}{(f+1)\mu} \mathbf{p}_i \cdot \mathbf{p}_j , \tag{7}$$

where K_c is the internal kinetic energy of the core, $\mathbf{p}_j = -i\hbar\partial/\partial\mathbf{x}_j$ is the momentum conjugate to $\mathbf{x}_j$, and $\mu = f/(f+1)m$ is the reduced mass of the core plus single-nucleon system. The interaction energy, V, is separated as

$$V = V_c + \sum_{i=1}^{n} \sum_{j=n+1}^{n+f} v_{ij} + \sum_{j>i=1}^{n} v_{ij} , \tag{8}$$

where v is the nucleon-nucleon potential and V_c is the interaction energy of the core. Combining Eqs. (7) and (8) for the total Hamiltonian we get

$$H = H_c + \sum_{i=1}^{n} h_i + \sum_{j>i=1}^{n} \left[v_{ij} + \frac{1}{(f+1)\mu} \mathbf{p}_i \cdot \mathbf{p}_j \right] , \tag{9}$$

with

$$h_i = \frac{1}{2\mu} \mathbf{p}_i^2 + \sum_{j=n+1}^{n+f} v_{ij} . \tag{10}$$

Here the spatial dependence of $H_c = K_c + V_c$ is expressed

in terms of ρ_c, and the radial dependence of v_{ij} in the last term of Eq. (9) is given by $\mathbf{r}_i-\mathbf{r}_j=\mathbf{x}_i-\mathbf{x}_j$. The h_i of Eq. (10) expresses the Hamiltonian for the relative motion between the single valence nucleon and the core. A key assumption of the cluster-orbital shell model is to approximate h_i with a single-particle Hamiltonian $\hat{h}_i=(1/2\mu)\mathbf{p}_i^2+U_i$, where U_i is assumed to be independent of ρ_c. Equation (9) then takes a desired form in which the coordinates $\mathbf{x}_i$ and ρ_c decouple

$$H=H_c+\sum_{i=1}^{n}\left[\frac{1}{2\mu}\mathbf{p}_i^2+U_i\right]+\sum_{j>i=1}^{n}\left[v_{ij}+\frac{1}{(f+1)\mu}\mathbf{p}_i\cdot\mathbf{p}_j\right]. \tag{11}$$

The Hamiltonian (11), basic to the cluster-orbital shell model, has the form similar to that used in the conventional shell model. Namely, it consists of the core Hamiltonian, the single-particle energy part of the valence nucleons and the interaction part among the valence nucleons. The only formal difference is that the last part includes the $\mathbf{p}_i\cdot\mathbf{p}_j$ term. The term arises because of the use of the nonorthogonal coordinates $\mathbf{x}_i$. In Sec. II B we discuss the relationship between h_i and $\hat{h}_i$.

The total angular momentum $\mathbf{J}$ of the system is given by

$$\mathbf{J}=\mathbf{J}_c+\mathbf{L}+\sum_{i=1}^{n}\mathbf{s}_i\ , \tag{12}$$

where $\mathbf{J}_c$ is the angular momentum of the core, $\mathbf{L}$ the orbital angular momentum of the valence nucleons, and $\mathbf{s}_i$ is the spin of the ith valence nucleon. When a given orthogonal transformation brings the set of coordinates $\mathbf{r}_1,\ldots,\mathbf{r}_n,\mathbf{R}_c$ to a set of coordinates $\boldsymbol{\eta}_1,\ldots,\boldsymbol{\eta}_n,\mathbf{R}$ and the matrix connecting $\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_n$ with $\boldsymbol{\eta}_1,\ldots,\boldsymbol{\eta}_n$ is denoted by A, i.e., $\boldsymbol{\xi}=A\boldsymbol{\eta}$, then we have

$$\begin{aligned}\mathbf{L}&=\sum_{j=1}^{n}\boldsymbol{\eta}_j\times\left[-i\hbar\frac{\partial}{\partial\boldsymbol{\eta}_j}\right]\\&=\sum_{j=1}^{n}\boldsymbol{\eta}_j\times\left[-i\hbar\sum_{k=1}^{n}A_{kj}\frac{\partial}{\partial\boldsymbol{\xi}_k}\right]\\&=\sum_{j=1}^{n}\boldsymbol{\xi}_j\times\left[-i\hbar\frac{\partial}{\partial\boldsymbol{\xi}_j}\right]\\&=\sum_{j=1}^{n}\mathbf{x}_j\times\mathbf{p}_j\ ,\end{aligned} \tag{13}$$

where $\boldsymbol{l}_i=\mathbf{x}_i\times\mathbf{p}_i$ is the orbital angular momentum of the ith valence nucleon relative to the core. Inserting Eq. (13) into Eq. (12) assures that the total angular momentum is given by

$$\mathbf{J}=\mathbf{J}_c+\sum_{i=1}^{n}(\boldsymbol{l}_i+\mathbf{s}_i)\ , \tag{14}$$

and that the ordinary rule of coupling angular momenta for the valence nucleons can also be applied in the cluster-orbital shell model.

One remarkable advantage of our formalism is that it is completely free from any spurious center-of-mass motion regardless of the excitations carried by the valence nucleons because the coordinates $\mathbf{x}_i$ are all a kind of relative coordinates. We finally note that properly normalized total wave functions are given by

$$\left[\frac{n+f}{f}\right]^{3/4}\Psi(\rho_c\sigma_c\tau_c,\mathbf{x}_1\cdots\mathbf{x}_n\sigma_1\cdots\sigma_n\tau_1\cdots\tau_n)\ , \tag{15}$$

when the function Ψ is normalized as

$$\sum_{\sigma_c\tau_c\sigma_1\cdots\sigma_n\tau_1\cdots\tau_n}\int|\Psi|^2d\rho_c\,d\mathbf{x}_1\cdots d\mathbf{x}_n=1\ , \tag{16}$$

where σ and τ stand for spin and isospin coordinates. The necessity of the factor $[(n+f)/f]^{3/4}$ of Eq. (15) is clear from Eq. (6).

B. The core plus single-nucleon system

It is essential for the cluster-orbital shell model that the Hamiltonian for the relative motion of the single valence nucleon, Eq. (10), can be approximated with an effective single-particle Hamiltonian, $\hat{h}=(1/2\mu)\mathbf{p}^2+U$. When the core is assumed to be inert, the wave function for the single valence nucleon plus core system can be written as

$$\Psi=\mathcal{A}[\Psi(\rho_c\sigma_c\tau_c)u_{lj}(x)[Y_l(\hat{\mathbf{x}})\times\chi_{1/2}]_{jm}]\ , \tag{17}$$

where $\Psi(\rho_c\sigma_c\tau_c)$ is the appropriate core wave function and $\mathcal{A}$ stands for the antisymmetrization operator. The equation of motion for an unknown relative motion function u_{lj} is derived from the resonating group method[6,7]

$$\langle\Psi(\rho_c\sigma_c\tau_c)[Y_l(\hat{\mathbf{x}})\times\chi_{1/2}]_{jm}|H_c+h-E|\mathcal{A}[\Psi(\rho_c\sigma_c\tau_c)u_{lj}(x)[Y_l(\hat{\mathbf{x}})\times\chi_{1/2}]_{jm}]\rangle=0\ . \tag{18}$$

Separating the direct and exchange parts in Eq. (18) we obtain for $f_{lj}(x)=xu_{lj}(x)$,

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}+\frac{\hbar^2}{2\mu}\frac{l(l+1)}{x^2}+V_{lj}(x)-\frac{\hbar^2k^2}{2\mu}\right]f_{lj}(x)+\int_0^\infty K_{lj}(x,x')f_{lj}(x')dx'=0\ , \tag{19}$$

with the so-called direct potential

$$V_{lj}(x)=\left\langle\Psi(\rho_c\sigma_c\tau_c)[Y_l(\hat{\mathbf{x}})\times\chi_{1/2}]_{jm}\left|\sum_{k=2}^{f+1}v_{1k}\right|\Psi(\rho_c\sigma_c\tau_c)[Y_l(\hat{\mathbf{x}})\times\chi_{1/2}]_{jm}\right\rangle\ , \tag{20}$$

and the exchange kernel

$$K_{lj}(x',x'')=\left\langle \Psi(\rho_c\sigma_c\tau_c)\frac{\delta(x-x')}{x'}[Y_l(\hat{\mathbf{x}})\times\chi_{1/2}]_{jm}|(H_c+h-E)(\mathcal{A}-1)|\Psi(\rho_c\sigma_c\tau_c)\frac{\delta(x-x'')}{x''}[Y_l(\hat{\mathbf{x}})\times\chi_{1/2}]_{jm}\right\rangle , \tag{21}$$

where $(1/2\mu)\hbar^2k^2=E-\langle\Psi(\rho_c\sigma_c\tau_c)|H_c|\Psi(\rho_c\sigma_c\tau_c)\rangle$ is the energy of the valence nucleon. Equation (19) suggests that the sought single-particle potential U_{lj} is formally given by an nonlocal potential

$$U_{lj}=V_{lj}(x)\delta(x-x')+K_{lj}(x,x') . \tag{22}$$

Although the nonlocality of the potential does not cause any difficulty in our formalism, its energy dependence through the E term of Eq. (21) makes cluster-orbital shell model calculations complicated. At present we have to assume that the energy dependence is not so strong that it can be neglected, at least in the energy region of interest.

For the Hamiltonian of Eq. (11) the total wave function, Eq. (15), reduces to the product form

$$\left[\frac{n+f}{f}\right]^{3/4}\Psi(\rho_c\sigma_c\tau_c)\Psi(\mathbf{x}_1\cdots\mathbf{x}_n\sigma_1\cdots\sigma_n\tau_1\cdots\tau_n) , \tag{23}$$

when the explicit inclusion of antisymmetrization is neglected between the core nucleons and the valence nucleons. To estimate how big the effect of antisymmetrization is, let us consider the simplest example of the norm kernel for a system of a single nucleon plus the core with a doubly closed-shell configuration, e.g., $N+{}^4$He, $N+{}^{16}$O, and $N+{}^{40}$Ca. When the core nucleons occupy the harmonic-oscillator levels with the number of oscillator quanta Q up to $Q=0$, $1,\ldots,Q_c$, the norms are given by[7,8]

$$\begin{aligned}(N_Q)^2&\equiv\langle\Psi(\rho_c\sigma_c\tau_c)u_l^{\{Q0\}}(\xi)[Y_l(\hat{\boldsymbol{\xi}})\times\chi_{1/2}]_{jm}|\mathcal{A}|\Psi(\rho_c\sigma_c\tau_c)u_l^{\{Q0\}}(\xi)[Y_l(\hat{\boldsymbol{\xi}})\times\chi_{1/2}]_{jm}\rangle\\&=\begin{cases}0 & \text{for } Q\le Q_c\\ \displaystyle\sum_{Q'=Q_c+1}^{Q}\left[-\frac{1}{f}\right]^{Q-Q'}\left[\frac{f+1}{f}\right]^{Q'}\frac{Q!}{(Q-Q')!Q'!} & \text{for } Q>Q_c ,\end{cases}\end{aligned} \tag{24}$$

where $u_l^{\{Q0\}}(\xi)$ is the radial part of the harmonic-oscillator wave function. The vanishing norms for $Q\le Q_c$ follow from the Pauli principle and imply that the relative motion functions of the valence nucleon must be orthogonal to the occupied orbits of the core. Table I lists the values of $[N_Q]^2$ for the cases of $N+{}^4$He, $N+{}^{16}$O, and $N+{}^{40}$Ca. We see that they are close to unity for a system with the heavier core or for $Q\ge 3$ even in the $N+{}^4$He case. Since the deviation from unity is due to the exchange effect of the norm kernel, we understand that the effect of antisymmetrization becomes less significant as the single-particle wave functions of the valence nucleons contain less components with small Q.

III. THE DISCRETIZATION OF CONTINUUM STATES

A. The momentum-bin discretization

One of the chief motivations for developing the cluster-orbital shell model is the recognition that the valence nucleons in the exotic nuclei are so weakly bound that their motion extends farther compared to the motion of the nucleons in stable nuclei. In this case the single-particle Hamiltonian, $\hat{h}=(1/2\mu)\mathbf{p}^2+U$, may have only few or even no bound-state spectra. For example, if we consider ^{6}He as a system of the ^{4}He core plus two neutrons, the single neutron moving around the ^{4}He core does not have any bound state in ^{5}He, so that we in general have to deal with the continuum spectrum to construct square integrable orbitals of ^{6}He, which is bound against the neutron separation.

One of the methods for constructing square integrable orbitals from the continuum is the momentum-bin discretization,[9] which makes use of the so-called eigendifferentials.[10] The method has extensively been used in three-body models of deuteron-induced reactions[11] with very successful results. Because the generalization of the nonlocal case is trivial, we assume that U_{lj} is local in the following. We also assume that the valence nucleon is a neutron. It is straightforward to generalize

TABLE I. Norms of the core plus single-nucleon system as a function of the number of the oscillator quanta Q.

	Q					
	1	2	3	4	5	6
^{4}He$+N$	1.25	0.9375	1.0156	0.9961	1.0010	0.9998
^{16}O$+N$	0	1.1289	0.9878	1.0010	0.9999	1.0000
^{40}Ca$+N$	0	0	1.0769	0.9961	1.0002	1.0000

to the proton case. We solve the equation

$$\hat{h}_{lj}f_{lj}(k,x)\equiv\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}+\frac{\hbar^2}{2\mu}\frac{l(l+1)}{x^2}+U_{lj}\right]f_{lj}(k,x)$$
$$=\frac{\hbar^2k^2}{2\mu}f_{lj}(k,x) \tag{25}$$

with the asymptotic behavior

$$f_{lj}(k,x)\rightarrow\left[\frac{2}{\pi}\right]^{1/2}\sin\left[kx-\frac{l}{2}\pi+\delta_{lj}\right]\quad\text{as}\quad x\rightarrow\infty\ , \tag{26}$$

where δ_{lj} is the scattering phase shift. It is easy to show that $u_{lj}=1/xf_{lj}$ satisfies the normalization

$$\int_0^\infty u_{lj}(k,x)u_{lj}(k',x)x^2dx=\delta(k-k')\ . \tag{27}$$

The momentum space of the valence nucleon is divided into bins. The interval of the ith bin is denoted by $[k_{i-1},k_i]$, and we construct the eigendifferential from the bin

$$\phi_{lj}^{(i)}(x)=\left[\frac{1}{N_i}\right]^{1/2}\int_{k_{i-1}}^{k_i}w_i(k)u_{lj}(k,x)dk \tag{28}$$

with

$$N_i=\int_{k_{i-1}}^{k_i}[w_i(k)]^2dk\ , \tag{29}$$

where $w_i(k)$ is an appropriate weight function. Equation (27) assures that $\phi_{lj}^{(i)}$ belonging to the different bins are orthogonal, i.e.,

$$\langle\phi_{lj}^{(i)}|\phi_{lj}^{(i')}\rangle=\delta_{ii'}\ . \tag{30}$$

Thus $\phi_{lj}^{(i)}$ are square integrable and serve as a basis set of the orbitals for the valence nucleon. Furthermore $\phi_{lj}^{(i)}$ belonging to the different bins have no off-diagonal matrix elements of the single-particle Hamiltonian

$$\langle\phi_{lj}^{(i)}[Y_l(\hat{\mathbf{x}})\times\chi_{1/2}]_{jm}|\hat{h}|\phi_{lj}^{(i')}[Y_l(\hat{\mathbf{x}})\times\chi_{1/2}]_{jm}\rangle=\epsilon_i\delta_{ii'} \tag{31}$$

with

$$\epsilon_i=\frac{1}{N_i}\int_{k_{i-1}}^{k_i}\frac{\hbar^2k^2}{2\mu}[w_i(k)]^2dk\ . \tag{32}$$

When there is no resonance in the ith momentum bin the weight function is chosen as unity, i.e., $w_i(k)=1$, then

$$\epsilon_i=\frac{\hbar^2}{2\mu}\left[\left[\frac{k_i+k_{i-1}}{2}\right]^2+\tfrac{1}{12}(k_i-k_{i-1})^2\right]\ . \tag{33}$$

For the resonant bin the weight function is usually chosen as[12]

$$w(k)=\left|\frac{\frac{i}{2}\Gamma}{\frac{1}{2\mu}\hbar^2k^2-\epsilon_r+\frac{i}{2}\Gamma}\right|\ , \tag{34}$$

where ϵ_r is the resonance energy and Γ its width. The choice of narrower size of bins is in general preferable for making the set of basis states more complete but leads to slower damping of the tail of the basis wave functions.

B. The diagonalization method of discretization

In this method that is sometimes called the pseudostate method,[13] we try to diagonalize the single-particle Hamiltonian in a finite set of square integrable basis functions $\phi^{(1)},\phi^{(2)},\ldots,\phi^{(N)}$ such as harmonic-oscillator functions, Gaussian functions, or local Gaussian functions. The solution of the secular equation

$$\sum_{k=1}^{N}[\langle\phi^{(i)}[Y_l(\hat{\mathbf{x}})\times\chi_{1/2}]_{jm}|\hat{h}|\phi^{(k)}[Y_l(\hat{\mathbf{x}})\times\chi_{1/2,jm}\rangle-\epsilon\langle\phi^{(i)}|\phi^{(k)}\rangle]C_k=0 \tag{35}$$

provides us with an orthogonal basis $\sum_i C_i\phi^{(i)}$. The discretization is achieved through the process of truncating the set of basis functions to the finite number N. The asymptotic form of the solution of Eq. (35) is usually ignored and the eigenvalues with the exception of resonance energies are quite dependent on the choice of the basis functions and N. A comparative study[14] of both methods of discretization has recently been done for the deuteron breakup continuum in deuteron-nucleus collisions.

IV. APPLICATION TO THE He ISOTOPES

We have applied the cluster-orbital shell model to discuss the properties of the He isotopes, ^{6}He-^{8}He assuming that they consist of the ^{4}He core and the valence neutrons. The resonating group method study of the ^{4}He+nucleon system was done by Kanada *et al.*[15] and a local, energy-independent neutron-^{4}He potential was determined in terms of a superposition of Gaussians. We use this local potential to generate the square integrable basis functions.

We have tested the two methods of continuum discretization described in the previous section. In the momentum-bin discretization method, Eq. (25) is solved with the Fox-Goodwin method using the mesh size of 0.05 fm. The k integration in Eq. (28) is done with a trapezoidal rule of the size of $(k_i-k_{i-1})/40$. The $p_{3/2}$ phase-shift calculation indicates a resonance at $\epsilon_r=0.85$ MeV ($k_r=0.18$ fm^{-1}) and the width $\Gamma=0.71$ MeV. These are in good agreement with experimental data of ^{5}He, namely $\epsilon_r=0.89$ MeV and $\Gamma=0.60\pm0.2$ MeV.[16] Although the calculation shows that the partial cross section of the $p_{1/2}$ channel reaches a maximum at $\epsilon=4.57$ MeV, the peak is so broad that we do not consider it a resonance. No other channels show a resonant behavior.

We have discretized the momentum interval of 0.03–1.23 fm^{-1} for the $p_{3/2}$ channel and chosen the bin [0.03,0.43] as the resonant bin. For the interaction acting between the valence neutrons we have first used the Hasegawa-Nagata (HN) No. 1 central potential[17], which has no component in odd states. For three $p_{3/2}$ discretized orbits constructed from the bins [0.03,0.43], [0.43,0.83], and [0.83,1.23], the 0^+ energy of ^{6}He relative to ^{4}He is 0.38 MeV, which is to be compared to the experimental value, −0.976 MeV.[18] If we use a smaller size of bins, $\Delta k = 0.2$ fm^{-1}, in the interval of 0.43–1.23 fm^{-1}, the resulting energy is 0.35 MeV, i.e., there is no significant gain in binding energy. The mixing in of the $p_{1/2}$ and $s_{1/2}$ discretized states constructed from the bin [0.03,0.63] decreases the energy to 0.26 MeV, but the mixing of the $d_{3/2}$ and $d_{5/2}$ channels is negligible. At this moment we have decided to restrict the single-particle orbits to the three lowest $p_{3/2}$ orbits, one $p_{1/2}$ orbit and one $s_{1/2}$ orbit. In the diagonalization method of discretization we have used Gaussian functions

$$\phi_l(a,x)=\left[\frac{2^{l+2}a^{l+3/2}}{(2l+1)!!\pi^{1/2}b^3}\right]^{1/2} \times\left[\frac{x}{b}\right]^l \exp\left[-\frac{a}{2}\left[\frac{x}{b}\right]^2\right], \tag{36}$$

where b is the length scale parameter chosen as 1.8 fm and a is the variational parameter. The discretized orbits are obtained as a superposition of Eq. (36) with different a. When solving Eq. (35) for the $s_{1/2}$ channel we note that there is one $s_{1/2}$ forbidden state occupied by the core nucleons. The occupied $s_{1/2}$ state corresponds to the wave function with $a_0=1.8144$,[15] so that the basis wave function $\phi_0(a,x)$ for the $s_{1/2}$ channel is replaced with

$$\phi_0(a,x)-\langle\phi_0(a_0,x)|\phi_0(a,x)\rangle\phi_0(a_0,x)$$

to make it orthogonal to the forbidden state. Since the semirealistic interaction such as the HN potential may lead to the underbinding of ^{6}He and in addition requires heavy computational effort in evaluating interaction matrix elements for general basis wave functions, we use the delta function interaction, $-v_0\delta(\mathbf{x}_1-\mathbf{x}_2)$, to simplify the calculation in the following.

In the momentum-bin discretization we use the bins [0.03,0.43], [0.43,0.63], and [0.63,0.83] for the $p_{3/2}$ channel, [0.03,0.63] for both the $p_{1/2}$ and $s_{1/2}$ channels. Their single-particle energies are 0.85, 7.37, 13.90, 3.60, and 3.60 MeV. Figure 2 displays the corresponding single-particle wave functions. The lowest $p_{3/2}$ wave function constructed from the resonant bin has most of the component at shorter distance, while the other $p_{3/2}$ and the $p_{1/2}$, $s_{1/2}$ wave functions have considerably longer tails. Actually, to show the slow damping of the wave functions at larger distance, we note that the norm integrals of the five states are 0.98, 0.91, 0.91, 0.96, and 0.98 even when integrated up to 40 fm. In the diagonalization method of discretization we have tested several choices of a and list results obtained with the use of seven a's, $a=0.20$, 0.36, 0.648, 1.168, 2.096, 3.776, and 6.8. The resulting single-particle energies of the three $p_{3/2}$, one

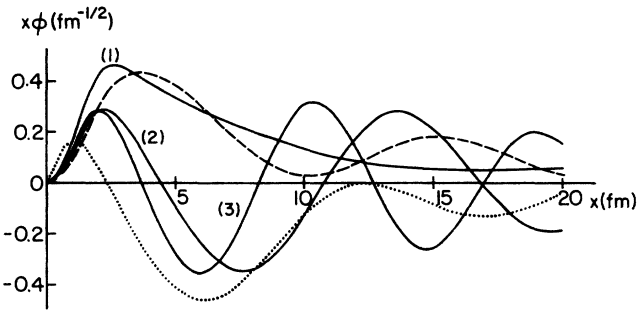

FIG. 2. The single-particle wave functions in the momentum-bin discretization. The three solid curves labeled (1), (2), and (3) denotes the $p_{3/2}$ orbits in the order of increasing single-particle energy, while the dashed and dotted curves denote the $p_{1/2}$ and $s_{1/2}$ orbits, respectively.

$p_{1/2}$, and one $s_{1/2}$ states are 0.98, 4.48, 14.10, 2.31, and 2.20 MeV. Their single-particle wave functions are shown in Fig. 3. Each curve of Fig. 3 seems to correlate with the corresponding curve of Fig. 2, but, of course, damps more rapidly than the latter. Table II lists the energies of ^{6}He ($0^+,2^+$), ^{7}He ($\frac{3}{2}^-$), and ^{8}He(0^+) relative to ^{4}He obtained by diagonalizing the Hamiltonian of Eq. (11) in the space including all the possible states that are constructed from three $p_{3/2}$, one $p_{1/2}$, and one $s_{1/2}$ single-particle orbits. The strength of the delta interaction is determined to fit the 0^+ energy of ^{6}He. The calculated energies of ^{7}He and ^{8}He, though good qualitatively, are deficient by 0.5–1.0 MeV. This lack of energy will at least partly be accounted for if we had two-nucleon interaction such that reproduces the 2^+ energy of ^{6}He as well. Table II also lists the matter root-mean-square (rms) radii that are calculated according to the expression

$$\begin{aligned}(R_{\rm rms})^2&=\frac{1}{n+f}\sum_{i=1}^{n+f}\left[\mathbf{r}_i-\left[\frac{1}{n+f}\right]^{1/2}\mathbf{R}\right]^2\\&=\frac{f}{n+f}[R_{\rm rms}(\text{core})]^2+\frac{1}{n+f}\left[1-\frac{1}{n+f}\right]\sum_{i=1}^{n}\mathbf{x}_i^2\\&\quad-\frac{2}{(n+f)^2}\sum_{j>i=1}^{n}\mathbf{x}_i\cdot\mathbf{x}_j\ .\end{aligned} \tag{37}$$

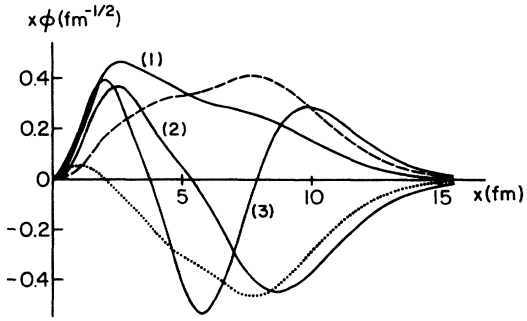

FIG. 3. The single-particle wave functions in the diagonalization method of discretization. See also the caption of Fig. 2.

TABLE II. Energies and root-mean-square radii of the He isotopes calculated with the discretization methods of momentum bin and diagonalization. The strength v_0 of the δ interaction is set to 805 in the k-bin method and 650 MeV fm^3 in the diagonalization method.

	J^π	$E-E(^4\text{He})$ (MeV)			R_{rms} (fm)		
		k bin	Diagonal	Experiment (Ref. 18)	k bin	Diagonal	Experiment (Ref. 2)
^{6}He	0^+	−0.98	−0.99	−0.976	4.78	2.29	2.48±0.03
	2^+	1.20	1.55	0.824			
^{7}He	$\frac{3}{2}^-$	0.00	0.22	−0.535	5.50	2.78	
^{8}He	0^+	−1.78	−1.88	−3.113	5.89	2.60	2.52±0.03

Here

$$[R_{\text{rms}}(\text{core})]^2=\frac{1}{f}\sum_{i=n+1}^{n+f}\left[\mathbf{r}_i-\left[\frac{1}{f}\right]^{1/2}\mathbf{R}_c\right]^2$$

is replaced with the experimental value of $(1.57\,\text{fm})^2$.[2] Apparently, the single-particle orbits obtained with the momentum-bin discretization give too big rms radii to be compared to experiment. On the contrary, the single-particle orbits of the diagonalization method better reproduces the sizes of the He isotopes. The dominant component of the wave functions is of course $(1p_{3/2})^n$ configuration. The percentages of the state in the diagonalization method of discretization are 76, 76, and 57 % for ^{6}He(0^+), ^{7}He($\frac{3}{2}^-$), and ^{8}He(0^+), respectively. The excitation of the valence neutrons to the second lowest $p_{3/2}$ orbit is next most important, and the $p_{1/2}$ and $s_{1/2}$ orbits play a less important role.

Figure 4 compares the neutron density distributions of the He isotopes. For the sake of calculational convenience we have used the density operator that measures the density at points relative to the center of mass of the core

$$\rho_N(\mathbf{x})=\sum_{\text{neutron}}\delta\left[\mathbf{r}_i-\left[\frac{1}{f}\right]^{1/2}\mathbf{R}_c-\mathbf{x}\right]. \tag{38}$$

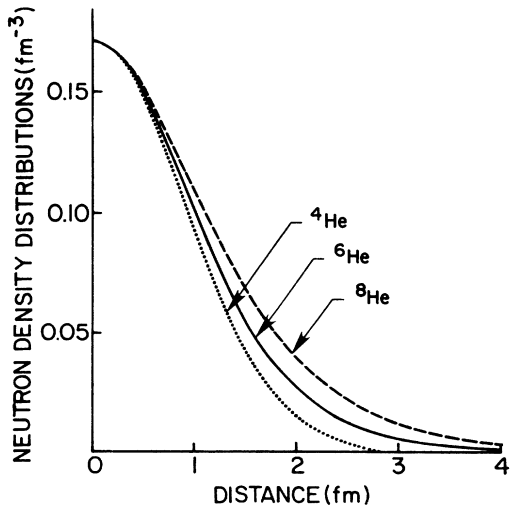

FIG. 4. Neutron density distributions of the He isotopes.

The neutron density distribution of the ^{4}He core is assumed to be given by

$$\rho_N^{(\text{core})}(\mathbf{x})=2\left[\frac{\alpha}{\pi}\right]^{3/2}\exp(-\alpha\mathbf{x}^2), \tag{39}$$

where the value of α is set to 0.609 fm^{-2} to reproduce the rms radius of ^{4}He. Since the proton density distribution of the He isotopes in our model is considered the same as the neutron density distribution of ^{4}He, we note that the asymmetry in the neutron-proton density distributions is enhanced at larger distance as the neutron-proton ratio increases. Recently, it has been suggested[19] that the excess neutrons in neutron-rich nuclei may have a collective motion against the core nucleus and that in particular there is the possibility of the enhancement of the electric dipole strength at lower energy than the normal dipole oscillation energy.

V. SUMMARY

We have formulated the cluster-orbital shell model for a system made up of a core plus several valence nucleons in order to study the structure of exotic nuclei like neutron-rich nuclei. We have arrived essentially at the same form as the conventional shell model but our formalism has the advantage that it is completely free from the spurious center-of-mass motion and has flexibility in employing such single-particle wave functions that are compatible with the underlying potential between the core and the valence nucleon. We would be able to use optical potentials for the potential but instead best construct them by the resonating group method used in microscopic nuclear cluster model studies. Our formalism thus utilizes the essential ingredients of both the shell and cluster models. We have neglected the excitation of the core in this paper. The generalization to the case that includes the core excitation is, however, not hard in principle. In this case we have to treat a coupled channel version of Eq. (18) to define the single-particle potential U, which now depends on the specific channels of the core excitation.

The model has been applied to the simplest example, the He isotopes. One of the most interesting questions here includes how the valence nucleon orbits of ^{6}He and ^{8}He are related to that of ^{5}He, which is unbound against the neutron separation. One new feature is that the available single-particle spectrum is a continuum and special

techniques have to be used to construct square integrable basis functions. The continuum single neutron spectrum outside ^{4}He has been discretized by two methods, the momentum-bin discretization method and the diagonalization method. Although the former seems to be in principle more natural and sound than the latter, we think there remains some problems for the former before it can be used as a practical tool in structure studies. For example, we have to find out a way to have the amplitudes of single-particle wave functions damp at large distance when we use small momentum bins. Although we can explain the empirical fact that ^{7}He is less bound and ^{8}He is more bound than ^{6}He, the binding energies are still too small by 0.5–1.0 MeV. The calculated matter radii of the He isotopes are in good agreement with experiment.

Migdal[20] argued that under certain circumstances two particles outside a well form a bound state, even in the case when their attraction is insufficient for the formation of a bound state outside the well but is sufficient to give rise to a resonance level with energy close to zero. His argument is based on the equation to determine the eigenvalue E and the eigenfunction Ψ.

$$\langle \Psi_\alpha | H' | \Psi \rangle = \int dE_\beta \frac{\langle \Psi_\alpha | H' | \Psi_\beta \rangle}{E - E_\beta} \langle \Psi_\beta | H' | \Psi \rangle \ , \qquad (40)$$

where Ψ_α denotes the noninteracting two-particle state in the state $\alpha=(k_1,k_2)$ and the energy E_α, and H' is the two-particle interactions. Although he tried to solve Eq. (40) approximately by the use of factorized form of scattering wave functions at low energies, the continuum discretization described in this paper should in principle facilitate a convenient basis of square integrable wave functions for solving Eq. (40).

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