Analyzing multiparticle reactions. II. Exactly solvable production models

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A relativistic Lippmann-Schwinger equation for production reactions is developed. To obtain such equations a relativistic generalization of the Schrödinger and Lippmann-Schwinger equations is given, which, when applied to a Fock space appropriate for particles of mass m and zero spin, describes production reactions. Separable potentials are used for the production channels, which then incorporate the correct inelasticity in the elastic channel. Exact solutions to the relativistic Lippmann-Schwinger equations are given for potentials that are separable on all subspaces of the full Fock space.

I. INTRODUCTION

In the preceding paper (hereafter called I), a method was presented for taking model $2\rightarrow n$ partial-wave amplitudes and making them unitary. The idea was to view model $2\rightarrow n$ partial-wave amplitudes as elements of a partial-wave Hilbert space and compute the lengths of the partial-wave amplitudes. These lengths were then modified in such a way as to satisfy exact inelastic unitarity.

While such a procedure is quite general, there is no dynamical input. It is desirable to have a formalism in which production reactions are incorporated into a dynamical set of equations, with potentials generating the production reactions. Strictly speaking, it is not possible to do this in the context of many-body nonrelativistic quantum mechanics, because the mass is superselected in any quantum-mechanical system whose natural (kinematic) invariance group is the Galilei group. While it is possible to have bound clusters break up into subclusters, the number of fundamental particles making up the clusters cannot change in the reaction.

One usually thinks of quantum field theory as providing the correct context in which to describe particle production. However, it is generally difficult to maintain exact unitarity when dealing with perturbative solutions of a quantum field theory. In fact, one of the goals of this and succeeding papers is to show how to take any set of perturbative scattering amplitudes and modify them so as to make them exactly unitary. This is done in the context of a relativistic quantum mechanics, in which the perturbative amplitude for elastic scattering is used to form a separable relativistic potential.

Relativistic quantum mechanics have a long history, starting with the fundamental papers by Dirac[1] and Bakmjian and Thomas.[2] More recently relativistic quantum theories have been applied to a variety of problems in intermediate energy nuclear physics.[3] While it is not difficult to formulate a relativistic quantum mechanics, in the absence of any locality requirements, it is necessary to impose cluster requirements[4] on the potentials to have a sensible theory.

The main goal of this paper is to show how to formulate a relativistic quantum mechanics for production reactions in such a way as to circumvent the cluster requirement; this is done by setting those potentials connecting multiparticle states with multiparticle states having more than two particles to zero. If further the 2 to n potentials are chosen to be separable, the dynamical equations for the (possibly) infinite degrees of freedom are reduced to two (or if desired a finite number of) degrees of freedom. There is no loss in generality in choosing separable 2 to n potentials, as it will be shown that such potentials correspond to the most general multiparticle partial-wave amplitudes. The two-particle dynamical equations have extra terms in them, corresponding to the production channels. If the $2\rightarrow 2$ potentials are also separable, the relativistic equations can be solved exactly; Sec. V discusses these solutions as well as showing how to make separable potentials from perturbative $2\rightarrow 2$ scattering amplitudes.

The way in which a relativistic quantum mechanics is constructed is to translate certain properties of a nonrelativistic many-body Schrödinger equation into a group theoretical language, with the underlying group being the Galilei group. In particular, the free Green's function for the many-body system is shown to be related to a Casimir operator of the Galilei group. After briefly reviewing nonrelativistic quantum mechanics from this point of view in Sec. II, the translation to a relativistic quantum mechanics, with the underlying Hilbert space now a Fock space and the underlying group in the Poincaré group, is given in Sec. III. Thus, tensor products of irreducible representations of the Poincaré group constitute the starting point for the relativistic quantum mechanics, and relativistic wave equations, such as the Klein Gordon or Dirac equations, are not used. The reduction of equations with infinite degrees of freedom to equations with a finite number is given in Sec. IV.

For simplicity, in Secs. III and IV only a model world consisting of spinless particles of mass m called pions is considered. However, once the forms of the solutions of the exactly solvable models are given, the generalization to spinless particles of different mass will be evident.

II. REVIEW OF NONRELATIVISTIC QUANTUM MECHANICS

In this section we consider a simple nonrelativistic system consisting of n spinless particles interacting via a potential denoted by V. The Hilbert space is $L^2(\mathbb{R}^{3n})$ and the action of the Galilei group on this space is inherited from the one-particle action:[5]

$$\begin{aligned}
&(U_{\mathbf{a}}\phi_n)(\mathbf{p}_1,\ldots,\mathbf{p}_n)=\exp(i\textstyle\sum\mathbf{p}_i\cdot\mathbf{a})\phi_n(\mathbf{p}_1,\ldots,\mathbf{p}_n)\ ,\\
&(U_{\mathbf{v}}\phi_n)(\mathbf{p}_1,\ldots,\mathbf{p}_n)=\phi_n(\mathbf{p}_1+m_1\mathbf{v},\ldots,\mathbf{p}_n+m_n\mathbf{v})\ ,\\
&(U_R\phi_n)(\mathbf{p}_1,\ldots,\mathbf{p}_n)=\phi_n(R^{-1}\mathbf{p}_1,\ldots,R^{-1}\mathbf{p}_n)\ ,\\
&(U_t\phi_n)(\mathbf{p}_1,\ldots,\mathbf{p}_n)\\
&\quad=\exp\left[i\left[\frac{p_1^2}{2m_1}+\ldots+\frac{p_n^2}{2m_n}\right]t\right]\phi_n(\mathbf{p}_1,\ldots,\mathbf{p}_n)\ .
\end{aligned}\tag{1}$$

Here $\mathbf{a}$ is a space translation, t a time translation, R a rotation and $\mathbf{v}$ a Galilei boost; taken together these ten transformations give the action of the Galilei group on n-particle wave functions ϕ_n. m_i, $i=1,\ldots,n$, is the mass of the ith particle.

The infinitesimal transformations generated by Eq. (1) give the observables $\mathbf{P}$, the momentum operator, $\mathbf{X}$, the position operator, $\mathbf{L}$, the angular momentum operator, and H, the energy operator for the n noninteracting particles. One of the Casimir operators that can be formed from these generators is $C=H-P^2/2M$ with $M=\sum m_i$; C commutes with the Galilei group action, Eq. (1).

The potential V which produces the interactions among the particles must commute with $U_{\mathbf{a}}$, $U_{\mathbf{v}}$, and U_R, but not C (or U_t) to guarantee Galilei invariance (if V commuted with C there would be no scattering). A simple way to construct potentials that commute with $U_{\mathbf{a}}$, $U_{\mathbf{v}}$, and U_R is to make a change of variables, from $\mathbf{p}_1,\ldots,\mathbf{p}_n$ to Jacobi variables $\mathbf{P},\mathbf{q}_1,\ldots,\mathbf{q}_{n-1}$, with $\mathbf{P}=\sum\mathbf{p}_i$. With these new variables the transformation properties become

$$\begin{aligned}
&(U_{\mathbf{a}}\phi_n)(\mathbf{P},\mathbf{q}_i)=\exp(i\mathbf{P}\cdot\mathbf{a})\phi_n(\mathbf{P},\mathbf{q}_i)\ ,\\
&(U_{\mathbf{v}}\phi_n)(\mathbf{P},\mathbf{q}_i)=\phi_n(\mathbf{P}+M\mathbf{v},\mathbf{q}_i)\ ,\\
&(U_R\phi_n)(\mathbf{P},\mathbf{q}_i)=\phi_n(R^{-1}\mathbf{P},R^{-1}\mathbf{q}_i)\ ,\\
&(U_t\phi_n)(\mathbf{P},\mathbf{q}_i)=\exp\left[i\left[\frac{P^2}{2M}+\sum\frac{q_i^2}{2\mu_i}\right]t\right]\phi_n(\mathbf{P},\mathbf{q}_i)\ ,
\end{aligned}\tag{2}$$

where μ_i are reduced masses. From these transformations it is clear that if the kernel of V does not depend on $\mathbf{P}$, V will commute with $U_{\mathbf{a}}$ and $U_{\mathbf{v}}$. Furthermore, with these new variables the Casimir operator action is

$$\begin{aligned}
(C\phi_n)(\mathbf{P},\mathbf{q}_i&=\left[H-\frac{P^2}{2M}\right]\phi_n(\mathbf{P},\mathbf{q}_i)\\
&=\sum\frac{q_i^2}{2\mu_i}\phi_n(\mathbf{P},\mathbf{q}_i)\ ,
\end{aligned}\tag{3}$$

which is exactly the relative kinetic energy. Thus, the free Green's function can be written as

$$G_0(z)=\frac{1}{z-C},\quad z\text{ complex}\ .\tag{4}$$

There are a number of different ways in which V can be made to commute with U_R. If the kernel of V depends only on scalar products of the $\mathbf{q}_i$, then V will commute with U_R. A more systematic way to construct rotationally invariant potentials is to make use of the fact that U_R acts on ϕ_n [Eq. (2)] in such a way as to rigidly rotate all the $\mathbf{q}_i$. New variables can be chosen which rotate the $\mathbf{q}_i$ as a rigid body. The rotation variables which rotate the rigid body, when suitably transformed, become the angular momentum and spin projection variables. Such a change of variables corresponds to decomposing the Hilbert space $L^2(\mathbb{R}^{3n})$, viewed as carrying a reducible representation of the Galilei group, into a direct sum and integral of irreducible representations. (For details of this decomposition, including the case when the particles have spin, see Ref. 5.)

If a potential V is given which commutes with $U_{\mathbf{a}}$, $U_{\mathbf{v}}$, and U_R, the Schrödinger equation becomes

$$(C+V)\psi=E\psi,\quad \psi\in\mathcal{H}\ ,\tag{5}$$

while the Lippmann-Schwinger equation is

$$\psi=\phi+G_0V\psi,\quad G_0^{-1}(E)\phi=0\ .\tag{6}$$

A simple instance when the Lippmann-Schwinger equation can be solved exactly is for separable potentials which are written as

$$V=\sum_i u_i\otimes u_i^\dagger, u_i\in\mathcal{H}\ .$$

In order that V commute with $U_{\mathbf{a}}$, $U_{\mathbf{v}}$, and U_R, it is necessary that the u_i be invariant with respect to these transformations. With a separable potential the Lippmann-Schwinger equation becomes

$$\psi^+=\phi+G_0(E+)\left[\sum_i u_i\otimes u_i^\dagger\right]\psi^+\ ,$$

which has a solution $\psi^+=\phi+\sum G_0u_iA_i$, where $A_i=(u_i,\psi^+)$ satisfies a matrix equation which can be inverted to give $A_i=\sum_j M_{ij}^{-1}(E)B_j$, $B_j=(u_j,\phi)$. The transition matrix is then

$$\begin{aligned}
T_{fin}&=(\phi_f,V\psi_{in}^+)\\
&=\sum_j(\phi_f,u_j)(u_j,\psi_{in}^+)\\
&=\sum_j(\phi_f,u_j)A_j\\
&=\sum_{j,k}(\phi_f,u_j)M_{jk}^{-1}(E)(u_k,\phi_{in})
\end{aligned}$$

and it is exactly this type of solution which will be of interest in the relativistic case, when the Lippmann-Schwinger equation is a relativistic Lippmann-Schwinger equation, the Hilbert space is a many-particle Fock space, the kinematical group is the Poincaré group, and V is a separable potential that induces transitions in the many-particle Fock space.

III. RELATIVISTIC QUANTUM MECHANICS AND PRODUCTION PROCESSES

We consider in this section a model world consisting only of spinless, uncharged particles of mass m called pions. The wave functions for such particles transform as irreducible representations of the Poincaré group labeled by mass m and zero spin and are elements of the one-particle Hilbert space $L^2(\mathsf{R}^3)$:

$$(U_{a,\Lambda}\phi)(p)=e^{ip\cdot a}\phi(\Lambda^{-1}p),\quad \phi\in L^2(\mathsf{R}^3)\ , \tag{7}$$

where a is a four translation, and Λ a Lorentz transformation. $p\cdot a$ is the Lorentz invariant inner product given by $p\cdot a=Ea_0-\mathbf{p}\cdot\mathbf{a}$, with $a=(a_0,\mathbf{a})$. Actually ϕ depends only on the three-momentum $\mathbf{p}$, since $E=+(m^2+\mathbf{p}^2)^{1/2}$.

For production reactions, the appropriate Hilbert space is the symmetric Fock space $\mathcal{S}(L^2(\mathsf{R}^3))$, defined by

$$\mathcal{S}(L^2(\mathsf{R}^3))=\sum_{n=0}^{\infty}\oplus[L^2(\mathsf{R}^3)\otimes\ldots\otimes L^2(\mathsf{R}^3)]_n^{\mathrm{sym}}\ , \tag{8}$$

where $[\ ]_n^{\mathrm{sym}}$ is an n-fold symmetrized tensor product of $L^2(\mathsf{R}^3)$. A nonnormalizable basis in the n-particle subspace is given by n creation operators $a^\dagger(p)$ acting on the vacuum state:

$$|p_1,\ldots,p_n\rangle=a^\dagger(p_1)\ldots a^\dagger(p_n)|0\rangle\ , \tag{9}$$

where $|0\rangle$ corresponds to the one-dimensional $n=0$ subspace of $\mathcal{S}(L^2(\mathsf{R}^3))$. The transformation properties of n-particle wave functions and states are given by

$$\begin{aligned}(U_{a,\Lambda}\phi_n)(p_1,\ldots,p_n)&\\ =\exp\left[i\sum_{j=1}^{n}p_j\cdot a\right]&\phi_n(\Lambda^{-1}p_1,\ldots,\Lambda^{-1}p_n)\\ U_{a,\Lambda}|p_1,\ldots,p_n\rangle=\exp\left[i\Lambda\sum_{j=1}^{n}p_j\cdot a\right]&|\Lambda p_1,\ldots,\Lambda p_n\rangle\ .\end{aligned} \tag{10}$$

As in the nonrelativistic case the infinitesimal generators of the transformations (10) give observables such as the momentum and angular momentum operators; in particular one of the Casimir operators is the mass operator,

$$\begin{aligned}&M^2=P_0^2-\mathbf{P}\cdot\mathbf{P}\ ,\\ &M^2\phi_n=(p_1+\ldots+p_n)^2\phi_n\ .\end{aligned} \tag{11}$$

Interactions in the many-particle Fock space will be governed by a "potential" V that connects the different n-particle subspaces. To construct potentials that commute with U_a, it is simplest to introduce a boost $B(p)$, a Lorentz transformation whose inverse carries all the n-particle momentum vectors p_i to the center of mass frame, where the momenta are p_i^*, satisfying $\sum\mathbf{p}_i^*=\mathbf{0}$. A boost is defined by

$$\begin{aligned}&p=B(p)\begin{pmatrix}\sqrt{s}\\ \mathbf{0}\end{pmatrix},\quad s=(p_1+\ldots+p_n)^2\\ &p_i^*=B^{-1}(p)p_i,\quad \mathbf{p}=\sum\mathbf{p}_i\ .\end{aligned} \tag{12}$$

The action of the Poincaré group on the new variables p and p_i^* is given by

$$(U_{a,\Lambda}\phi_n)(p,p_i^*)=e^{ip\cdot a}\phi_n(\Lambda^{-1}p,R_W^{-1}\mathbf{p}_i^*)\ , \tag{13}$$

where R_W is a Wigner rotation,

$$R_W=B^{-1}(p)\Lambda B(\Lambda^{-1}p)\ .$$

As in the nonrelativistic case, if the kernel of the potential V is independent of $\mathbf{p}$, then V will commute with U_a. From the transformation properties of ϕ_n under a Lorentz transformation Λ, it is clear that all the $\mathbf{p}_i^*$ are rotated as a rigid body by R_w. This again suggests making a change of variables to a body fixed frame of reference. The rotation which transforms between the body fixed frame and a space fixed frame, where the momentum vectors are $\mathbf{p}_i^*$, $\sum_i\mathbf{p}_i^*=\mathbf{0}$ can itself be transformed to give the angular momentum j and angular momentum projections along the z axis of the body fixed frame (r) and space fixed frame (σ). Such variables are the natural variables to use when decomposing the n-particle subspace of the Fock space into irreducible representations. Details of these transformations (including particles with spin) are given in Ref. 6.

An n-particle wave function is now written as $\phi_n(\mathbf{p}\sigma sjy_n)$, where the y_n are a set of variables describing the internal configurations of the $n-$particle system. With these variables the Poincaré transformations are

$$(U_{a,\Lambda}\phi_n)(\mathbf{p}\sigma sjy_n)=e^{ip\cdot a}\sum_{\sigma'}D_{\sigma\sigma'}(R_w)\phi_n(\Lambda^{-1}p,\sigma' sjy_n)\ . \tag{14}$$

To ensure relativistic invariance the kernels of potentials acting on elements of $\mathcal{S}$ should depend only on the variables s, j, and y_n. Assuming, as is generally done, that the potentials are diagonal in the angular momentum j, the kernel of a potential connecting an n to n' particle subspace is written as

$$\begin{aligned}(V^{n\to n'}\phi_n)(\mathbf{p}\sigma sjy_{n'})&\\ =\int_{nm}^{\infty}d\sqrt{s'}\int d\mu(y_n')V_j(sy_{n'},s'y_n')&\phi_n(\mathbf{p}\sigma' sjy_n')\ .\end{aligned} \tag{15}$$

Using Eqs. (14) and (15) it is easy to see that $[U_a,V]=0$, but in general V will not commute with U_{a_0} or U_Λ because the s dependence of the four-vector $p=((\mathbf{p}^2+s)^{1/2},\mathbf{p})$.

Let u_n be an element of the n-particle subspace of $\mathcal{S}$ that does not depend on $\mathbf{p}$ or σ; then from Eq. (14) it is seen that u_n will be invariant under the action of $U_{a,\Lambda}$ (that is $U_{a,\Lambda}u_n=u_n$). This suggests defining new "partial-wave" Hilbert spaces $\mathcal{H}^{sj}$, in which elements u_n not depending on $\mathbf{p}$ or σ have the norm

$$\|u_n\|^2(sj)=\int d\mu(y_n)|u_n(sjy_n)|^2\ , \tag{16}$$

which is the same norm as for partial-wave amplitudes given in I; the measure $d\mu(y_n)$ is given in Eq. (A8) in the Appendix of I. Since the variables $\mathbf{p},\sigma$ only describe the n-particle system as a whole, they will be dropped and the norm of an element in the reduced Fock space becomes

$$\|u_n\|^2_{\text{reduced}} = \sum_{n=2}^{\infty}\sum_{j=0}^{\infty}\frac{2j+1}{8\pi^2}\int_{nm}^{\infty} d\sqrt{s_n}\,\|u_n\|^2(s_n,j) < \infty \ . \quad (17)$$

In this reduced Fock space V, as given by Eq. (15), will automatically commute with $U_{a,\Lambda}$ (since the action $U_{a,\Lambda}$ on u_n is trivial) but not with M^2. Then a relativistic (time-independent) Schrödinger equation can be written as

$$(M+V)\psi=\sqrt{s}\,\psi \ , \quad \psi\in\mathcal{S}_{\text{reduced}} \quad (18)$$

and the corresponding relativistic Lippmann-Schwinger equation is given by

$$\psi=\phi+G_0(s)V\psi \ , \quad G_0^{-1}(s)\phi=0 \ , \quad (19)$$

where

$$G_0=\begin{bmatrix} G_2 & & \\ & G_3 & \\ & & \ddots \end{bmatrix} ,$$

and on each n-particle subspace,

$$G_n(s)=1/(s-M^2) \ .$$

Note there is no one-particle Green's function, and the sum in (15) begins at the two-particle subspace.

It is important to see how the potential V acting on $\mathcal{S}$ differs from the nonrelativistic case. Depending on the choice of V, there will be subspaces of $L^2(\mathbb{R}^{3n})$, the nonrelativistic Hilbert space, which correspond to bound subsystems of the n-particle system; in particular there may be n-particle bound states which are orthogonal to the scattering sectors. For each V these bound subsystems must be investigated separately.

Implicit in our formulation of relativistic quantum mechanics is the idea that there is only one "bound state," namely the one-particle sector of the Fock space corresponding to a physical particle of mass m and zero spin and charge. Only those potentials which generate this one stable particle are allowed in the Lippmann-Schwinger equation. This constraint on the potential is most easily satisfied by demanding that the only pole of the $2\to 2$ partial-wave amplitude be that corresponding to a particle of mass m and zero spin. It will be obvious how such a constraint can be satisfied in the simple solvable model discussed in Sec. V.

IV. RELATIVISTIC QUANTUM MECHANICS AND PRODUCTION REACTIONS

To solve the Lippmann-Schwinger equation for production reactions, potentials must be given satisfying Eq. (15) which connect different particle number subspaces. If these potentials connect arbitrarily large particle number subspaces, the Lippmann-Schwinger equations will describe an infinite degree of freedom system. However, if the potentials are zero beyond a certain particle number, the system will have finite degrees of freedom.

The main result of this section will be to show that it is always possible to describe production reactions by separable potentials, in such a way that the (potentially) infinite degrees of freedom Lippmann-Schwinger equations reduce to equations with finite degrees of freedom, with all the inelastic thresholds correctly built into the equations.

For simplicity we consider two particles interacting via the potential $V^{2\to 2}$, which connects the two-particle to two-particle subspace; then a potential V which connects the entire Fock space can be written as

$$V=\begin{bmatrix} V^{2\to 2} & u_2\otimes u_3^\dagger \ , & u_2\otimes u_4^\dagger \ , & \cdots \\ u_3\otimes u_2^\dagger & & & \\ u_4\otimes u_2^\dagger & & 0 & \\ \vdots & & & \end{bmatrix} . \quad (20)$$

The u_k, $k=2,3,\ldots$ are elements of the k particle reduced Fock space. The Lippmann-Schwinger equations for such a potential are given by

$$\begin{aligned} &\psi_2=\phi_2+G_2\left[V^{2\to 2}\psi_2+u_2\sum_{k=3}^{\infty}(u_k,\psi_k)\right] , \\ &\psi_k=G_k u_k(u_2,\psi_2) \ , \quad k=3,4,\ldots ; \end{aligned} \quad (21)$$

the ϕ_k, $k=3,4,\ldots$ are assumed to be zero, since it is not possible (at present) to produce three or more particle initial scattering states. Substituting ψ_k into ψ_2 gives

$$\begin{aligned} \psi_2&=\phi_2+G_2\left[V^{2\to 2}\psi_2+u_2\sum_{k=3}^{\infty}(u_k,G_k u_k)(u_2,\psi_2)\right] \\ &=\phi_2+G_2\left[V^{2\to 2}+u_2\otimes u_2^\dagger\sum_{k=3}^{\infty}g_k\right]\psi_2 \ , \end{aligned} \quad (22)$$

where

$$g_k\equiv(u_k,G_k u_k)=\int_{nm}^{\infty} d\sqrt{s}\,\frac{\|u_k\|^2(sj)}{s_0-s+i\epsilon} \ .$$

If $u_2(sj)=0$, Eq. (22) reduces to a Lippmann-Schwinger equation for ψ_2, driven by the potential $V^{2\to 2}$. However, if $u_2(sj)\neq 0$, then each term $u_2\otimes u_k^\dagger$ will generate a new channel at $\sqrt{s}=km$. The term $u_2\otimes u_2^\dagger\sum_{k=3}^{\infty}g_k$ in Eq. (22) represents the inelasticity, whereby flux from the initial two-particle system is flowing into the various inelastic channels.

Given Eq. (22), the various amplitudes are

$$\begin{aligned} A^{2\to 2}&=(\phi_2^f,V\psi^{in}) \\ &=(\phi_2^f,V^{2\to 2}\psi_2^{in})+\sum_{k=3}^{\infty}(\phi_2^f,u_2)(u_k,\psi_k^{in}) \\ &=(\phi_2^f,V^{2\to 2}\psi_2^{in})+\sum_{k=3}^{\infty}(\phi_2^f,u_2)g_k(u_2,\psi_2^{in}) \\ A^{2\to n}&=\sum_k(\phi_n^f,V\psi_2^{in}) \\ &=(\phi_n^f,u_k)(u_2,\psi_2^{in}) \ . \end{aligned} \quad (23)$$

Since (u_2,ψ_2^{in}) depends only on s and j and u_n is an arbi-

trary element of the reduced n-particle subspace and has the variables associated with partial-wave amplitudes discussed in I, any model partial-wave amplitude can be used for u_k and the inelasticity will automatically be put into the $2\to 2$ amplitude via the term

$$\sum_{k=3}^{\infty}(\phi_2^f,u_2)g_k(u_2,\psi_2^{in}) .$$

Thus, from a phenomenological point of view, to incorporate inelastic channels there is no loss in generality in assuming that $V^{n\to n'}=0$ for $n,n'>2$; this means that production reactions can always be included in any relativistic Lippmann-Schwinger equation. In the final section we assume that $V^{2\to 2}$ is also separable, and show that under such circumstances, the Lippmann-Schwinger equation can be solved exactly.

V. EXACTLY SOLVABLE PRODUCTION MODELS

The $V^{2\to 2}$ potential is needed as input to solve the relativistic Lippmann-Schwinger equation; there is a simple way to make $V^{2\to 2}$ separable, using perturbative elastic scattering amplitudes. Let $\mathcal{A}(sj,r_1r_2,r_1'r_2')$ be the partial-wave amplitude of any perturbative elastic scattering amplitude. For particles with spin, r_1 and r_2 (r_1' and r_2') are the invariant spin projections of the initial (final) system, and vary between $-j_1$ ($-j_2$) and j_1 (j_2), respectively.

Now choose the kernel of the $V^{2\to 2}$ potential to be

$$V_j^{2\to 2}(s,r_1r_2;s',r_1'r_2')=\sum_{r_1''r_2''}\mathcal{A}(sj,r_1r_2,r_1''r_2'')\times\mathcal{A}^*(s'j,r_1'r_2',r_1''r_2'')=\sum_i v_i\otimes v_i^\dagger ,\quad v_i\in\mathcal{H}_2^{(sj)} ; \tag{24}$$

then Eq. (22) becomes

$$\psi_2=\phi_2+G_2\left[\sum_i v_i(v_i,\psi_2)+\sum_{k=3}^{\infty}u_2(u_k,\psi_k)\right]$$
$$=\phi_2+\sum_i G_2v_i(v_i,\psi_2)+G_2u_2\sum g_k(u_2,\psi_2) ,$$
$$(u_2,\psi_2)=(u_2,\phi_2)+\sum_i g_{2i}(v_i,\psi_2)+g_2\sum g_k(u_2,\psi_2) , \tag{25}$$
$$(v_j,\psi_2)=(v_j,\phi_2)+\sum_i g_{ji}(v_i,\psi_2)+g_{j2}\sum g_k(u_2,\psi_2) ,$$

with $g_{ji}=(v_j,G_2v_i)$, $g_{2i}=(u_2,G_2v_i)$, $g_{j2}=(v_j,G_2u_2)$, and $g_2=(u_2,G_2u_2)$, as before. It is possible to solve for (u_2,ψ_2) and (v_j,ψ_2) and insert these solutions into the equation for ψ_2 [Eq. (25)], which will then give an exact solution.

For the pion Fock space of Sec. III, $V^{2\to 2}$ will be of the form $v_1\otimes v_1^\dagger$, with only one term in the sum, Eq. (24); then

$$(u_2,\psi_2)=\frac{1}{D}[(1-g_{11})(u_2,\phi_2)-g_{21}(v_1,\phi_2)] ,$$
$$(v_1,\psi_2)=\frac{1}{D}\left[\left[1-g_2\sum g_k\right](v_1,\phi_2)-g_{12}\sum g_k(u_2,\phi_2)\right] , \tag{26}$$

where

$$D(s,j)=1-(g_{11}g_2+g_ng_{21})\sum g_k .$$

Substituting into Eq. (23) then gives an explicit solution for the $2\to 2$ and $2\to n>2$ scattering amplitudes.

It should be noted that there are other ways of choosing V so as to be able to get an exact solution to the Lippmann-Schwinger equations. For example, choose V' so that

$$V'=\begin{bmatrix} 0 & u_2\otimes u_3^\dagger & 0 & \cdots & \\ u_3\otimes u_2^\dagger & 0 & u_3\otimes u_4^\dagger & 0 & \\ 0 & u_4\otimes u_3^\dagger & 0 & & \ddots \\ \ddots & & & \ddots & \end{bmatrix} . \tag{27}$$

Though it is somewhat more complicated to solve for ψ_k, $k=2,3,\ldots$, the general form of the scattering amplitude will again be of the form

$$A^{2\to n}=(\phi_n^f,u_n)f_{n2}(s,j)(u_2,\phi_2^{in}) ; \tag{28}$$

in fact, for all separable potentials associated with infinite degree of freedom systems, the general form of the production amplitudes is given by Eq. (28), where $f_{n2}(sj)$ depends on the form of the separable potential and guarantees that the amplitudes will be properly unitary.

VI. CONCLUSION

We have shown how to write a Lippmann-Schwinger equation that is properly relativistic and includes particle production. For the example discussed in Sec. IV, where two particles interact via a two-body relativistic potential $V^{2\to 2}$, adding separable potentials of the form $u_2\otimes u_k^\dagger$ incorporates production channels for as many number of particles as desired. Further there is no loss in generality in assuming the production potentials to be of the form $u_2\otimes u_k^\dagger$. For let the $2\to n$ partial-wave amplitude be given as $\mathcal{A}^{2\to n}$. From Eq. (23) the $2\to n$ partial-wave amplitude is $u_n(u_2,\psi_n^{in})$. But (u_2,ψ_2^{in}) depends only on s and j, so that if u_n is chosen to be $\mathcal{A}^{2\to n}(u_2,\psi_2^{in})^{-1}$, then the $2\to n$ partial-wave amplitude of Eq. (23) will be $\mathcal{A}^{2\to n}$ and give all the correct thresholds in the $2\to 2$ amplitude. Using multiparticle data to get model $2\to n$ partial-wave amplitudes, as described in I, should then make it possible to extend $2\to 2$ phase-shift analyses beyond production thresholds.

Section V showed how to get exactly solvable solutions from separable potentials. Of particular interest here is the pion-nucleon system, which is dominated by the Δ resonance in the $j=\frac{3}{2}$ channel. It is straightforward to construct separable potentials $V^{\pi N\to\pi N}$, $V^{\pi N\to\pi\pi N}$, by writing

$$V^{2\to 2}=f^{\pi N\Delta}(sr_N)f^{\pi N\Delta^*}(s'r_N') ,$$
$$V^{2\to 3}=f^{\pi N\Delta}(s_{\pi N}r_N)g^{\pi\Delta\Delta}(sr_\Delta)f^{\pi N\Delta^*}(s'r_N')$$

and use the solution, Eq. (26). $f^{\pi N\Delta}$ and $g^{\pi\Delta\Delta}$ are vertex functions in partial-wave variables. Results of such a procedure will be reported in a succeeding paper. More

generally what has been shown is how to take any perturbative amplitudes for the elastic and inelastic channels and use them as input for the relativistic Lippmann-Schwinger equation. The solutions to these equations are unitary at all energies and have the correct threshold behavior.

It should be noted that the models described here can not be used for initial states of more than two particles, for the partial-wave amplitudes for $n' \to n$ reactions with $n, n' > 2$ will not have the correct cluster properties, whereby, when subsystems of particles are moved far away from one another, the amplitudes reduce to products of subsystem amplitudes. However, this is no problem for $2 \to n$ reactions, which are the only reactions for which there is experimental data; moreover, since the interactions are purely phenomenological, the separable potentials are not meant to be approximations to more fundamental strong interactions.

As is well known there is an intimate relationship between poles of $2 \to 2$ scattering amplitudes and bound states. For the model discussed in Sec. V, the only stable particle is the pion itself, with a mass given by m. The zeros of the denominator of Eq. (26) determine the bound states and so a (weak) constraint on the partial-wave elements u_n is that they give a zero at the mass m of the pi meson. If there are other stable particles such as the nucleon, then there must be zeroes at the masses of these particles also. Such a point of view is to be contrasted with the usual situation in quantum mechanics where the bound-state problem must be solved first before the scattering problem can be addressed.

One of the drawbacks to using the simple multiparticle partial-wave amplitudes given in Eq. (23) is that the elements u_n which form the multiparticle partial-wave amplitudes must be chosen so that the matrix elements of the Green's functions are well defined and can be calculated analytically. Such a requirement is not needed for the partial-wave amplitudes discussed in I. Since only single-particle distribution and multiplicity data are, for the most part, available, this suggests choosing u_n elements that are explicit functions of $\sqrt{s}$, j, and $\sqrt{s_x}$, the invariant mass and angular momentum of the overall system, and the invariant mass of the undetected cluster X, only. The various multiparticle systems are then distinguished by the range of $\sqrt{s_x}$ which can vary from the rest mass of the X cluster to $\sqrt{s} - m_c$, where m_c is the mass of the detected particle. With this simplification it is not difficult to construct models for which the Green's function matrix elements can be computed analytically.

As discussed in I this simplification of the u_n elements, in which they depend on three variables only, means thinking of the multiparticle reactions as two-body-like reactions in which $a + b \to c + X$, where c is the detected particle and X is the cluster of undetected particles. To use such a point of view to fit data, it is also necessary to include relativistic spin and internal symmetries such as isospin. From the way in which the partial-wave spaces were defined, it is clear that these extra degrees of freedom can readily be included. Then the cluster X will be described by its mass $\sqrt{s_x}$, its "spin" and spin projection j_x and m_x, where j_x can range between 0 and ∞, and its isospin I_x. Choosing the partial-wave elements to depend on such variables should make it relatively straightforward to compute Green's function matrix elements and use the phenomenological elements to fit data.

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[1]P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).
[2]B. Bakmjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953).
[3]See, for example, P. L. Chung, F. Coester, B. D. Keister, and W. N. Polyzou, Phys. Rev. C **37**, 2000 (1988).
[4]F. Coester and W. N. Polyzou, Phys. Rev. D **26**, 1348 (1982); S. N. Sokolov, Teor. Mat. Fiz. **23**, 355 (1975); for a review of cluster properties in relativistic quantum mechanics, see F. M. Lev, Fortschr. Phys. **31**, 75 (1983).
[5]R. E. Warren and W. H. Klink, J. Math. Phys. **11**, 1155 (1970).
[6]W. H. Klink, Phys. Rev. D **4**, 2260 (1971).