Analyzing multiparticle reactions. I. Unitarizing perturbative amplitudes

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In order to analyze multiparticle reactions in which a small number of particles are created, it is necessary to ensure that model multiparticle amplitudes are properly unitarized. Multiparticle partial-wave amplitudes can be thought of as elements in a multiparticle Hilbert space; it is shown how to modify the lengths of model partial-wave amplitudes so that they will satisfy inelastic unitarity conditions.

I. INTRODUCTION

It has long been desirable to have a generalization of a phase-shift analysis for multiparticle reactions. A phase-shift analysis is basically a method for approximating the partial-wave amplitude of a $2\rightarrow 2$ reaction from experimental data. The goal of a phase-shift analysis is to present experimental data in terms of the phase shifts and inelasticity parameter. These quantities are functions of relativistic invariants only, namely the invariant energy and angular momentum.

It is well known that there are a number of ambiguities involved in extracting the phase-shift and inelasticity parameter from $2\rightarrow 2$ reaction data.[1] The situation becomes much more complicated when dealing with production reactions; in contrast to $2\rightarrow 2$ reactions, whose amplitudes require only two variables, scattering amplitudes describing $2\rightarrow n$ production reactions, where there are $n>2$ particles in the final state, require $3n-4$ variables for a complete description of the reaction. However, experimentalists cannot hope to carry out a complete analysis of a $2\rightarrow n$ reaction, as this would require complicated correlation experiments over large regions of phase space.

Nevertheless, there are considerable data available on multiparticle reactions, generally in the form of multiplicity data or single-particle distribution data.[2] Such data are the simplest generalization of $2\rightarrow 2$ reaction data; it is the goal of this and the following paper to present methods for analyzing such data. The motivation for this analysis is to analyze the pi-nucleon and K-nucleon systems, but the formalism to be presented in these two papers will not be specifically tied to these two systems, but will rather present a general formalism, valid for any production reactions.

Though there are considerable production data available, there seem to be no models that are able to fit these data. Up to about fifteen years ago there was considerable experimental and theoretical effort devoted to trying to fit and understand multiparticle reaction data; for example, the book of Perl[2] devotes several chapters to these topics. However, from the point of view of quantum chromodynamics (QCD), the kinematic region involving the production of a few particles (say, ten or less) is expected to be very complicated, so interest in trying to describe and understand low-multiplicity data has waned.

One of the reasons that it is difficult to use model amplitudes to fit production data, particularly those amplitudes coming from a perturbative field theory, is that such amplitudes are not unitary. Yet a particularly striking feature of multiparticle data is the way in which channels open, contribute significantly to the cross section, and then die out as the beam energy is increased. In dealing with multiparticle data, it is very important that model amplitudes be unitary. One of the goals of this paper is to show how model production scattering amplitudes can be made unitary. Manifestations of unitarity are particularly evident in total cross section data, where the total number of particles [both charged and (hopefully) uncharged] in a reaction are detected, but there is no knowledge of the particle's direction or energy. Plotted as a function of beam energy, one can see the cross sections of the various channels rising and falling in such a way as to satisfy unitarity.

Total cross section data typically come from bubble chambers. The other type of data that will be of interest in this paper is single-particle distribution data, in which one outgoing particle is detected, so that its direction and momentum are known. Since all the other particles are undetected, such a reaction is typically written as $a+b\rightarrow c+X$, where c is the detected particle and X denotes all the undetected particles. Experiments in which one particle is detected are the simplest generalizations of $2\rightarrow 2$ reactions, in that the variables needed to describe the kinematic configurations of the particles are the beam energy, which can be chosen as the invariant mass, $\sqrt{s}$, of the system $[s=(p_A+p_B)^2]$, the laboratory scattering angle θ_{lab} between the beam direction and the outgoing direction of particle c, and the relativistic energy (or momentum) in the laboratory, E_{lab} of c.

It is a straightforward kinematic exercise to show that these variables can be transformed to more natural center-of-mass variables. These variables have often been chosen to be the longitudinal and perpendicular momentum components of c in the center of mass; the distribution functions are then called Peyrou plots.[3] However, for the purposes of unitarizing model amplitudes, it is more useful to choose as variables the center-of-mass scattering angle θ^*, and the invariant mass $\sqrt{s_x}$, of the undetected particles $[s_x=(\sum_{i\epsilon x}p_i)^2$, where p_i is the four momentum of the ith undetected particle].

To keep the enumeration of channels as simple and general as possible, we will write $a+b\rightarrow c+X$ as $2\rightarrow n$, and specifically think of particle c as the first of the n particles while X contains the remaining $n-1$ particles. As different channels open, they are enumerated by n. For example, in the pi-nucleon system $2\rightarrow 2$ means $\pi N\rightarrow \pi N$, $2\rightarrow 3$ means $\pi N\rightarrow \pi\pi N$, $2\rightarrow 4$ means $\pi N\rightarrow \pi\pi\pi N$, and so forth. Thus, for a given reaction $a+b\rightarrow c+X$ (or $2\rightarrow n$) there is a distribution function $D_n(s,\theta^*,s_x)$; usually, however, the undetected particles do not come from one channel. Then it is necessary to sum over all possible open channels to get the distribution function measured by experimentalists, namely

$$D(s,\theta^*,s_x)=\sum_n D_n(s,\theta^*,s_x)\ .$$

If $D_n(s,\theta^*,s_x)$ were known, the multiplicity would be given by integrating over θ^* and s_x. Though only $D(s,\theta^*,s_x)$ is generally measured, by also using multiplicity data, in which the open channels are known, inferences can be made about $D_n(s,\theta^*,s_x)$.

The goal of this paper is to show how to take model multiparticle amplitudes (often arising from perturbative quantum field theory) and unitarize them so they can be used to try to fit multiplicity and single-particle distribution data. The reactions that motivate this work are the pi-nucleon and K-nucleon systems, primarily the pi-nucleon system in which one has reactions of the type $\pi N\rightarrow N+(n-1)\pi$. Section II gives the general analysis while Sec. III works out a simple example of a multiparticle amplitude, and then discusses other model amplitudes more suitable for fitting data.

II. MULTIPARTICLE PARTIAL-WAVE AMPLITUDES

As stated in the Introduction the reactions of interest in this paper are of the form $a+b\rightarrow c+X$, where c is the detected particle and X denotes the cluster of undetected particles. The scattering amplitude for such a $2\rightarrow n$ reaction has $3n-4$ independent variables, three of which are chosen as $s=(p_a+p_b)^2$, the angle θ^* between the detected particle and the beam direction in the center-of-mass system (the notation of Ref. 2 is used, in which center-of-mass quantities are denoted by asterisks), and $\sqrt{s_x}$, the invariant mass of the undetected particles comprising the X cluster. The transform of θ^* gives the angular momentum, so as discussed in the Appendix, a $2\rightarrow n$ partial-wave amplitude can be written as $\mathcal{A}^{2\rightarrow n}(sjy_n)$, where y_n denotes a set of variables including s_x which describe the final-state configuration.

If the $2\rightarrow n$ partial-wave amplitude were known, the total inelastic n-particle cross section would be given by

$$\sigma_n(s)=\sum_j \int d\mu(y_n)|\mathcal{A}^{2\rightarrow n}(sjy_n)|^2 = \sum_j \|\mathcal{A}^{2\rightarrow n}\|^2(sj)\ , \tag{1}$$

where $\|\mathcal{A}^{2\rightarrow n}\|$ is the length of the partial-wave amplitude, defined by

$$\|\mathcal{A}^{2\rightarrow n}\|^2(s,j)\equiv \int d\mu(y_n)|\mathcal{A}^{2\rightarrow n}(s,j,y_n)|^2\ ; \tag{2}$$

$d\mu(y_n)$ is the measure associated with the variables y_n. Its concrete form depends on the variables chosen; a method for deriving the measure along with a justification for calling $\|\mathcal{A}^{2\rightarrow n}\|$ defined in Eq. (2) the length of the partial-wave amplitude is given in the Appendix.

Also, if $\mathcal{A}^{2\rightarrow n}$ were known, the single-particle distribution function would be given by

$$\begin{aligned} D_n(s,\theta^*,s_x) &= \int d\mu(y_n)\delta(s_x-s_x(y_n))|A^{2\rightarrow n}(s,\theta^*,y_n)|^2 \\ &= \int d\mu(y_n)\delta(s_x-s_x(y_n))\sum_{J,j,j'} P_J(\cos\theta^*)\langle J0|j0j'0\rangle \mathcal{A}^{2\rightarrow n}(s,j,y_n)\overline{A^{2\rightarrow n}}(s,j',y_n) \\ &= \sum_J \mathcal{B}_n(s,J,s_x)P_J(\cos\theta^*)\ , \end{aligned} \tag{3}$$

where

$$\begin{aligned} \mathcal{B}_n(s,J,s_x) &= \sum_{j,j'} \langle J0|j0j'0\rangle \int d\mu(y_n)\delta(s_x-s_x(y_n))\mathcal{A}^{2\rightarrow n}(s,j,y_n)\overline{A^{2\rightarrow n}(s,j',y_n)} \\ &= \sum_{j,j'} \langle J0|j0j'0\rangle (\mathcal{A}^{2\rightarrow n}\delta(s_x-s_x(y_n)),\mathcal{A}^{2\rightarrow n})(s,j,j')\ . \end{aligned} \tag{4}$$

$\langle J0|j0j'0\rangle$ is a SU(2) Clebsch-Gordan coefficient and $s_x(y_n)$ means the invariant mass of the X cluster as a function of the y_n variables. The partial-wave inner product, discussed in the Appendix, is given by

$$(\mathcal{A}^{2\rightarrow n},\mathcal{A}^{2\rightarrow n})(s,j,j')=\int d\mu(y_n)\overline{\mathcal{A}^{2\rightarrow n}}(s,j',y_n)\mathcal{A}^{2\rightarrow n}(s,j,y_n)\ . \tag{5}$$

The partial-wave scattering amplitude $\mathcal{A}^{2\rightarrow n}$ is in general not known; but if some model partial-wave amplitude, $\mathcal{A}^{2\rightarrow n}_{\text{model}}$ (or its associated amplitude $A^{2\rightarrow n}_{\text{model}}$) is given, then using Eqs. (1) and (3) the multiplicity and single-particle distribution functions can be calculated. But generally model amplitudes will not satisfy unitarity requirements, so with channels opening and closing as the beam energy of the reacting particles is increased, there is little hope that model amplitudes will adequately fit multiplicity or single-particle distribution data.

The point of this paper is to show how model multiparticle amplitudes can be made unitary. The starting point is to introduce the strongly interacting scattering operator, S, which satisfies the unitarity condition $SS^\dagger = S^\dagger S = I$ on the appropriate Fock space. Denoting the $a+b$ two-particle system by "2," we want to evaluate $\langle 2'|S^\dagger S|2\rangle = \langle 2'|2\rangle$, from which unitarity conditions on the multiparticle partial-wave amplitudes can be extracted. This is most easily done using "partial-wave" variables, in which $\mathbf{p}_a, \mathbf{p}_b \to \mathbf{P}$, $\sqrt{s}$, and j,σ, where $\mathbf{P}$ is the total momentum and j,σ are the angular momentum and spin projection of the two-particle system. From relativistic invariance it follows that the $2\to 2$ partial-wave amplitude can depend only on s and j.

The Appendix shows that the unitarity condition for partial-wave amplitudes can be written as

$$\eta^2(s,j) + \sum_{n=3}^{n_{\max}(s)} \int_{m_n}^{\infty} d\sqrt{s_n}\,\delta(\sqrt{s_n} - \sqrt{s}\,)\|\mathcal{A}^{2\to n}\|^2(s_n,j) = 1 \ ; \tag{6}$$

η is the $2\to 2$ inelasticity parameter, $n_{\max}(s)$ denotes the maximum channel number allowed relative to the beam energy $\sqrt{s}$, and $m_{(n)} = m_1 + \cdots + m_n$. Equation (6) says that the squared lengths of the partial-wave amplitudes must sum to one. Such a condition can be satisfied by imagining an infinite-dimensional sphere, in which each dimension is a new channel. Then the various lengths can be parametrized by

$$\eta(s,j) = |\cos\alpha_3||\cos\alpha_4| \cdots = \prod_{k=3}^{\infty} |\cos\alpha_k|$$

$$\|\mathcal{A}^{2\to 3}\|(s,j) = |\sin\alpha_3||\cos\alpha_4| \cdots = |\sin\alpha_3| \prod_{k=4}^{\infty} |\cos\alpha_k|$$

$$\vdots$$

$$\|\mathcal{A}^{2\to n}\|(s,j) = |\sin\alpha_n||\cos\alpha_{n+1}| \cdots = |\sin\alpha_n| \prod_{k=n+1}^{\infty} |\cos\alpha_k| \ , \tag{7}$$

where the angles $\alpha_3, \alpha_4 \cdots$ are functions of s and j, of the form

$$\alpha_n = \theta(\sqrt{s} - m_{(n)}) \times \text{ functions of } s \text{ and } j \ ,$$

with $\theta(x)$ the step function.

Thus, if $\sqrt{s}$ is above the two-body threshold, but below the three-body threshold, then $\alpha_3 = \alpha_4 = \cdots = 0$ and $\eta = 1$, $\|\mathcal{A}^{2\to n}\| = 0$ for all $n = 3,4,\ldots$. Between the three- and four-body thresholds $\alpha_3 = \alpha_3(s,j) \neq 0$, but $\alpha_4 = \alpha_5 = \cdots = 0$. Then $\eta = |\cos\alpha_3(s,j)|$ and $\|\mathcal{A}^{2\to 3}\| = |\sin\alpha_3(s,j)|$. Continuing in this way it is seen that the unitarity condition, Eq. (6), is automatically satisfied for all $\sqrt{s} \geq m_1 + m_2$.

To determine the dependence of α_k on s and j, we return to the fact that in general $\mathcal{A}^{2\to n}_{\text{model}}$ will not satisfy the unitarity condition, Eq. (6). However, since the $2\to n$ partial-wave amplitude has a norm associated with it, it is possible to define a unit length $2\to n$ partial-wave amplitude by

$$\hat{\mathcal{A}}^{2\to n}_{\text{model}} = \frac{\mathcal{A}^{2\to n}_{\text{model}}}{\|\mathcal{A}^{2\to n}_{\text{model}}\|} \ ; \tag{8}$$

now set

$$\mathcal{A}^{2\to n}_{\text{unit}} \equiv \|\mathcal{A}^{2\to n}\| \hat{\mathcal{A}}^{2\to n}_{\text{model}} \ , \tag{9}$$

where

$$\|\mathcal{A}^{2\to n}\| = |\sin\alpha_n| \prod_{k=n+1}^{\infty} |\cos\alpha_k| \ .$$

By construction $\mathcal{A}^{2\to n}_{\text{unit}}$ will automatically satisfy the unitary condition, Eq. (6).

We assume that $\mathcal{A}^{2\to n}_{\text{model}}$ has the correct threshold behavior as the $2\to n$ channel is opening up. The threshold behavior of $\mathcal{A}^{2\to n}_{\text{unit}}$ will differ from that of $\mathcal{A}^{2\to n}_{\text{model}}$ because of the length factor $\|\mathcal{A}^{2\to n}_{\text{model}}\|$. If α_n is chosen to be

$$\alpha_n(s,j) = \theta(\sqrt{s} - m_{(n)}\|\mathcal{A}^{2\to n}_{\text{model}}\|(s,j) \ , \tag{10}$$

then near threshold, where $\sin\alpha_n$ is approximately α_n, the unitary partial-wave amplitude $\mathcal{A}^{2\to n}_{\text{unit}}$ will agree with $\mathcal{A}^{2\to n}_{\text{model}}$. But as soon as $\sqrt{s}$ is well above threshold, $\mathcal{A}^{2\to n}_{\text{unit}}$ will differ from $\mathcal{A}^{2\to n}_{\text{model}}$ (in its length dependence, but not in its "directional" dependence).

Notice that in this formulation, the inelasticity parameter $\eta = \prod_{k=3}^{\infty} |\cos\alpha_k|$ is not determined by $\mathcal{A}^{2\to 2}_{\text{model}}$, but by $\|\mathcal{A}^{2\to n}_{\text{model}}\|$. However, the phase shift for the $2\to 2$ reaction is still determined by $\mathcal{A}^{2\to 2}_{\text{model}}$.

From the $2\to n$ unitary partial-wave amplitudes, the multiplicity can be calculated [from Eq. (1)] to be

$$\text{mult}_n(s) = \sum_j \|\mathcal{A}^{2\to n}_{\text{unit}}\|^2(s,j)$$

$$= \sum_j \left[\sin\alpha_n \prod_{k=n+1}^{\infty} \cos\alpha_k \right]^2 (s,j) \ , \tag{11}$$

and in a similar, though more complicated fashion, the single-particle distribution function can be determined through the dependence of $\mathcal{B}_n(s,J,s_x)$ defined in Eq. (4):

$$\mathcal{B}_n(s,J,s_x)=\sum_{j,j'}(\mathcal{A}^{2\to n}_{\text{unit}},\mathcal{A}^{2\to n}_{\text{unit}})(sjj's_x)\langle J0|j0j'0\rangle$$

$$=\sum_{j,j'}\frac{\|\mathcal{A}^{2\to n}\|}{\|\mathcal{A}^{2\to n}_{\text{model}}\|}(sj')\frac{\|\mathcal{A}^{2\to n}\|}{\|\mathcal{A}^{2\to n}_{\text{model}}\|}(sj)(\mathcal{A}^{2\to n}_{\text{model}}\delta(s_x-s_x(y_n)),\mathcal{A}^{2\to n}_{\text{model}}(sjj's_x)\langle J0|j0j'0\rangle\ , \tag{12}$$

where $\|\mathcal{A}^{2\to n}\|(s,j)$ is given by Eq. (7).

III. MODEL MULTIPARTICLE PARTIAL-WAVE AMPLITUDES

The previous section has shown that if arbitrary elements from each n-particle partial-wave Hilbert space are chosen as the $2\to n$ partial-wave amplitudes, then there is a procedure for unitarizing these elements for all values of $\sqrt{s}$ above threshold. The question to be addressed in this section is how one might choose model $2\to n$ partial-wave amplitudes.

An obvious way of generating $2\to n$ partial-wave amplitudes is to take production Feynman diagrams or resonance models and calculate their partial-wave amplitudes. For example, if particles a and b react to produce particle c plus a resonance R, which subsequently decays into particles 2 and 3, the partial-wave amplitude can be written as

$$\mathcal{A}^{2\to 3}(sjy_3)=\sum_{k=-j_R}^{+j_R}\mathcal{A}^{2\to 2}_k(sjs_R)d^{j_R}_{k_0}(\theta_{c2})B_{M_R}(s_R)\ . \tag{13}$$

$\mathcal{A}^{2\to 2}_k(sjs_R)$ is the partial-wave amplitude for the $a+b\to c+R$ reaction and the variables y_3 are θ_{c2}, the angle between particles c and 2 in the rest frame of the resonance $(\mathbf{p}_2+\mathbf{p}_3=\mathbf{0})$, $s_R=(p_2+p_3)^2$, and M_R and j_R, the mass and spin of the resonance, respectively. $\mathcal{A}^{2\to 2}_k$ can be approximated by a one-particle exchange diagram, in which case it will be related to Q_j type functions with somewhat complicated kinematic arguments.[4] Also, if some of the particles (say c and 3) are identical, the partial-wave amplitudes must be suitably symmetrized, which can also be done with the aid of rotations in the partial-wave spaces.[5] In any event it is clear that Feynman diagrams can be used to obtain multiparticle partial-wave amplitudes; the problem is that partial-wave amplitudes of even simple Feynman diagrams are quite complicated, and in particular, computing their lengths probably cannot be done analytically.

For the purpose of this paper, which is to present a formalism that allows one to fit multiplicity and distribution data, the partial-wave amplitudes need only depend on s, j, and s_x. If we imagine a and b reacting to produce c and a new "particle" of mass $\sqrt{s_x}$ (it should also have spin j_x and spin projection σ_x, but internal spin variables are ignored in these two papers), then there will be a model partial-wave amplitude for $a+b\to c+X$, in which only the desired variables appear. Then the length of $\mathcal{A}^{2\to n}(s,j,s_x)$, using the measure derived in the Appendix, is

$$\|\mathcal{A}^{2\to n}\|^2(s,j)=\int_{m_x^2}^{(\sqrt{s}-m_c)^2}\frac{p_c^*}{p^*}ds_xR_{n-1}(s_x)|\mathcal{A}^{2\to n}(s,j,s_x)|^2\ . \tag{14}$$

p^* is the momentum for the initial system, given implicitly by

$$\sqrt{s}=[(p^*)^2+m_a^2]^{1/2}+[(p^*)^2+m_b^2]^{1/2}\ ,$$

while p_c^* is the momentum of c,

$$\sqrt{s}=[(p_c^*)^2+m_c^2]^{1/2}+[(p_c^*)^2+s_x]^{1/2}\ .$$

A simple model $2\to n$ partial-wave amplitude, determined mostly by the threshold behavior, is

$$\mathcal{A}^{2\to n}(s,j,s_x)=\left[\frac{E_c^*}{R_{n-1}(s_x)}\right]^{1/2}(p_c^*)^{j+1/2}\ ; \tag{15}$$

such a partial-wave amplitude is certainly not unitary. Its length is given by

$$\|\mathcal{A}^{2\to n}\|^2(s,j)=\frac{1}{p^*}\int ds_xE_c^*(p_c^*)^{2j+2}$$

$$=\frac{2\sqrt{s}}{p^*}\int_0^{\lambda^{1/2}/2\sqrt{s}}dp_c^*(p_c^*)^{2j+3}$$

$$=\frac{\lambda^{j+2}(s,m_c^2,m_x^2)}{(2l+4)p^*(2\sqrt{s})^{2j+3}}\ ; \tag{16}$$

the factors $[E_c^*/R_{n-1}(s_x)]^{1/2}$ have been chosen solely so the length integration can be done explicitly. $R_{n-1}(s_x)$ is a phase space factor defined after Eq. (A8) in the Appendix, $\lambda(s,m_c^2,m_x^2)$ is the triangle function, $\lambda=(s+m_c^2-m_x^2)^2-4sm_c^2$, and m_x is the sum of the masses of the particles in the X cluster; it should be noted that m_x is the only variable that distinguishes between the various n-particle final states.

Once the length of the $2\to n$ partial-wave amplitude is known, the factors that modify the length to make the partial-wave amplitudes unitary can be calculated from Eq. (10):

$$\alpha_n(s,j)=\theta(\sqrt{s}-(m_c+m_x))\frac{\lambda^{j+2}(s,m_c^2,m_x^2)}{(2l+4)p^*(2\sqrt{s})^{2j+3}}\ . \tag{17}$$

More generally, by broadening the class of functions it should be possible to fit multiplicity and single-particle distribution functions.

IV. CONCLUSION

There is a great deal of multiparticle data available, particularly multiplicity and single-particle distribution data, for systems such as the $\pi-N$ and $K-N$ systems. Because of the multiparticle nature of the final states, these data cannot be analyzed with a generalized phase-shift analysis, even though the single-particle distribution data are very similar to $2\to 2$ reaction data. It is almost hopeless to try to look for model independent ways of presenting multiparticle data, because of the many variables of a multiparticle amplitude and because of the complicated correlation experiments that are required to do a complete multiparticle experiment.

In such a situation it seems best to take model multiparticle amplitudes and use them to make predictions concerning multiplicity and single-particle distribution data. However, model multiparticle amplitudes do not generally satisfy unitarity requirements, and hence cannot be used to fit even the simplest multiparticle data.

What we have shown in this paper is how to unitarize model multiparticle amplitudes. Given a model multiparticle amplitude there is an associated multiparticle partial-wave amplitude that can be viewed as an element of a multiparticle "partial-wave" Hilbert space, labeled by the invariant mass $\sqrt{s}$ and angular momentum j of the multiparticle system. This means that multiparticle partial-wave amplitudes have a length which depends on s and j. Unitarity then forces the partial-wave amplitude to have a length between 0 and 1 for all values of s above threshold and j.

If partial-wave amplitudes are given for all the open channels, then unitarity states that the sum of all the lengths squared plus the square of the inelasticity parameter must equal one. By choosing a convenient parametrization for the lengths of partial-wave amplitudes, model partial-wave amplitudes that are not unitary can be made unitary. In the succeeding paper we will show that this parametrization also appears in exactly solvable models of production amplitudes.

With unitarized multiparticle partial-wave amplitudes, it is possible to confront experimental data. Roughly speaking the multiplicity data is related to the lengths of the partial-wave amplitudes, while the single-particle distribution data is related to the "direction" of the multiparticle partial-wave amplitudes, as seen in Eq. (12).

Model multiparticle partial-wave amplitudes need not come from perturbative field theory. From the point of view of this paper, they are elements of a partial-wave Hilbert space with length less than one. In fact, Sec. III showed that as far as fitting data is concerned, it is probably better to choose elements of partial-wave spaces that depend only on those variables that ignore the internal structure of the X cluster. Thus, our generalization of a phase-shift analysis to multiparticle reactions not only incorporates exact inelastic unitarity, but uses those variables describing the final-state configurations that correspond to experimentally relevant variables.

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APPENDIX: PHASE SPACE MEASURES AND PARTIAL-WAVE AMPLITUDES

If $\phi(\mathbf{p})$ is a momentum space wave function, its norm is defined by

$$\|\phi\|^2=\int \frac{d^3p}{E}|\phi(\mathbf{p})|^2<\infty,\quad E=+(\mathbf{p}^2+m^2)^{1/2}\ ; \tag{A1}$$

then $\langle \mathbf{p}'|\mathbf{p}\rangle = E\delta^3(\mathbf{p}'-\mathbf{p})$. An n-particle wave function ϕ_n, has a norm given by

$$\|\phi_n\|^2=\int \frac{d^3\mathbf{p}_1}{E_1}\cdots\frac{d^3\mathbf{p}_n}{E_n}|\phi_n(\mathbf{p}_1,\ldots,\mathbf{p}_n)|^2<\infty \tag{A2}$$

with

$$\langle \mathbf{p}_1',\ldots,\mathbf{p}_n'|\mathbf{p}_1,\ldots,\mathbf{p}_n\rangle=\prod_{i=1}^{n}E_i\delta^3(\mathbf{p}'-\mathbf{p}_i)\ .$$

The measure for two-particle variables is given by

$$\begin{aligned}\int\frac{d^3p_1}{E_1}\frac{d^3p_2}{E_2}&=\int\frac{d^3p_1}{E_1}\frac{d^3p_2}{E_2}d^4p\,\delta^4(p-p_1-p_2)\\&=\int d^4p\frac{d^3\mathbf{p}_1^*}{E_1^*}\frac{d^3\mathbf{p}_2^*}{E_2^*}\delta^4\left[\begin{pmatrix}\sqrt{s}\\ \mathbf{0}\end{pmatrix}-p_1^*-p_2^*\right]\\&=\int d^4p\frac{p^*}{\sqrt{s}}d\hat{p}^*\\&=\sum_{j\sigma}\frac{2j+1}{4\pi}\int d^4p\frac{p^*}{\sqrt{s}}\ .\end{aligned} \tag{A3}$$

Here p_i^* are the four-momentum vectors of particles $i=1,2$ in their center-of-mass frame (we follow the notation of Perl, Ref. 2 who denotes all center-of-mass quantities with an asterisk); $\hat{p}^*$ is the unit vector direction of particle 1 and p^* is its magnitude. p^* is related to $\sqrt{s}$ by

$$\sqrt{s}=(p^{*2}+m_1^2)^{1/2}+(p^{*2}+m_2^2)^{1/2}\ .$$

With "partial-wave" variables $pj\sigma$, two-particle states are normalized to

$$\langle p'j'\sigma'|pj\sigma\rangle=\delta^4(p'-p)\frac{\sqrt{s}}{p^*}\frac{4\pi}{2j+1}\delta_{j'j}\delta_{\sigma'\sigma}\ . \tag{A4}$$

The inelasticity parameter and phase shift are defined by

$$\langle p'j'\sigma'|S|pj\sigma\rangle =\delta^4(p'-p)\frac{\sqrt{s}}{p^*}\frac{4\pi}{2j+1}\delta_{j'j}\delta_{\sigma'\sigma}\eta(sj)e^{2i\delta_j(s)}\ . \tag{A5}$$

Similarly, the $\mathcal{A}^{2\to 2}$ partial-wave amplitude is

$$\langle p'j'\sigma'|T|pj\sigma\rangle =\delta^4(p'-p)\frac{\sqrt{s}}{p^*}\frac{4\pi}{2j+1}\delta_{j'j}\delta_{\sigma'\sigma}\mathcal{A}^{2\to 2}(sj)\ ; \tag{A6}$$

since in partial-wave variables all the delta functions are written explicitly, T and S are related by $S=I-iT$, so that

$$\eta e^{2i\delta}=1-i\mathcal{A}^{2\to 2}\ . \tag{A7}$$

For multiparticle states the invariant volumes can be written as

$$\int \frac{d^3\mathbf{p}_1}{E_1}\cdots\frac{d^3\mathbf{p}_n}{E_n}=\int d^4p\frac{d^3\mathbf{p}_1^*}{E_1^*}\cdots\frac{d^3\mathbf{p}_n^*}{E_n^*}\delta^4\left[\begin{bmatrix}\sqrt{s}\\0\end{bmatrix}-\sum_{i=1}^{n}p_i^*\right]$$
$$=\sum_{j\sigma}\frac{2j+1}{4\pi}\int d^4p\frac{1}{\sqrt{s}}\int d\mu(y_n)=\sum_{j\sigma}\frac{2j+1}{4\pi}\int_{m_n}^{\infty}d\sqrt{s_n}\int\frac{d^3p}{(s_n+p^2)^{1/2}}\int d\mu(y_n)\ , \qquad (A8)$$

where $\int d\mu(y_n)$ is an integral (sum) over the remaining $3n-6$ variables. For example, for the model discussed in Sec. III, where only the variables s,j,s_x appear in the partial-wave amplitude, the phase space integration becomes

$$\int \frac{d^3\mathbf{p}_1}{E_1}\cdots\frac{d^3\mathbf{p}_n}{E_n}=\int d^4p\frac{d^3\mathbf{p}_1^*}{E_1^*}\left[\frac{d^3\mathbf{p}_2^*}{E_2^*}\cdots\frac{d^3\mathbf{p}_n^*}{E_n^*}\delta^4\left[\begin{bmatrix}\sqrt{s}\\0\end{bmatrix}-p_1^*-\sum_{i=2}^{n}p_i^*\right]\right]$$
$$=\int d^4p d\hat{p}_1^*\frac{p_1^*}{2}\frac{ds_x}{\sqrt{s_n}}R_{n-1}(s_x)=\sum_{j\sigma}\frac{2j+1}{4\pi}\int d\sqrt{s_n}\int\frac{d^3p}{(s_n+p^2)^{1/2}}\int\frac{p_1^*}{2}ds_xR_{n-1}(s_x)\ ,$$

so that

$$\int d\mu(y_n)=\int\frac{p_1^*}{2}ds_xR_{n-1}(s_x)\ ,$$

where $R_{n-1}(s_x)$ is the phase space integral

$$\int\frac{d^3\mathbf{p}_2}{E_2}\cdots\frac{d^3\mathbf{p}_n}{E_n}\delta^4\left[p-\sum_{i=2}^{n}p_i\right]\ ,$$

with $p^2=s_x$; p_1^* is the momentum of particle 1 in the overall center of mass for the n-particle system and is given by

$$\sqrt{s_n}=(p_1^{*2}+m_1^2)^{1/2}+(p_1^{*2}+s_x)^{1/2}\ ,$$

with $s_n=(p_1+\cdots+p_n)^2$.

With these conventions the normalization of n-particle states becomes

$$\langle p'j'\sigma'y_n'|pj\sigma y_n\rangle=\delta^4(p'-p)\frac{4\pi}{2j+1}\frac{\sqrt{s}}{\mu(y_n)}\delta^{3n-6}(y_n'-y_n)\ ,$$

where $d\mu(y_n)=\mu(y_n)^{3n-6}dy_n$.

This change of variables from individual momenta to "partial-wave" variables can be used to define n-particle partial-wave Hilbert spaces $\mathcal{H}_n^{sj}$, with an inner product given by

$$(u,u')(s,j)=\int d\mu(y_n)\overline{u(sjy_n)}u'(sjy_n)\ ; \qquad (A9)$$

it is this definition which is used to define the lengths of partial-wave amplitudes.[6]

The $2\to n$ partial-wave amplitude is defined by

$$\langle n|T|2\rangle=\delta^4(p'-p)\frac{4\pi}{2j+1}\frac{\sqrt{s}}{\sqrt{p^*}}\delta_{j'j}\delta_{\sigma'\sigma}\mathcal{A}^{2\to n}(sjy_n) \qquad (A10)$$

and has a norm

$$\|\mathcal{A}^{2\to n}\|^2(s,j)=\int d\mu(y_n)|\mathcal{A}^{2\to n}(sjy_n)|^2<\infty\ . \qquad (A11)$$

Then $\langle n|S|2\rangle=-i\langle n|T|2\rangle$.

Using the unitarity condition $S^\dagger S=I$ gives

$$\langle 2''|2'\rangle=\langle 2''|S^\dagger S|2'\rangle=\sum_{n=2}^{n_{\max}(s)}\int\langle 2''|S^\dagger|n\rangle\langle n|S|2'\rangle$$
$$=\int\langle 2|S|2''\rangle^*\langle 2|S|2'\rangle+\sum_{n=3}^{n_{\max}(s)}\int\langle n|S|2''\rangle^*\langle n|S|2'\rangle\ . \qquad (A12)$$

Now the left-hand side of (A12) is given, for partial-wave variables, by Eq. (A4). The right-hand side has contributions from two-particle and $n>2$ particle intermediate states:

$$\int\langle 2|S|2''\rangle^*\langle 2|S|2'\rangle=\eta^2\delta^4(p''-p')\frac{4\pi}{2j+1}\frac{\sqrt{s}}{(p^*)^{1/2}}\delta_{j''j'}\delta_{\sigma''\sigma'}\ ,$$
$$\int\langle n|S|2''\rangle^*\langle n|S|2'\rangle=\int d^4p\frac{1}{\sqrt{s}}\sum\frac{2j+1}{4\pi}\int d\mu(y_n)\langle n|T|2''\rangle^*\langle n|T|2'\rangle$$
$$=\delta^4(p''-p')\frac{4\pi}{2j+1}\frac{\sqrt{s}}{(p^*)^{1/2}}\delta_{j''j'}\delta_{\sigma''\sigma'}\int_{m_n}^{\infty}d\sqrt{s_n}\delta(\sqrt{s_n}-\sqrt{s})\|\mathcal{A}^{2\to n}\|^2(s_n j)\ ,$$

from which follows that

$$\eta^2+\sum_{n=3}^{n_{\max}(s)}\int_{m_{(n)}}^{\infty}d\sqrt{s_n}\delta(\sqrt{s_n}-\sqrt{s})\|\mathcal{A}^{2\to n}\|^2(s_n,j)=1\ ,$$

which is the starting point for Eq. (6).

[1]See, for example, R. J. Cence, *Pion-Nucleon Scattering* (Princeton University Press, Princeton, 1964).

[2]See, for example, M. Perl, *High Energy Hadron Physics* (Wiley, New York, 1974), especially Chaps. 7 and 8.

[3]M. Perl, *High Energy Hadron Physics* (Wiley, New York, 1974), p. 142.

[4]S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966).

[5]W. H. Klink, Phys. Rev. D **4**, 2260 (1971).

[6]W. H. Klink, Nucl. Phys. **B77**, 56 (1974).