ARTICLES

W-matrix method for the representation of the scattering T matrix: Analytical example

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The original W-matrix approach, introduced recently by the Bonn group, represents the T matrix by a rank-one separable term. The method is exactly half on shell, but a nonzero remainder R occurs fully off shell. For potentials for which the scattering phase shift has a zero, the remainder can be non-negligible, even for energies far from the zero. We present expressions for W and R in terms of the scattering K matrix for the case that the potential is separable of finite rank. We show that the rank of R is one less than the rank of the potential. Then, for a rank-two analytical S-wave example of the potential, which is designed to produce a zero in the phase shift, we examine numerically the fully off-shell properties of the W method. We find pathologies at both negative and positive energies, in that the remainder R gives a significant off-shell contribution even far from the singularity.

I. INTRODUCTION

In the description of the two-body system proposed recently by Bartnik, Haberzettl, and Sandhas,[1] and referred to as the W-matrix approach, a single nonsingular inhomogeneous equation in momentum space gives the solution of the two-body problem for both the continuum and the bound state regimes. The W matrix, that is the solution of this nonsingular equation, allows an exact representation of the full-off-shell T matrix consisting of a separable term of rank one plus a remainder that is a real function vanishing both on and half on the energy shell. Since the bound-state pole and the scattering cut information are contained in the separable term, it was suggested to neglect the remainder and to use this rank-one term as a separable approximation of the transition matrix in the Alt, Grassberger, and Sandhas (AGS) three-body equations. This procedure indeed gave excellent results for three-body bound-state and continuum calculation[2] using the Malfliet-Tjon (I–III) potential. However, recent calculations by Gibson, Pearce, and Payne,[3] while confirming these results for a Malfliet-Tjon (I–III) potential, found one case where the neglect of the remainder led to large errors in the three-body binding energy. This controversial result has motivated us to study the properties of the W-matrix representation in a simple solvable potential model. The aim of this paper is to study, within the finite-rank potential model, the full-off-shell properties of the W-matrix representation with particular attention to the remainder term. In the next section we establish the various connections among W matrix, T matrix, K matrix, and Jost function for the case where the input potential is itself separable of rank N. These relations have not yet been obtained previously, and they make more transparent the nature of the approximation. The analysis shows that with a separable interaction of rank N, the W matrix remainder term is of rank $N-1$. In particular, for a rank-one potential, the remainder term is identically zero everywhere. The treatment is devoted only to those relations that are relevant for the discussion of the fully analytical example given in Sec. III. This consists in a very simple rank two, S-wave potential model which contains both an attractive and a repulsive term. We compare the exact transition matrix, the W matrix rank one approximation and the influence of the remainder for both positive and negative energies. A mean-square value of the remainder, calculated by a double integration over the full-off-shell momenta is defined for both positive and negative-energy regimes in order to give some insight into the quality of the rank-one approximation. Furthermore, it reveals the occurrence of unphysical singularities in correspondence of zeros of the on-shell reactance K matrix. These pathologies, connected to the Noyes-Kowalski representation of the T matrix, were pointed a out long time ago in the literature,[4] and several authors[5] proposed suitable modifications avoiding these singularities by increasing the rank of the approximation. Within our model calculation, we show that the method suggested in Ref. 1 is not free from these kinds of divergencies. Conclusions and final comments are contained in Sec. IV.

II. THEORETICAL CONSIDERATIONS

As given in Ref. 1, the W matrix is defined by the following inhomogeneous nonsingular integral equation in momentum space:

$$W_l(p,p';E)=V_l(p,p')p'^{-l} + \int \frac{V_l(p,q)q^{-l}-V_l(p,k)k^{-l}}{E-q^2} \times W_l(q,p';E)q^{l+2}dq \ . \quad (2.1a)$$

In the following, we assume $\hbar^2/2m=1$ and the energy is expressed in fm^{-2}. For scattering states this means $k^2=E$. For the sake of simplicity, we found it convenient to define a *reduced W* matrix

$$w_l(p,p';E)=W_l(p,p';E)p'^l \ . \tag{2.1b}$$

Note that both W and w are real and not symmetric matrices.

According to Ref. 1, the T matrix is decomposed into a rank-one separable piece Ts, plus a remainder term R, namely:

$$T_l(p,p';E^+)=Ts_l(p,p';E^+)+R_l(p,p';E) \ . \tag{2.2}$$

The quantity Ts is given in terms of w as follows:

$$Ts_l(p,p';E^+)=\frac{w_l(p,k;E)w_l(p',k;E)}{w_l(k,k;E)\mathbf{F}_l(E^+)} \ , \tag{2.3}$$

where

$$\mathbf{F}_l(E^+)=1-\int_0^\infty \frac{w_l(q,k;E)}{E^+-q^2}\left[\frac{q}{k}\right]^l q^2dq \tag{2.4}$$

is the Jost function. The remainder is obtained as the difference between T and Ts, it is real, vanishes half on shell, and is given by:

$$R_l(p,p';E)=w_l(p,p';E)-\frac{w_l(p,k;E)w_l(k,p';E)}{w_l(k,k;E)} \ . \tag{2.5}$$

We find it more convenient to deal with the reactance matrix K instead of the transition matrix T. The (real) matrix K is correspondingly decomposed into a rank-one separable term Ks, and a remainder R:

$$K_l(p,p';E)=Ks_l(p,p';E)+R_l(p,p';E) \ , \tag{2.6}$$

where

$$Ks_l(p,p';E)=\frac{w_l(p,k;E)w_l(p',k;E)}{w_l(k,k;E)\mathcal{R}\mathbf{F}_l(E^+)} \ , \tag{2.7}$$

and where R is the same as in Eq. (2.2) [with $\mathcal{R}\mathbf{F}_l(E^+)$ we mean the real part of $\mathbf{F}_l(E^+)$].

The preceding expression involves the half-on-shell values of w. By making use of the half-on-shell relations between w and K, namely:

$$K_l(p,k;E)=\frac{w_l(p,k;E)}{\mathcal{R}(\mathbf{F}_l(E^+))} \ , \tag{2.8}$$

one obtains the full-off-shell expression for Ks:

$$Ks_l(p,p';E)=\frac{K_l(p,k;E)K_l(k,p';E)}{K_l(k,k;E)} \ , \tag{2.9}$$

and correspondingly;

$$Ts_l(p,p';E)=\frac{Ks_l(p,p';E)}{1+ik\frac{\pi}{2}K_l(k,k;E)} \ . \tag{2.10}$$

It should be noted that when the on-shell value of K (and hence w) goes through zero, the expression of Ks acquires a singularity. Since K is finite, R also becomes singular, so as to cancel the singularity in Ks. This pathology occurs when the scattering phase shift δ goes through zero,[6,7] as can be seen from the relation

$$tg\delta_l=-k\frac{\pi}{2}K_l(k,k;E)=-k\frac{\pi}{2}\frac{w_l(k,k;E)}{\mathcal{R}(\mathbf{F}_l(E^+))} \ . \tag{2.11}$$

Now the idea is to assume as an ansatz that the input potential is itself separable, namely:

$$V(p,p')=\sum_{i,j=1}^{N} u_i(p)\lambda_{ij}u_j(p') \ . \tag{2.12}$$

From now on the angular momentum label l is not indicated explicitly, unless necessary. In order to make equations more transparent, it is convenient to use shorthand notations defined for example in Ref. 8, namely: $[u(p)\rangle$ is a column vector whose elements are $u_i(p)$, $\langle u(p)]$ is a row vector,

$$\langle u(p)][u(p')\rangle=\sum_{i=1}^{N} u_i(p)u_i(p')$$

is a scalar, and $[u(p)\rangle\langle u(p')]$ is a matrix whose elements are $u_i(p)u_j(p')$. Note that this is a rank-one matrix. $[P(E)]$ is a matrix whose elements are $P_{ij}(E)$.

With this notation Eq. (2.12) may be written as follows:

$$V(p,p')=\langle u(p)][\lambda][u(p')\rangle \ . \tag{2.13}$$

Then the solutions of the integral equations of scattering corresponding respectively to the T, K, and w matrices are separable with the same rank, and they can be expressed in terms of the matrices M, P, and Y as follows:

$$T(p,p';E^+)=\langle u(p)][M(E^+)]^{-1}[u(p')\rangle \ , \tag{2.14}$$

$$K(p,p';E)=\langle u(p)][P(E)]^{-1}[u(p')\rangle \ , \tag{2.15}$$

$$w(p,p';E)=\langle u(p)][Y(E)]^{-1}[u(p')\rangle \ . \tag{2.16}$$

Denoting by $[t]$ the inverse of the matrix $[\lambda]$

$$[t]=[\lambda]^{-1} \ , \tag{2.17}$$

one obtains the following relations:

$$[M(E^+)]=[t]-\int_0^\infty \frac{[u(q)\rangle\langle u(q)]}{E^+-q^2}q^2dq \ , \tag{2.18}$$

$$[P(E)]=[t]-\mathcal{P}\int_0^\infty \frac{[u(q)\rangle\langle u(q)]}{E-q^2}q^2dq \ , \tag{2.19}$$

$$[Y(E)]=[t]-\int_0^\infty \frac{[u(q)\rangle-[u(k)\rangle\left[\frac{q}{k}\right]^l}{E-q^2}\times\langle u(q)]q^2dq \ . \tag{2.20}$$

The matrices M, P, and Y are closely related to each other, as will now be shown. The integrals in Eqs. (2.18) and (2.20) may be written as follows

$$[M(E^+)]=[P(E)]+ik\frac{\pi}{2}[u(k)\rangle\langle u(k)] \ , \tag{2.21}$$

$$[Y(E)]=[P(E)]+[u(k)\rangle\langle b(k)] \ , \tag{2.22}$$

where $\langle b]$ is the Hilbert transform of $\langle u](q/k)^l$, namely:

$$\langle b(k)]=\mathcal{J}\int_0^\infty \frac{\langle u(q)]\left[\frac{q}{k}\right]^l}{k^2-q^2}q^2dq \ . \tag{2.23}$$

It is worthwhile to note that the elements of the vector $\langle b(k)]$ may be rewritten as

$$b_{nl}(k)=-k\frac{\pi}{2}\hat{u}_{nl}(k) \ , \tag{2.24a}$$

where $\hat{u}_{nl}(k)$ is the representation in momentum space of the potential form factor $v_{nl}(r)$ transformed with respect to the irregular solution $G_l(kr)$ of the free Schrödinger equation

$$\hat{u}_{nl}(p)=\frac{1}{p}\sqrt{2/\pi}\int_0^\infty G_l(pr)v_{nl}(r)r\,dr \ , \tag{2.25}$$

while in Eq. (2.12) $u_{nl}(k)$ is the momentum-space representation of $v_{nl}(r)$ transformed with respect to the regular solution $F_l(kr)$

$$u_{nl}(p)=\frac{1}{p}\sqrt{2/\pi}\int_0^\infty F_l(pr)v_{nl}(r)r\,dr \ . \tag{2.26}$$

In order to prove Eq. (2.24a), which can be written more explicitly as follows

$$\mathcal{J}\int_0^\infty \frac{\left[\frac{q}{k}\right]^l qdq}{q^2-k^2}\int_0^\infty F_l(qr)v_{nl}(r)r\,dr = \frac{\pi}{2}\int_0^\infty G_l(kr)v_{nl}(r)r\,dr \tag{2.24b}$$

one has to use the completeness property of the spherical Bessel functions[9] together with Eq. (27) of Ref. 1. As a consequence of Eqs. (2.24) the matrix $[Y(E)]$ is equal to

$$[\hat{M}(E)]=[P(E)]-k\frac{\pi}{2}[u(k)\rangle\langle\hat{u}(k)] \ ,$$

which is the Fredholm matrix with regular boundary condition [see Eqs. (3.9a) and (3.10a) of Ref. 8]. By contrast the matrix $[M(E^+)]$ corresponds to the physical outgoing boundary condition, while $[P(E)]$, real part of $[M(E^+)]$, has stationary boundary conditions. The relations (2.14)–(16) suggest that R can be related to a matrix $[R_a]$ according to

$$R(p,p';E)=\langle u(p)][R_a][u(p')\rangle \ . \tag{2.27}$$

By making use of Eqs. (2.6), (2.9), and (2.15) one obtains

$$[R_a]=[P(E)]^{-1}-\frac{[P(E)]^{-1}[u(k)\rangle\langle u(k)][P(E)]^{-1}}{\langle u(k)][P(E)]^{-1}[u(k)\rangle} \ . \tag{2.28}$$

It will now be demonstrated that the determinant of $[R_a]$ is equal to zero. This can be seen by noting that Eq. (2.28) is of the form

$$[A]=[B]+c[x\rangle\langle y] \ . \tag{2.29a}$$

For this type of matrices, given by a nonsingular part $[B]$ plus a rank-one piece $[x\rangle\langle y]$, the following identities hold:[8]

$$[A]^{-1}=[B]^{-1}-c\frac{[B]^{-1}[x\rangle\langle y][B]^{-1}}{1+c\langle y][B]^{-1}[x\rangle} \ , \tag{2.29b}$$

$$\det[A]=\det[B](1+c\langle y][B]^{-1}[x\rangle) \ . \tag{2.29c}$$

By comparing Eq. (2.28) with (2.29a), and making use of (2.29c), it follows that $\det[R_a]$ vanishes.

The fact that $\det[R_a]$ is equal to zero, implies that if V (and consequently T and K) are rank-N operators, then the remainder R is rank-$(N-1)$ (at most). In particular if V is rank one, then $R=0$, and the approximation is exact full off shell. In other words in case of a rank-N potential, the W-matrix approach separates T (or K) in two parts: Ts (or Ks) of rank-one, and R of rank $N-1$, null on and half on the energy shell.

By application of the rule (2.29b) to the expression (2.21) we may calculate the determinant of $[M]$, namely:

$$\det[M(E^+)]=\det[P(E)]\left[1+ik\frac{\pi}{2}K(k,k;E)\right] \ . \tag{2.30}$$

Since P and K are real, this implies that the imaginary part of $\det[M]$ is proportional to K, and so the Eq. (2.11) is consistent with the well-known relation

$$\tan\delta=-\frac{\mathcal{T}\det[M(E^+)]}{\mathcal{R}\det[M(E^+)]} \ . \tag{2.31}$$

The case of negative energies will be discussed next. In that case the singularity of the denominator of Eq. (2.1a) disappears, and k can be chosen as an independent parameter.[1] The transition and reactance matrices coincide in the case of negative energies, and for a separable interaction we may write:

$$K(p,p';\varepsilon)=\langle u(p)][P(\varepsilon)]^{-1}[u(p')\rangle \ . \tag{2.32}$$

Both M and P are now given by

$$[M(\varepsilon)]=[P(\varepsilon)]=[t]+\int_0^\infty \frac{[u(q)\rangle\langle u(q)]}{\varepsilon+q^2}q^2dq \ . \tag{2.33}$$

where $\varepsilon=\alpha^2=-E$. The w matrix becomes now:

$$w(p,p';k,\varepsilon)=\langle u(p)][Y(k,\varepsilon)]^{-1}[u(p')\rangle \ , \tag{2.34}$$

with

$$[Y(k,\varepsilon)]=[P(\varepsilon)]+[u(k)\rangle\langle b(k,\varepsilon)] \ , \tag{2.35}$$

and with

$$\langle b(k,\varepsilon)]=-\int_0^\infty \frac{\langle u(q)]\left[\frac{q}{k}\right]^l}{\varepsilon+q^2}q^2dq \ . \tag{2.36}$$

Finally Eqs. (2.6), (2.9), and (2.10) are substituted by the following:

$$Ks(p,p';k,\varepsilon)=\frac{K(p,k;\varepsilon)K(k,p';\varepsilon)}{K(k,k;\varepsilon)} \ , \tag{2.37}$$

$$R(p,p';k,\varepsilon)=K(p,p';\varepsilon)-\frac{K(p,k;\varepsilon)K(k,p';\varepsilon)}{K(k,k;\varepsilon)}$$

$$=w(p,p';k,\varepsilon)-\frac{w(p,k;k,\varepsilon)w(k,p';k,\varepsilon)}{w(k,k;k,\varepsilon)} \ . \tag{2.38}$$

III. A NUMERICAL EXAMPLE

In the following, we analyze the behavior of R in the simple case of a rank two, S-wave potential. For the form factors u in momentum space we assume the following expressions:

$$u_1(p)=\sqrt{2/\pi}\frac{1}{\beta_1^2+p^2}; \quad u_2(p)=\sqrt{2/\pi}\frac{2\beta_2}{(\beta_2^2+p^2)^2} \ . \tag{3.1}$$

We assume attractive and repulsive diagonal strengths

$$t_{11}=-0.1 \ (\text{fm}^3) \ ;$$

$$t_{22}=0.05 \ (\text{fm}^5) \ ;$$

$$t_{12}=t_{21}=0 \ (\text{fm}^4) \ ,$$

and the following range parameters:

$$\beta_1=1; \quad \beta_2=0.5 \ (\text{fm}^{-1}) \ .$$

In Eq. (3.1) $u_1(p)$ is the well-known Yamaguchi form factor, while $u_2(p)$ is a generalized Yamaguchi, as suggested in Ref. 10. For positive energies the model gives the following analytical P-matrix elements:

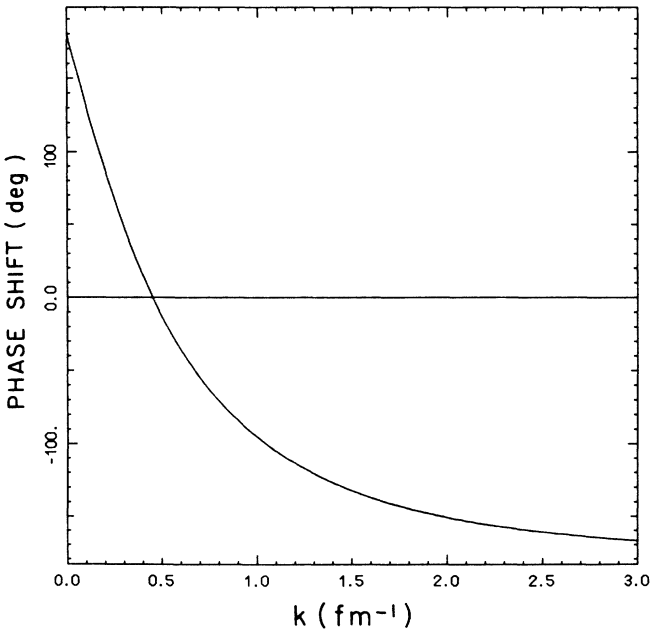


FIG. 1. Scattering phase-shift versus momentum k for the rank-two separable potential described in Sec. III.

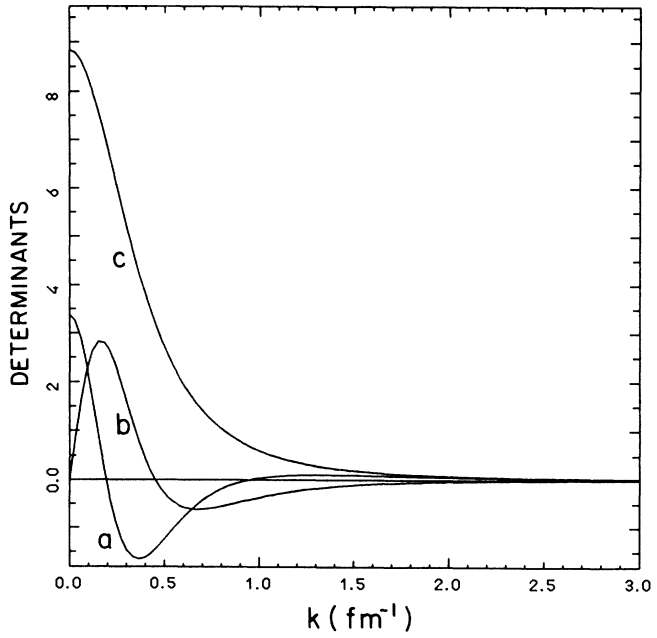


FIG. 2. Real (line a) and imaginary (line b) parts of the physical outgoing Fredholm determinant versus momentum k. The regular Fredholm determinant is shown by line c.

$$P_{11}=t_{11}+\frac{\beta_1^2-k^2}{2\beta_1(\beta_1^2+k^2)^2} \ , \tag{3.2a}$$

$$P_{22}=t_{22}+\frac{5\beta_2^6-15\beta_2^4k^2-5\beta_2^2k^4-k^6}{4\beta_2^3(\beta_2^2+k^2)^4} \ ; \tag{3.2b}$$

$$P_{12}=t_{12}+\frac{\beta_1\beta_2^2(\beta_1+2\beta_2)-k^2(\beta_1^2+2\beta_1\beta_2+3\beta_2^2)-k^4}{(\beta_1^2+k^2)(\beta_1+\beta_2)^2(\beta_2^2+k^2)^2} \ . \tag{3.2c}$$

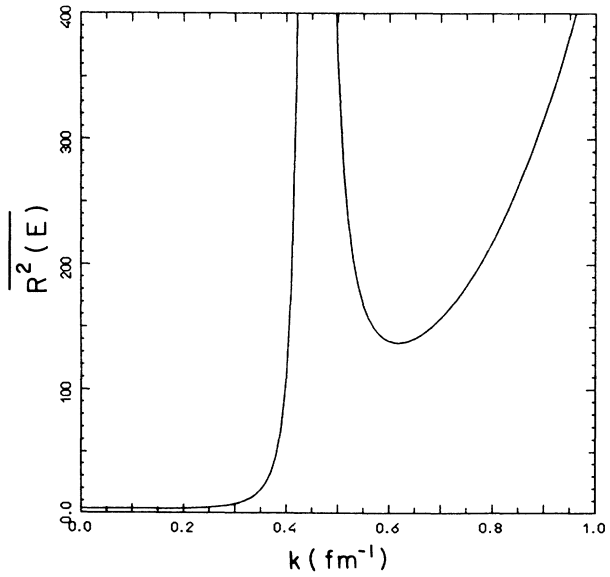


FIG. 3. Mean-square remainder, defined in Eq. (3.4), for positive energies, as a function of the on-shell momentum k.

The coordinate space form factors v, which are related to the momentum form factors u according to Eq. (2.26), are

$$v_1(r)=\frac{e^{-\beta_1 r}}{r}, \quad v_2(r)=e^{-\beta_2 r} . \tag{3.3}$$

To have an idea of the overall (on and off-energy-shell) agreement between $K(p,p';E)$ and $Ks(p,p';E)$, we calculate the mean-square remainder, defined as follows:

$$\overline{R^2(E)}=\int_0^\infty \int_0^\infty R^2(p,p';E)dp\,dp' . \tag{3.4}$$

Within our model calculation, this quantity can be evaluated by means of analytic expressions for both positive and negative energy regimes. The explicit expressions are given in the Appendix.

For negative energies the same potential model gives the following P-matrix elements:

$$P_{11}=t_{11}+\frac{1}{2\beta_1(\beta_1+\alpha)^2} ; \tag{3.5a}$$

$$P_{22}=t_{22}+\frac{5\beta_2^2+4\beta_2\alpha+\alpha^2}{4\beta_2^3(\beta_2+\alpha)^4} ,$$

$$P_{12}=t_{12}+\frac{\beta_1+2\beta_2+\alpha}{(\beta_1+\alpha)(\beta_1+\beta_2)^2(\beta_2+\alpha)^2} . \tag{3.5b}$$

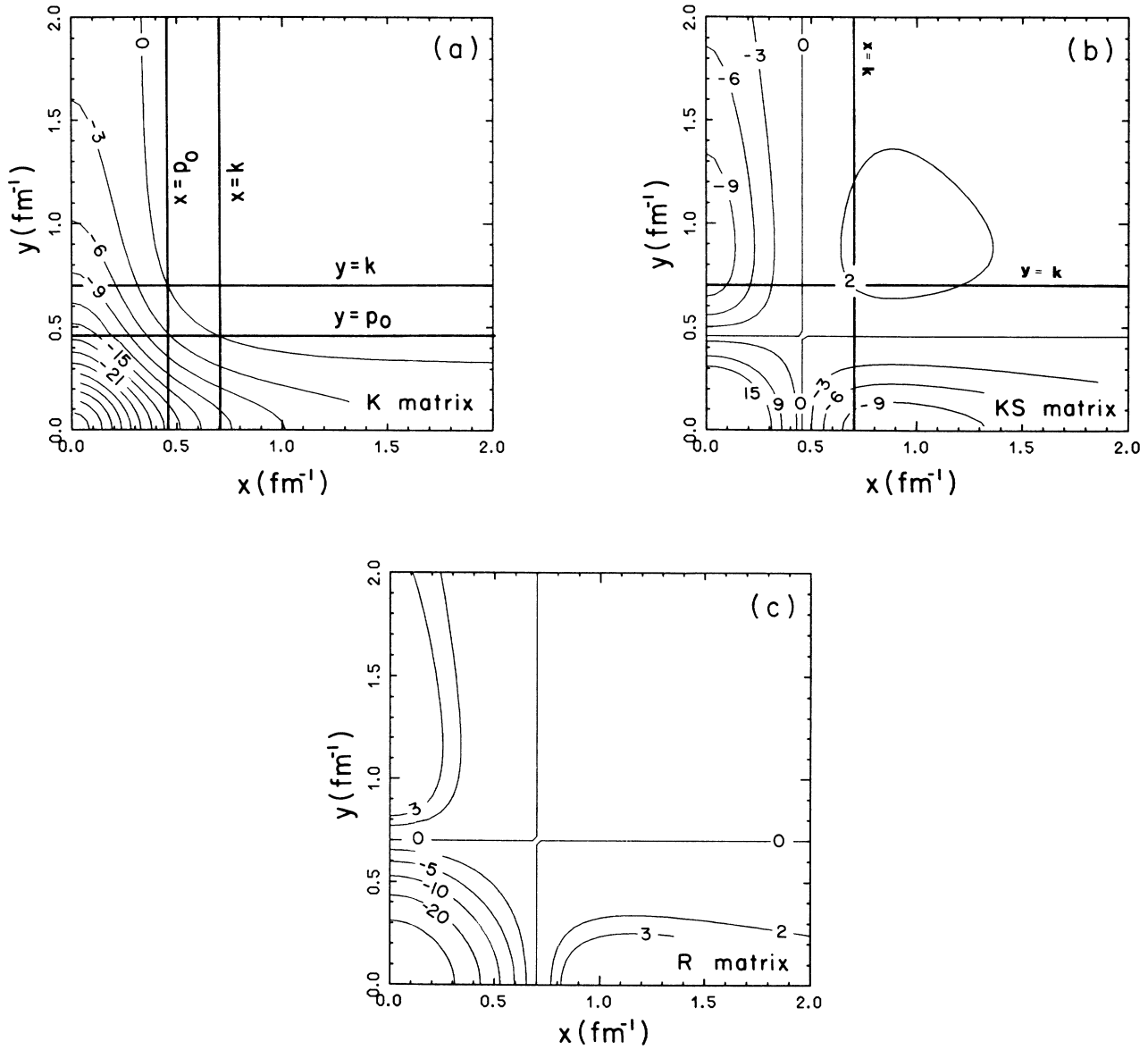


FIG. 4. Full-off-shell behavior of the reactance matrix K for the rank-two analytical potential described in Sec. III, for a positive energy. The on-shell momentum has the value $k=0.7$ fm^{-1}. The x and y axes represent values of the off-shell momenta, in units of fm^{-1}. The exact result is illustrated in (a). The half-on-shell momentum p_0 where the K matrix vanishes, as well as the on-shell k value, are illustrated in the figure by the straight lines. The rank-one approximation to K, given by Eq. (2.9), is illustrated in (b). The remainder R is shown in (c).

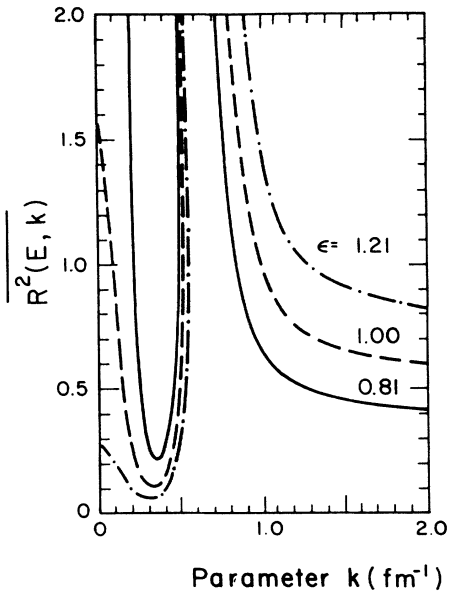


FIG. 5. The mean-square remainder for negative energies, as defined in Eq. (3.6). The parameter k, on the abscissa, is no longer related to the energy. The negative energies $-\varepsilon$ are indicated in the figure.

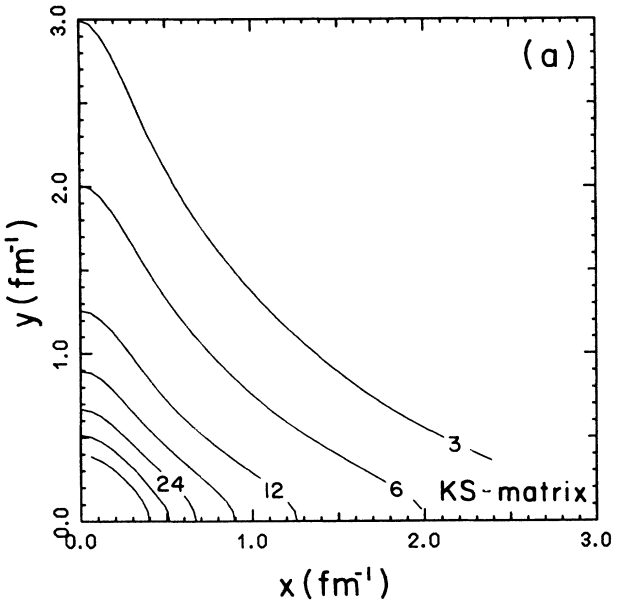
The calculation of the mean-square remainder, defined as in Eq. (2.41), namely

$$\overline{R^2(k,\varepsilon)}=\int_0^\infty\int_0^\infty R^2(p,p';k,\varepsilon)dp\,dp' , \tag{3.6}$$

is here more important in order to fix the free parameter k.

Figure 1 shows the phase shift δ as a function of the incident momentum k. One sees that it goes through zero at $k=0.450$ fm^{-1}, and hence the on-shell value of K is also zero at this point. At a higher momentum ($k=6.312$ fm^{-1}), the phase shift goes through a resonance which is due to the nonlocality of the potential model. Since we are not interested here in discussing effects arising from nonlocality we will not analyze the behavior of the phase shift in this higher momentum region. These effects have been widely discussed in the literature.[11] The Fig. 2 shows the real and imaginary part of det$[M(E^+)]$ versus k. The function det$[Y(E)]$ is also shown, and is non null in the considered energy region. This excludes the possibility that the pathology at $k=0.450$ fm^{-1} is due to nonlocality effects. Observe that, according to Eqs. (2.11), (2.31), δ, $K(k,k;E)$, and $\mathcal{T}$det$[M(E^+)]$ go to zero at the same point, while $\mathcal{R}$(det$[M(E^+)]$) is different from zero at this point.

Since we expect that R has a singularity at the zero of



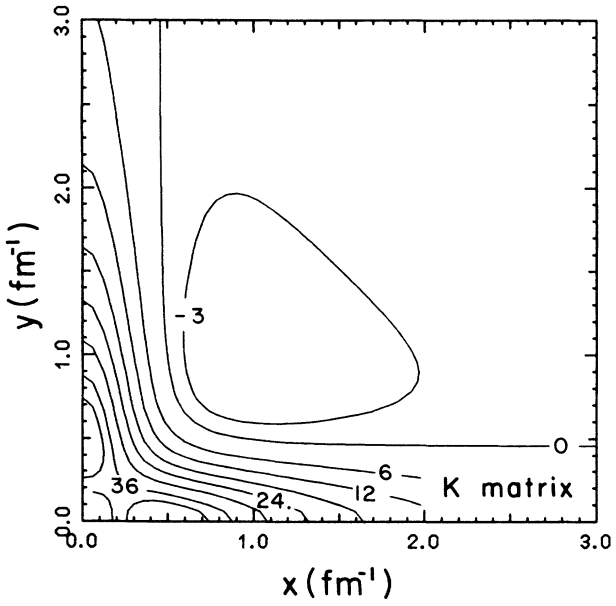


FIG. 6. Full-off-shell behavior of the exact reactance matrix K at a negative energy, $\varepsilon=1$. The notation is the same as in Fig. 4.

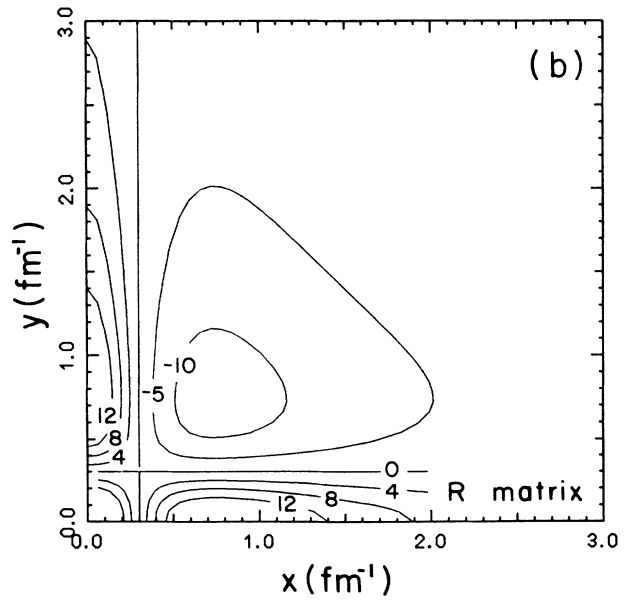


FIG. 7. Full-off-shell behavior of the rank-one approximant to K (a) and the exact value of the remainder (b) for a negative energy $\varepsilon=1$ and the parameter $k=0.3$ fm^{-1}.

K, the average value of R will show a similar singularity. This is illustrated in Fig. 3, where the divergence of $\overline{R^2}$ in the point $k=0.450$ fm^{-1}, is clearly seen.

Figure 4 shows K,Ks, and R as functions of the momenta p,p'. The physical momentum k has been chosen to have the value of 0.7 fm^{-1}, where the mean square R goes through a minimum. We are therefore not too close to the singularity. Note that the contour lines for Ks and R have a very different structure from those of K, and that R is not everywhere negligible with respect to K. According to Eq. (2.5), $R(p,p';E)$ is zero when either p or p' equal k. This is indeed observed to be the case in Fig. 4(c), where for either p or p' equal to $k=0.7$ fm^{-1}, R vanishes. On the other hand, the zero-value lines of $Ks(p,p';E)$ occur at the points where either p or p' are equal to p_0, defined by the condition that

$$K(p_0,k;E)=K(k,p_0;E)=0 .$$

The value of p_0 is found to be equal to 0.454 fm^{-1}, according to Fig. 4(a). The values of p_0 and k are thus seen to play an important role in the behavior of Ks and R, and hence they are indicated by heavy solid lines in Fig. 4(a). We have repeated the calculations choosing the physical energy k very close to k_0 [defined by $K(k_0,k_0;E_0)=0$]. In this case p_0 approaches k_0 together with k, and the values of both R and Ks increase with opposite sign, and tend to cancel each other so that K can remain finite. In the vicinity of the zero lines, the gradients of R and Ks also become very large and opposite in sign.

A similar type of difficulty also occurs at negative energies. The mean-square remainder, illustrated for three values of ε in Fig. 5, becomes very large for values of the parameter k in the interval from 0.5 to 1.2 fm^{-1}. According to Eq. (2.38) a singularity in R is expected to occur when $K(k_0,k_0;\varepsilon)=0$. From the plot of $K(p,p';\varepsilon=1)$, shown in Fig. 6, one can see that the value of k_0 lies near 0.6 fm^{-1}, where indeed $R^2(\varepsilon=1)$ becomes very large. The values of $\overline{R^2}$ go through a minimum in the vicinity of $k=0.3$ fm^{-1}. For this choice of the parameter k, a plot of the contour lines of Ks and R for $\varepsilon=1$ are shown in Figs. 7(a) and 7(b), respectively. Again it is seen that there are large regions in the momentum-space plane where R is not negligible in comparison to K.

IV. CONCLUSIONS

In this paper we have examined the W-matrix method, giving general expressions which relate W and R to T, K, and the Jost function F. We have also extended these results to the case that the potential is of finite rank. In this case interesting relations are derived among the above-mentioned functions and the Fredholm matrices defined relative to different boundary conditions. One important result is that for the case of rank one potential the remainder R vanishes identically, while for rank-N potential R is an operator of rank $(N-1)$. As Kowalski and Osborn have already pointed out, this method runs into a difficulty in the vicinity of a zero of the diagonal values of the K matrix. We have illustrated this difficulty for a rank-two separable potential. The difficulty found by Gibson *et al.*[3] appears to be of a different nature than the one which occurs in the vicinity of a zero of the phase shift. In that paper the singlet S Reid soft core potential was modified by making the intermediate range part of the potential slightly more attractive [by replacing $-1650.6(e^{-4x}/z)$ by $1815.66(e^{-4x}/z)$, where $z=0.7r$]. This modification increases the phase shifts and displaces the zero of the phase shift to larger momenta (larger than 1.74 fm^{-1}). Gibson *et al.* conjecture that the problem with the W-matrix method may be due to the presence of the strong repulsive core.

Since this study was completed, an improved W-matrix procedure was developed by Haberzettl,[12] which provides a separable representation of T of rank higher than one, by an iterative procedure. It is very possible that the difficulties encountered by the rank one W method are eliminated by this extension to higher ranks. This would be a welcome feature since the W-matrix approach has very nice on-shell and half-on-shell properties. However, since the extension of the W method is no longer of rank one, it becomes desirable to compare it with other kinds of separable expansions of low rank, available in the literature. In particular, the effect of the presence of strongly repulsive cores should then also be examined. These investigations are, however, beyond the scope of this study.

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APPENDIX

In this appendix we give the explicit expressions of the mean-square remainder $\overline{R^2}$ referring to the model of Sec. III, for both scattering and bound-state regimes. We start from Eqs. (2.20) and (2.21), which in the case of a rank-two potential is given by

$$R(p,p')=\frac{u_1(p)u_1(p')u_2^2(k)+u_2(p)u_2(p')u_1^2(k)-u_1(k)u_2(k)[u_1(p)u_2(p')+u_2(p)u_1(p')]}{u_1^2(k)P_{22}+u_2^2(k)P_{11}-2u_1(k)u_2(k)P_{12}} . \quad \text{(A1)}$$

Then, $\overline{R^2}$ becomes

$$\overline{R^2}=\frac{\left[u_1^2(k)\int u_2^2(p)dp+u_2^2(k)\int u_1^2(p)dp\right]^2+4u_1^2(k)u_2^2(k)\int u_1^2(p)u_2^2(p)dp}{[u_1^2(k)P_{22}+u_2^2(k)P_{11}-2u_1(k)u_2(k)P_{12}]^2}$$
$$-\frac{4u_1(k)u_2^3(k)\int u_1(p)u_2(p)dp\int u_1^2(p)dp+4u_1^3(k)u_2(k)\int u_1(p)u_2(p)dp\int u_2^2(p)dp}{[u_1^2(k)P_{22}+u_2^2(k)P_{11}-2u_1(k)u_2(k)P_{12}]^2}\ . \tag{A2}$$

Finally, by using the form factors (4.1) we obtain the explicit expression of $\overline{R^2}$:

$$\overline{R^2}=\left[\frac{\pi}{2}\frac{\dfrac{4\beta_2^2}{2\beta_1^3(\beta_2^2+k^2)^4}+\dfrac{5}{4\beta_2^5(\beta_1^2+k^2)^2}-\dfrac{4(\beta_1+2\beta_2)}{\beta_1\beta_2(\beta_1+\beta_2)^2(\beta_1^2+k^2)(\beta_2^2+k^2)^2}}{\dfrac{P_{22}}{(\beta_1^2+k^2)^2}+\dfrac{4\beta_2^2P_{11}}{(\beta_2^2+k^2)^4}-\dfrac{4\beta_2P_{12}}{(\beta_1^2+k^2)(\beta_2^2+k^2)^2}}\right]^2\ . \tag{A3}$$

For scattering states $\overline{R^2}=\overline{R^2}(E)$, since $k^2=E$ and the P-matrix elements are given by Eqs. (3.2); for negative energies $\overline{R^2}=\overline{R^2}(k,\varepsilon)$, since k has to be considered as a free parameter and the P-matrix elements are given by Eqs. (3.5).

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