PHYSICAL REVIEW C VOLUME 41, NUMBER 6 JUNE 1990

Off-shell form factors and low energy theorems for pion photoproduction

H. W. L. Naus
National Institute for Nuclear Physics and High Energy Physics, Section K (NIKHEF-K), P.O. Box 41882, NL-1009 DB Amsterdam, The Netherlands

J. H. Koch
National Institute for Nuclear Physics and High Energy Physics (NIKHEF-K) and Institute for Theoretical Physics, University of Amsterdam, Amsterdam, The Netherlands

J. L. Friar
Theory Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 2 January 1990)

The derivation of the low energy theorem for threshold photoproduction of neutral pions on a nucleon is reconsidered. Instead of considering explicitly the Born terms, as done in earlier derivations, a more general class of diagrams is separated from the total amplitude. Gauge invariance is imposed on the operator level and the partial conservation of the axial-vector current constraint is incorporated. Besides recovering the older results in this approach, relations for the off-shell behavior of the strong and electromagnetic vertices are derived.

I. INTRODUCTION

Recent accurate remeasurements[1,2] of the pion photoproduction amplitude for neutral pions near threshold have renewed the interest in the low energy theorems for this reaction.[3-9] This was due to the fact that the data as published suggest that these theorems fail. However, the comparison between experiment and the low energy theorem, derived for equal pion masses, is complicated by isospin breaking. Due to the different masses, the charged pion threshold lies a few MeV above the neutral pion threshold. The apparent violation of the low energy theorem is due to an analysis of the low energy data which attempts to account for this channel coupling. In Refs. 6 and 8, it was shown that other methods to include this coupling lead to different results. As we will see below, for no isospin breaking this coupling is included in the low energy theorem. The well-known Kroll-Ruderman theorem[10] states that the threshold amplitude to zeroth order in the photon energy is model independent. It can be obtained by the low energy limit of Born diagrams, evaluated with the physical masses and coupling constants. While it yields a nonvanishing prediction for charged pions, the photoproduction amplitude for neutral pions is zero in this limit. It was later realized that another constraint, obtained from the hypothesis of the partial conservation of the axial-vector current (PCAC), yields the linear terms, which contain the first nonvanishing contribution to the production amplitude of a π^0 on a nucleon. For neutral pions, even the quadratic terms can be obtained at threshold in this way. The situation for the (γ,π) amplitude is somewhat analogous to the low energy theorem for Compton scattering, where the amplitude to first order in the photon energy is determined in a model-independent fashion. While the Compton amplitude involves two conserved currents, the photopion amplitude involves the conserved electromagnetic current and one almost conserved axial-vector current, whose nonconservation, however, is given by PCAC. This makes it possible to also derive a similar low energy prediction.

In the derivation of the extended low energy theorem for pion photoproduction by the Baenst,[11] one starts by splitting the total amplitude into known Born terms and the unknown rest (see also Ref. 12). It is then shown that current conservation and PCAC can be used to express higher-order terms of the total amplitude through known physical properties: mass, magnetic moment, strong coupling constant and charge. In this paper, we derive the low energy theorem by using a more general basis which allows us to extend the earlier results of de Baenst.[11] Rather than splitting the amplitude into Born terms and 'rest', we follow the method of Gell-Mann and Goldberger.[13] We split the diagrams into class A, which we deal with explicitly, consisting of those in which pion and photon vertices are separated by a single nucleon or pion. The rest, class B, consists of diagrams which cannot be reduced in this fashion. A reaction such as photoproduction on a nucleon necessarily involves an intermediate nucleon off its mass shell. This implies that the strong and electromagnetic vertex operators as well as the form factors in class A have a much more complicated structure than the free vertices in the Born terms. We enforce gauge invariance on the operator level by using the Ward-Takahashi[14] identity, rather than on the amplitude level by imposing the weaker constraint of current conservation. Of course, the latter constraint is necessary but might not be sufficient to arrive at the most general result. We show in our approach explicitly for the case of π^0 photoproduction on the nucleon how the model-dependent terms disappear when PCAC is used, which justifies the earlier calculations (e.g., de Baenst[11]) in which on-shell form factors were assumed. An interesting by-product of this result is that the PCAC hypothesis yields relations between the model-dependent off-shell behavior of the vertices and class B terms. In the general framework we use, all "rescattering" corrections at the π^0 threshold are included. We here only consider the π^0 threshold limit. We assume that the masses of charged and neutral pions are the same. A consequence of this assumption is that no cusp appears at the $\pi^+ n$ threshold. Our main purpose here is to reconsider the low energy theorem using strong gauge invariance and by explicitly considering not only the (on-shell) Born terms, but a more general class of amplitudes than heretofore used. We will make no assumptions about the pion-nucleon coupling and our derivation includes pseudoscalar and pseudovector coupling as special cases.

Other approaches to study the low energy behavior of the photoproduction amplitude are current algebra, combined with soft pion techniques[15-17] and chiral Lagrangian models.[18] A recent review of the field can be found in Ref. 19. As an extension, chiral symmetry-breaking effects have been studied which could lead to violations of the low energy theorem.[7,20,21] For studies on the quark level, see, e.g., Refs. 22 and 23. Chiral symmetry constraints derived from QCD for low energy reactions involving pions are discussed in the recent paper by Donoghue and Holstein.[24] Explicit calculations of the rescattering of the pion on the nucleon have been carried out by Araki,[9] Nozawa, Lee, and Blankleider,[8] and Yang.[6] Other attempts, e.g., Ref. 25, to explain the data have relied on introducing vector meson exchange, which were found to be small in most cases,[3] as are the anomaly contributions from Primakoff-like processes.[26] The latter effects, which break the PCAC assumption,[27] are not considered in this paper.

In the next section, we outline the ingredients of our calculation: the propagators, the general structure of the vertices and the different classes of production amplitudes, together with the constraints of "strong" gauge invariance on the operator level. In Sec. III we then examine the power-series expansion of the general production operator. We display the model-dependent terms in this expansion and show how they get cancelled when PCAC is used, resulting in constraints for the model-dependent off-shell form factors. Section IV contains a summary of our findings and conclusions.

II. THE PHOTOPRODUCTION PROCESS

A. Kinematics

We first consider the general photoproduction amplitude of a pion from a nucleon,

$$\gamma(k)+N(p)\rightarrow N'(p')+\pi(q) \ . \tag{2.1}$$

We restrict ourselves to real photons, $k^2=0$. For reasons explained later, we assume that the produced pion is not on its mass shell and that the initial and final nucleon masses are different.[11] The initial mass is taken to be the physical mass, M, while the mass after the pion-nucleon vertex is taken to be M'. Of course, four-momentum is conserved,

$$k+p=q+p' \ . \tag{2.2}$$

In the CM frame the components of the four-vectors are given as

$$\begin{aligned} k&=(k_0,\mathbf{k}), \quad q=(q_0,\mathbf{q}) \ , \\ p&=(p_0,-\mathbf{k}), \quad p'=(p_0',-\mathbf{q}) \ , \end{aligned} \tag{2.3}$$

where

$$p_0=\sqrt{\mathbf{k}^2+M^2}, \quad p_0'=\sqrt{\mathbf{q}^2+M'^2}, \quad k_0=|\mathbf{k}| \ . \tag{2.4}$$

Instead of the commonly used Mandelstam variables, we will later use the dimensionless scalar variables[11]

$$\nu=\frac{1}{2M^2}(p+p')\cdot k \ , \quad \nu_1=\frac{1}{2M^2}q\cdot k \ . \tag{2.5}$$

At threshold, where the pion is produced at rest in the CM frame, $\mathbf{q}=0$, these variables have the values

$$\begin{aligned} \nu^0&=\frac{1}{2M^2}(k_0p_0+k_0M'+k_0^2) \ , \\ \nu_1^0&=\frac{1}{2M^2}q_0k_0 \ . \end{aligned} \tag{2.6}$$

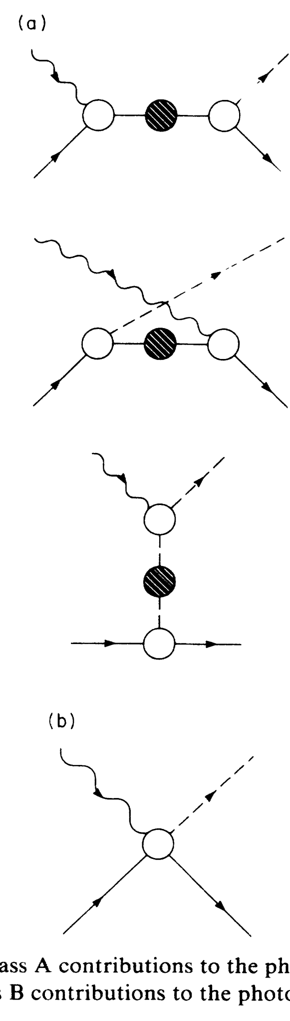

FIG. 1. (a) Class A contributions to the photoproduction amplitude. (b) Class B contributions to the photoproduction amplitude.

B. Class A and B diagrams

Following the approach in Ref. 13, we divide the possible diagrams contributing to the photoproduction amplitude into two classes. "Class A" consists of diagrams where the photon and pion vertex are separated by either a single nucleon or pion propagator. The possible contributions to this class are shown in Fig. 1. Note that, in contrast to the Born terms, these diagrams consist of the most general vertices and (off-shell) propagators. The first diagram in Fig. 1, the nucleon pole term A_1, has the form (we suppress isospin indices)

$$M_\mu^{A_1}(p',q;p,k)=\Lambda_5(p',p+k)S(p+k)\Gamma_\mu^{\rm irr}(p+k,p)$$
$$=\Lambda_5(p',p+k)S_0(p+k)\Gamma_\mu(p+k,p)\ . \qquad (2.7)$$

Here $\Gamma_\mu^{\rm irr}$ is the irreducible photon-nucleon vertex, used together with the fully dressed nucleon propagator, $S(p)$,

$$S(p)=\frac{1}{\not p-M-\Sigma(p)+i\varepsilon}\ , \qquad (2.8)$$

where $\Sigma(p)$ is the self-energy of the nucleon. The irreducible vertex is related to the reducible one, Γ_μ, through

$$S(p')\Gamma_\mu^{\rm irr}(p',p)S(p)=S_0(p')\Gamma_\mu(p',p)S_0(p)\ . \qquad (2.9)$$

To avoid double counting when using the reducible vertex, one has to work with the bare propagator for the intermediate nucleon,

$$S_0(p)=\frac{1}{\not p-M+i\varepsilon}\ . \qquad (2.10)$$

This yields the second form in Eq. (2.7). In this paper, we will work with the reducible electromagnetic vertex for the nucleon. Since the photon couples to an initial on-shell nucleon, but the intermediate nucleon is off its mass shell, this vertex has the form[28-30]

$$\Gamma_\mu u(\mathbf{p})=e\left[\gamma_\mu e_N+\frac{M+\not p+\not k}{2M}\frac{i\sigma_{\mu\nu}k^\nu}{2M}F_2^+(0,(p+k)^2,M^2)+\frac{M-(\not p+\not k)}{2M}\frac{i\sigma_{\mu\nu}k^\nu}{2M}F_2^-(0,(p+k)^2,M^2)\right]u(\mathbf{p})\ . \qquad (2.11)$$

In this equation e_N is the charge of the nucleon considered (in units of e). Other equivalent linear combinations of the Dirac-operators in Eq. (2.11) can be chosen. In the most general case, the form factors $F_2^\pm$ are functions of three scalar variables, e.g.,

$$F_2^\pm(k^2,p_f^2,p_i^2)\ . \qquad (2.12)$$

For the kinematics in diagram A_1, this dependence reduces to the form in Eq. (2.11).

The πNN vertex for both the initial (momentum p) and final nucleon (momentum p') off-shell can be written in the general form[31]

$$\Lambda_5(p',p)=\gamma_5 f_1+\gamma_5\frac{(\not p-M)}{M}f_2+\frac{(\not p'-M')}{M'}\gamma_5 f_3+\frac{(\not p'-M')}{M'}\gamma_5\frac{(\not p-M)}{M}f_4\ , \qquad (2.13)$$

where the form factors again depend on three scalar variables,

$$f_i=f_i(p^2,p'^2,(p-p')^2)\ . \qquad (2.14)$$

Using the quantities defined above, we can immediately also write down the structure of the term corresponding to diagram A_2 in Fig. 1.

$$M_\mu^{A_2}(p',q;p,k)=\Gamma_\mu^{\rm irr}(p',p'-k)S(p'-k)\Lambda_5(p'-k,p)=\Gamma_\mu(p',p'-k)S_0(p'-k)\Lambda_5(p'-k,p)\ . \qquad (2.15)$$

Here the electromagnetic vertex connects an off-shell nucleon to the final on-shell nucleon. Using space and time reversal invariance, this vertex is[28-30]

$$\bar u(\mathbf{p}')\Gamma_\mu=e\bar u(\mathbf{p}')\left[\gamma_\mu e_N+\frac{i\sigma_{\mu\nu}k^\nu}{2M}F_2^+(0,M'^2,(p'-k)^2)\frac{M'+\not p-\not k}{2M'}+\frac{i\sigma_{\mu\nu}k^\nu}{2M}F_2^-((0,M'^2,(p'-k)^2)\frac{M'-(\not p'-\not k)}{2M'}\right]\ . \qquad (2.16)$$

The form factors are related to the ones in Eq. (2.11) according to

$$F_2^\pm(0,M^2,p^2)=F_2^\pm(0,p^2,M^2)\ . \qquad (2.17)$$

Strictly speaking, the form factors could, in addition to the kinematical variables, also depend on the nucleon rest mass, i.e., be different for M and M'. As one can show, this does not change the final results and we will not consider this explicitly below.

Finally, the pion pole term of the class A diagrams, A_3

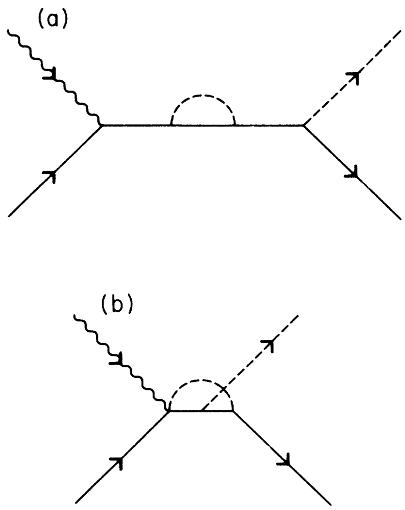

FIG. 2. (a) Example for rescattering contribution contained in class A terms. (b) Example for rescattering contribution contained in class B terms (for pseudovector coupling).

in Fig. 1, can be written as

$$M_\mu^{A_3}(p',q;p,k)=\Gamma_\mu^{\mathrm{irr},\pi}(q,q-k)\Delta(q-k)\Lambda_5(p',p)\ . \tag{2.18}$$

Note that this term only occurs for charged pion photoproduction. In Eq. (2.18) $\Gamma_\mu^{\mathrm{irr},\pi}$ is the irreducible electromagnetic vertex of the pion. Its most general form is[31,32]

$$\Gamma_\mu^{\mathrm{irr},\pi}(q',q)=e[(q+q')_\mu g^+((q'-q)^2,q'^2,q^2)+(q'-q)_\mu g^-((q'-q)^2,q'^2,q^2)]\ . \tag{2.19}$$

The intermediate pion propagator $\Delta(q)$ is given by

$$\Delta(q)=\frac{1}{q^2-m_\pi^2-\Pi(q^2)+i\varepsilon}\ , \tag{2.20}$$

where Π is the pion self-energy. Because we do not consider the production of an on-shell pion, we do not write this in terms of a reducible electromagnetic vertex, since it does not lead to a simpler structure as was the case for the nucleon pole diagrams A_1 and A_2. In the classification of Ref. 13, all other diagrams are called "class B". As we will discuss below, this class is subject to certain conditions due to gauge invariance and PCAC.

In concluding this section we would like to point out that rescattering corrections (π-N scattering and charge exchange after the π is produced) are included in this framework where isospin symmetry is not broken. For example, as illustrated in Fig. 2, they are present in the nucleon self energies in the class A terms which we consider separately; others are contained in class B, and are therefore included in this derivation of the low energy theorem. Moreover, this holds for resonant production and rescattering, which have been calculated separately and found to be relatively small, e.g., Refs. 15 and 33–35.

C. Gauge invariance

Instead of working with current conservation as a constraint for the pion photoproduction matrix element, we impose gauge invariance on the operator level. For the nucleon lines this means that we use the Ward-Takahashi identity[14]

$$(p'-p)^\mu\Gamma_\mu^{\mathrm{irr}}=ee_N[S^{-1}(p')-S^{-1}(p)]\ . \tag{2.21}$$

Similarly, we have for the vertex operator of a pion

$$(q'-q)^\mu\Gamma_\mu^{\mathrm{irr},\pi}=ee_\pi[\Delta^{-1}(q')-\Delta^{-1}(q)]\ . \tag{2.22}$$

In Eq. (2.22) e_π denotes the charge of the pion (in units of e).

As was first shown by Kazes,[31,36,37] gauge invariance then implies a relation for the total pion photoproduction operator analogous to the Ward-Takahashi identities, Eqs. (2.21) and (2.22). For example, for π^+ production, one has

$$k^\mu M_\mu(p',q;p,k)=e[\Delta^{-1}(q)\Delta(q-k)\Lambda_5(p',p)-\Lambda_5(p',p+k)S(p+k)S^{-1}(p)]\ . \tag{2.23}$$

With the known structure of the type A diagram for this process and the Ward-Takahashi identities, Eqs. (2.21) and (2.22), one obtains from the gauge constraint, Eq. (2.23), a condition for the up to now unspecified class B terms

$$k^\mu M_\mu^B(p',q;p,k)=e[\Lambda_5(p',p)-\Lambda_5(p',p+k)]\ . \tag{2.24}$$

Similarly, for π^0 production on a proton, which we will discuss later in this paper, one gets

$$k^\mu M_\mu(p',q;p,k)=e[S^{-1}(p')S(p'-k)\Lambda_5(p'-k,p)-\Lambda_5(p',p+k)S(p+k)S^{-1}(p)]\ , \tag{2.25}$$

and for the class B operator

$$k^\mu M_\mu^B(p',q;p,k)=e[\Lambda_5(p'-k,p)-\Lambda_5(p',p+k)]\ . \tag{2.26}$$

The analogous relations for π^0 photoproduction on a neutron, also considered below, are easily obtained: the right-hand sides of the expressions corresponding to Eqs. (2.25) and (2.26) are zero.

The above equations make clear that class B contains insertions of a photon into the πNN-vertex, which cannot be reduced. Gauge invariance yields a separate condition for these terms, e.g., Eqs. (2.24) and (2.26). The simplest example for a class B term is the contact or 'seagull' term for pseudovector πNN coupling in charged pion production.[36]

D. Connection with Born terms

The commonly used (and approximate) Born approach can be expressed in our general framework in the following way.

One takes the form factors appearing in the electromagnetic vertices, Eqs. (2.11) and (2.16), as constant, i.e.,

$$F_2^+(0,p_f^2,p_i^2)=F_2^-(0,p_f^2,p_i^2)=\kappa\ , \tag{2.27}$$

where κ is the anomalous magnetic moment of the on-shell nucleon:

$$F_2^+(0,M^2,M^2)=\kappa\ . \tag{2.28}$$

This yields the electromagnetic vertex used in the Born approach. The πNN vertex for the Born diagrams can be obtained from the general one for pseudoscalar coupling by taking all strong form factors f_i to be zero except

$$f_1=g(q^2)\ , \tag{2.29a}$$

where $g(q^2)$ is the strong pseudoscalar coupling form factor, whose on-shell value g is given by

$$\frac{g^2}{4\pi}=14.3\ . \tag{2.29b}$$

Pseudovector coupling results from *also* taking

$$f_2=\frac{M}{M'+M}g,\quad f_3=\frac{M'}{M'+M}g\ . \tag{2.30}$$

Note that the "seagull" or "contact" term is a class B term. It trivially satisfies the condition imposed by gauge invariance [e.g., Eq. (2.24) for π^+ production]. Finally in the pion electromagnetic vertex one takes

$$g^+=e_\pi,\quad g^-=0\ . \tag{2.31}$$

If we compare our class A terms with the Born terms of de Baenst[11] we encounter, besides the modifications due to the dependence of the off-shell scalar variables, extra model-dependent terms proportional to f_2, f_3, g^-, and $F_2^+-F_2^-$.

E. PCAC and pion production

We will now use the partially conserved axial-vector current (PCAC) hypothesis to provide an extra constraint for the photoproduction amplitude. As was first realized by Gaffney,[38] this makes it possible to extend the Kroll-Ruderman theorem.[10]

The matrix element for the photoproduction amplitude is

$$\mathcal{M}=\langle N(\mathbf{p}')|j_\pi^{\pm,0}|N(\mathbf{p}),\gamma(\mathbf{k})\rangle\ , \tag{2.32}$$

where $\pm,0$ are the isospin indices and j_π is the pion source,

$$(q^2-m_\pi^2)\phi^{\pm,0}=j_\pi^{\pm,0}\ . \tag{2.33}$$

Note that we take equal pion masses. PCAC relates the divergence of the axial-vector weak current to the strongly interacting pion field. Adler[39] extended this relation by including the electromagnetic field A_μ. Up to order e one has:

$$(i\partial^\mu+ee_\pi A^\mu)J_{5,\mu}^{\pm,0}=if_\pi m_\pi^2\phi^{\pm,0}\ , \tag{2.34}$$

where $J_{5,\mu}$ is the axial-vector current. These relations can be used for pion production as follows. Taking matrix elements of the operator in Eq. (2.34) between a final nucleon state and an initial state consisting of a nucleon and a photon and using the field equation for the pion field, Eq. (2.33), yields

$$\begin{aligned}\frac{f_\pi m_\pi^2}{m_\pi^2-q^2}\langle N(\mathbf{p}')|j_\pi^{\pm,0}|N(\mathbf{p}),\gamma(\mathbf{k})\rangle &= iq^\mu\langle N(\mathbf{p}')|J_{5,\mu}^{\pm,0}|N(\mathbf{p}),\gamma(\mathbf{k})\rangle\\ &\quad -ee_\pi\langle N(\mathbf{p}')|J_{5,\mu}^{\pm,0}A^\mu|N(\mathbf{p}),\gamma(\mathbf{k})\rangle\ .\end{aligned} \tag{2.35}$$

The left-hand side, which contains the pion production amplitude (the connection between the production amplitudes in isospin formalism and in terms of physical amplitudes is given in Appendix A), is defined only for virtual pions. Using the reduction formalism, the second term on the right-hand side can be related to the nucleon axial-vector current, see, e.g., Ref. 40. For the neutral pion, which we will discuss here, it is, of course, equal to zero. The first term on the right-hand side can be used to obtain the production amplitude in the "soft pion" limit in the following way. We first take the spatial momentum $\mathbf{q}$ of the pion to be zero (threshold limit), keeping the time component q_0 finite; the soft pion limit is then obtained by letting $q_0\to 0$. In this limit one cannot put the term proportional to q_μ equal to zero because the matrix element may diverge due to a nucleon pole. Several procedures to deal with this problem exist.[40] We will follow here de Baenst,[11] who introduced a fictitious mass difference ΔM between the external nucleons. In this case, the right-hand side of Eq. (2.35) vanishes in the soft pion limit because the contribution of the nucleon intermediate state remains finite, i.e.,

$$\lim_{q_0\to 0}\frac{q_0}{\Delta M}=0\ . \tag{2.36}$$

Therefore for neutral pion production of the nucleon where the external nucleons have different masses we have a vanishing amplitude in the soft pion limit:

$$\lim_{q_0\to 0}\mathcal{M}(\Delta M\neq 0,q_0)=0\ . \tag{2.37}$$

The physical amplitude is then obtained by at the end letting $\Delta M\to 0$.

III. EXPANSIONS AT THRESHOLD

The general form of the pion production amplitude can be written in terms of six invariant amplitudes, which are functions of the kinematical variables ν and ν_1 (Ref. 41):

$$\begin{aligned}\mathcal{M}=\frac{-ie}{2M}\varepsilon^\mu\bar{u}(\mathbf{p}')\gamma_5\Bigg[&\frac{q_\mu}{2M}A+\frac{(p+p')_\mu}{4M}B+\gamma_\mu C\\ &+\left[\frac{q_\mu}{2M}D+\frac{(p+p')_\mu}{4M}F+\gamma_\mu G\right]\frac{\not k}{2M}\Bigg]u(\mathbf{p}).\end{aligned} \tag{3.1}$$

This amplitude should be conserved; replacing ε by k in the last expression must yield zero. The earlier defined strong gauge invariance implies this "weak" gauge invariance; the opposite is not necessarily true. It should be stressed that due to this gauge invariance constraint only four of the six functions are linearly independent:

$$
\begin{aligned}
&2\nu_1 A(\nu,\nu_1)+\nu B(\nu,\nu_1)=0 , \\
&4C(\nu,\nu_1)+2\nu_1 D(\nu,\nu_1)+\nu F(\nu,\nu_1)=0 .
\end{aligned} \tag{3.2}
$$

This relation can be used to rewrite these "Ball" amplitudes, Eq. (3.1) in terms of the more conventional amplitudes defined by Chew, Goldberger, Low, and Nambu.[42] However, in the following we will use all six invariant amplitudes rather than eliminating two of them.

We will now calculate these invariant functions at threshold for π^0 production. Therefore we will only consider the isospin $+0$ amplitudes. For type A diagrams discussed in the previous section we start from the most general forms of the vertices and propagators. The contribution of the class B diagrams to these invariants can only be determined for the lowest-order terms in an expansion in the photon energy (which we assume to exist). For the total amplitude, class A and class B, we will then consider a power-series expansion in two parameters, ε and δ. They are defined as

$$
\delta=\frac{M'-M}{M}, \quad \varepsilon=\frac{q_0}{M} . \tag{3.3}
$$

This, together with the PCAC constraint, Eq. (2.37), will yield the well-known low energy theorem. To arrive at this result, we will make use of crossing symmetry for equal nucleon masses. In terms of the invariant amplitudes the crossing symmetry relations are given by

$$
\begin{aligned}
&A^{+0}(\nu,\nu_1)=-A^{+0}(-\nu,\nu_1) , \\
&B^{+0}(\nu,\nu_1)=B^{+0}(-\nu,\nu_1) , \\
&C^{+0}(\nu,\nu_1)=-C^{+0}(-\nu,\nu_1) , \\
&D^{+0}(\nu,\nu_1)=-D^{+0}(-\nu,\nu_1) , \\
&F^{+0}(\nu,\nu_1)=F^{+0}(-\nu,\nu_1) , \\
&G^{+0}(\nu,\nu_1)=G^{+0}(-\nu,\nu_1) .
\end{aligned} \tag{3.4}
$$

A. Invariant amplitudes of class A diagrams

We will use the following notations for the form factors

$$
\begin{aligned}
&{}^1f_i=f_i((p+k)^2,M'^2,q^2) , \\
&{}^2f_i=f_i(M^2,(p'-k)^2,q^2) , \\
&{}^3f_i=f_i(M^2,M'^2,(p-p')^2) , \\
&f_i=f_i(M^2,M'^2,q^2) ;
\end{aligned} \tag{3.5a}
$$

$$
\begin{aligned}
&{}^1F_2^{\pm}=F_2^{\pm}(0,(p+k)^2,M^2) , \\
&{}^2F_2^{\pm}=F_2^{\pm}(0,M'^2,(p'-k)^2) .
\end{aligned} \tag{3.5b}
$$

Diagram A_1 yields a contribution to the π production matrix element:

$$
\begin{aligned}
\mathcal{M}_{A_1}&=ie\bar{u}(\mathbf{p}')\left[\gamma_5\,{}^1f_1+\gamma_5\frac{\not{p}+\not{k}-M}{M}\,{}^1f_2\right]\frac{i}{\not{p}-\not{k}-M}\left[\gamma_\mu e_N+\frac{M+\not{p}+\not{k}}{2M}\frac{i\sigma_{\mu\nu}k^\nu}{2M}\,{}^1F_2^{+}+\frac{M-\not{p}-\not{k}}{2M}\frac{i\sigma_{\mu\nu}k^\nu}{2M}\,{}^1F_2^{-}\right]u(\mathbf{p}) \\
&=ie\bar{u}(\mathbf{p}')\left[\gamma_5\,{}^1f_1+\gamma_5\frac{\not{p}+\not{k}-M}{M}\,{}^1f_2\right]\frac{i}{\not{p}+\not{k}-M}\left[\gamma_\mu e_N+\frac{i\sigma_{\mu\nu}k^\nu}{2M}\,{}^1F_2^{+}+\frac{\not{p}+\not{k}-M}{2M}\frac{i\sigma_{\mu\nu}k^\nu}{2M}({}^1F_2^{+}-{}^1F_2^{-})\right]u(\mathbf{p}) ,
\end{aligned} \tag{3.6}
$$

which yields the following invariant amplitudes, defined in Eq. (3.1), at threshold

$$
\begin{aligned}
&A_1=\frac{-2e_N\,{}^1f_1}{\nu+\nu_1}, \quad B_1=\frac{-4e_N\,{}^1f_1}{\nu+\nu_1}, \quad C_1=-{}^1f_1\,{}^1F_2^{+}-2e_N\,{}^1f_2-{}^1f_2({}^1F_2^{+}-{}^1F_2^{-})(\nu+\nu_1) , \\
&D_1=\frac{2\,{}^1f_1\,{}^1F_2^{+}}{\nu+\nu_1}+2\,{}^1f_2({}^1F_2^{+}-{}^1F_2^{-}), \quad F_1=\frac{4\,{}^1f_1\,{}^1F_2^{+}}{\nu+\nu_1}+4\,{}^1f_2({}^1F_2^{+}-{}^1F_2^{-}) , \\
&G_1=\frac{2\,{}^1f_1(e_N+{}^1F_2^{+})}{\nu+\nu_1}+{}^1f_1({}^1F_2^{+}-{}^1F_2^{-})+2\,{}^1f_2\,{}^1F_2^{+} .
\end{aligned} \tag{3.7}
$$

For diagram A_2 we obtain in a similar way

$$
\begin{aligned}
&A_2=\frac{-2e_N\,{}^2f_1}{\nu-\nu_1}, \quad B_2=\frac{-4e_N\,{}^2f_1}{\nu-\nu_1} , \quad C_2={}^2f_1\,{}^2F_2^{+}+2e_N\frac{M}{M'}\,{}^2f_3-\left[\frac{M}{M'}\right]^2{}^2f_3({}^2F_2^{+}-{}^2F_2^{-})(\nu-\nu_1) , \\
&D_2=\frac{2\,{}^2f_1\,{}^2F_2^{+}}{\nu-\nu_1}-2\left[\frac{M}{M'}\right]^2{}^2f_3({}^2F_2^{+}-{}^2F_2^{-}) , \quad F_2=\frac{-4\,{}^2f_1\,{}^2F_2^{+}}{\nu-\nu_1}+4\left[\frac{M}{M'}\right]^2{}^2f_3({}^2F_2^{+}-{}^2F_2^{-}) , \\
&G_2=\frac{-2\,{}^2f_1\left[e_N+\dfrac{M'}{M}\,{}^2F_2^{+}\right]}{\nu-\nu_1}+\left[\frac{M}{M'}\right]{}^2f_1({}^2F_2^{+}-{}^2F_2^{-})+2\left[\frac{M}{M'}\right]{}^2f_3\,{}^2F_2^{+} .
\end{aligned} \tag{3.8}
$$

Diagram A_3 yields

$$A_3=\frac{-8M^2{}^3f_1g^+(0,q^2,q^2-4M^2\nu_1)}{q^2-4M^2\nu_1-m_\pi^2-\Pi(q^2-4M^2\nu_1)}\ ,$$
$$B_3=C_3=D_3=F_3=G_3=0\ . \tag{3.9}$$

B. Invariant amplitudes of class B diagrams

In order to fulfill (strong) gauge invariance we need class B terms, Eqs. (2.24) and (2.26), except for π^0 production on the neutron. In contrast to class A diagrams their form is not explicitly known. However, for π^0 production on the proton we can use Eq. (2.26) to expand the matrix element. Up to terms linear in k one has:

$$\mathcal{M}_\mu^B=-ie\bar{u}(\mathbf{p}')\left[\frac{\partial\Lambda_5}{\partial p'^\mu}+\frac{\partial\Lambda_5}{\partial p^\mu}-\frac{1}{2}k^\beta\frac{\partial^2\Lambda_5}{\partial p'^\beta\partial p'^\mu}+\frac{1}{2}k^\beta\frac{\partial^2\Lambda_5}{\partial p^\mu\partial p^\beta}\right]u(\mathbf{p})+ie\bar{u}(\mathbf{p}')S_\mu u(\mathbf{p})\ , \tag{3.10}$$

where S is an undetermined operator of order (k) and $k^\mu S_\mu=0$. In terms of the form factors in the strong vertex, Λ_5, Eq. (2.13), we obtain for the invariant amplitudes of the first part of class B (i.e., without S contribution):

$$\begin{aligned}
A_k&=4M^2f_1'-4M^2\,'f_1+4M^4(\nu+\nu_1)f_1''+4M^4(\nu-\nu_1)''f_1\ ,\\
B_k&=8M^2f_1'+8M^2\,'f_1+8M^4(\nu+\nu_1)f_1''-8M^4(\nu-\nu_1)''f_1\ ,\\
C_k&=2f_2-2\frac{M}{M'}f_3+2M^2(\nu+\nu_1)f_2'+2M^2\frac{M}{M'}(\nu-\nu_1)'f_3\ ,\\
D_k&=4M^2f_2'-4M^2\frac{M}{M'}\,'f_3\ ,\\
F_k&=8M^2f_2'+8M^2\frac{M}{M'}\,'f_3\ ,\\
G_k&=0\ ,
\end{aligned} \tag{3.11a}$$

with

$$f_i'=\frac{d}{d(t^2)}f_i(t^2,M'^2,q^2)\bigg|_{t^2=M^2}\ ,\qquad {}'f_i=\frac{d}{d(u^2)}f_i(M^2,u^2,q^2)\bigg|_{u^2=M'^2}\ , \tag{3.11b}$$

and similarly for the second derivatives. Before combining the B terms with type A, we rewrite these coefficients in a more convenient form. This modification neglects terms of order (k^2) and higher and yields the contributions denoted with the index K [in fact, for equal masses we have $C_k=C_K+O(\varepsilon^3)$]:

$$\begin{aligned}
A_K&=(-f_1+{}^1f_1)\frac{2}{\nu+\nu_1}+(-f_1+{}^2f_1)\frac{2}{\nu-\nu_1}\ ,\\
B_K&=(-f_1+{}^1f_1)\frac{4}{\nu+\nu_1}-(-f_1+{}^2f_1)\frac{4}{\nu-\nu_1}\ ,\\
C_K&=(f_2+{}^1f_2)-\frac{M}{M'}(f_3+{}^1f_3)\ ,\\
D_K&=(-f_2+{}^1f_2)\frac{2}{\nu+\nu_1}+\frac{M}{M'}(-f_3+{}^2f_3)\frac{2}{\nu-\nu_1}\ ,\\
F_K&=(-f_2+{}^1f_2)\frac{4}{\nu+\nu_1}-\frac{M}{M'}(-f_3+{}^2f_3)\frac{4}{\nu-\nu_1}\ ,\\
G_K&=0\ .
\end{aligned} \tag{3.12}$$

We combine the neglected terms of order (k^2) and higher together with S, the undetermined contribution, denoted by the index R. The class B matrix element therefore becomes

$$\mathcal{M}_\mu^B=\mathcal{M}_\mu^K+\mathcal{M}_\mu^R\ . \tag{3.13}$$

Now we sum up all "known" (i.e., expressable in terms of electromagnetic and strong form factors, which are partly model dependent) matrix elements: class A and the known part of (modified) class B. We present this result in terms of the isospin amplitudes $(+0)$ which are relevant for π^0 production. Here we use that the known part of class B for π^0 production on the neutron is zero. In the following, it is understood that F_2 is taken to be the isoscalar electromagnetic form factor for the 0 amplitudes and the isovector electromagnetic form factor for the + isospin components. From now on we will indicate the sum with an index T:

$$\mathcal{M}_\mu^T=\mathcal{M}_\mu^A+\mathcal{M}_\mu^K\ . \tag{3.14}$$

This yields the following amplitudes

$$A_T^{+0}=-f_1\frac{1}{\nu+\nu_1}-f_1\frac{1}{\nu-\nu_1},\quad B_T^{+0}=-f_1\frac{2}{\nu+\nu_1}+f_1\frac{2}{\nu-\nu_1}\ ,$$

$$C_T^{+0}=\frac{1}{2}(f_2-{}^1f_2)-\frac{1}{2}\frac{M}{M'}(f_3-{}^2f_3)-{}^1f_1\,{}^1F_2^+-{}^1f_2({}^1F_2^+-{}^1F_2^-)(\nu+\nu_1)+{}^2f_1\,{}^2F_2^+-\left[\frac{M}{M'}\right]^2{}^2f_3({}^2F_2^+-{}^2F_2^-)(\nu-\nu_1)\ ,$$

$$D_T^{+0}=(-f_2+{}^1f_2+2\,{}^1f_1\,{}^1F_2^+)\frac{1}{\nu+\nu_1}+2\,{}^1f_2({}^1F_2^+-{}^1F_2^-)$$
$$+\left[\frac{M}{M'}(-f_3+{}^2f_3)+2{}^2f_1\,{}^2F_2^+\right]\frac{1}{\nu-\nu_1}-2\left[\frac{M}{M'}\right]^2 f_3({}^2F_2^+-{}^2F_2^-)\,, \qquad (3.15)$$
$$F_T^{+0}=(-f_2+{}^1f_2+2\,{}^1f_1\,{}^1F_2^+)\frac{2}{\nu+\nu_1}+4{}^1f_2({}^1F_2^+-{}^1F_2^-)$$
$$-\left[\frac{M}{M'}(-f_3+{}^2f_3)+2\,{}^2f_1\,{}^2F_2^+\right]\frac{2}{\nu-\nu_1}+4\left[\frac{M}{M'}\right]^2 {}^2f_3({}^2F_2^+-{}^2F_2^-)\,,$$
$$G_T^{+0}=({}^1f_1+2\,{}^1f_1\,{}^1F_2^+)\frac{1}{\nu+\nu_1}+{}^1f_1({}^1F_2^+-{}^1F_2^-)+2\,{}^1f_2\,{}^1F_2^+$$
$$-\left[{}^2f_1+2\frac{M'}{M}\,{}^2f_1\,{}^2F_2^+\right]\frac{1}{\nu-\nu_1}+\frac{M}{M'}\,{}^2f_1({}^2F_2^+-{}^2F_2^-)+2\frac{M}{M'}\,{}^2f_3\,{}^2F_2^+\,.$$

Note that this amplitude constructed by using strong gauge invariance is weakly gauge invariant, Eq. (3.2), and crossing symmetric, Eq. (3.4), in all orders in k for zero mass difference.

C. The low energy theorem

We now expand the invariant amplitudes obtained in the preceding section with respect to two independent small parameters. The first one, ε, is used for expanding around the soft pion limit and is defined in Eq. (3.3). The second small parameter is δ which measures the mass difference between the initial and final nucleon, introduced in Eq. (3.3) to deal with the poles of the amplitude in the soft pion limit. The kinematical variables in Eq. (2.5) can be expressed in terms of these parameters

$$\nu+\nu_1=(\varepsilon+\delta)+\tfrac{1}{2}(\varepsilon+\delta)^2\,,$$
$$\nu-\nu_1=\tfrac{1}{2}(1+\delta)[(1+\varepsilon+\delta)-(1+\varepsilon+\delta)^{-1}]\,. \qquad (3.16)$$

At threshold, $\mathbf{q}=0$, and choosing the transverse gauge, $\varepsilon^0=0$, we obtain from the general form of the pion production amplitude, Eq. (3.1), the following matrix element

$$\mathcal{M}=\frac{-ie}{2M}\varepsilon^\mu\bar{u}(\mathbf{p}')\gamma_5\gamma_\mu[C+\tfrac{1}{2}(\delta+\varepsilon)G]u(\mathbf{p})\,, \qquad (3.17)$$

and, consequently, we have only to consider the C and G amplitudes.

We start with the contributions of the residual diagrams R, which do not contain poles in k. We therefore assume the following expansions

$$X_R=X_{00}+X_{10}\nu+X_{01}\nu_1+X_{11}\nu_1\nu\cdots\,, \qquad (3.18)$$

where X stands for the amplitudes C and G, and in general

$$X_{ik}=X_{ik}(\varepsilon,\delta)\,. \qquad (3.19)$$

The class of diagrams R has been defined in such a way that its matrix element is at least of order k; therefore we have

$$C_{00}=0\,. \qquad (3.20)$$

In fact, it is easy to see that also $A_{00}=0$, $B_{00}=0$. Expanding also the X_{ik} around $\varepsilon=0,\delta=0$ and keeping only linear terms in the matrix element, we obtain for the class R terms:

$$\mathcal{M}^R=\frac{-ie}{2M}(\varepsilon+\delta)\varepsilon^\mu\bar{u}(\mathbf{0})\gamma_5\gamma_\mu[C_{10}(0,0)+\tfrac{1}{2}G_{00}(0,0)]u(\mathbf{0})\,. \qquad (3.21)$$

Note that also the spinors have been expanded in ε and δ. However, as is shown in Appendix B, only the constant term contributes.

We also look at the expansion of the C_T^{+0} and G_T^{+0} terms. Since these terms contain pole contributions we must be careful in taking the limits $\varepsilon\to 0,\delta\to 0$; in fact the order in which one takes these limits is crucial. The expansion of C_T^{+0} yields only linear terms:

$$C_T^{+0}=\delta\bar{c}^{\,+0}+\varepsilon\tilde{c}^{\,+0}+\cdots\,, \qquad (3.22)$$

with

$$\bar{c}^{\,+0}=-M^2[f_2'(0)+{}'f_3(0)]$$
$$-[f_2(0)+f_3(0)][F_2^+(0)-F_2^-(0)]$$
$$-2M^2[f_1'(0)+{}'f_1(0)]F_2^+(0)\,, \qquad (3.23)$$
$$\tilde{c}^{\,+0}=-2M^2f_1(0)[{}'F_2^+(0)+F_2^{+\prime}(0)]+\bar{c}^{\,+0}\,.$$

We have used the abbreviations

$$f_i(0)=f_i(M^2,M^2,0)\,,$$
$$F_2^\pm(0)=F_2^\pm(0,M^2,M^2)\,, \qquad (3.24)$$

and similarly for the derivatives.

For G_T we only need the lowest power in ε and δ. It depends on the order in which the limits are taken:

$$\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} G_T^{+0}=\bar{g}^{\,+0}, \quad \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} G_T^{+0}=\tilde{g}^{\,+0}, \tag{3.25}$$

where

$$\begin{aligned}\bar{g}^{\,+0}&=2M^2[f_1'(0)+'f_1(0)][2F_2^+(0)+1]\\&\quad+2F_2^+(0)[f_2(0)+f_3(0)]-2f_1(0)F_2^-(0)\,,\\ \tilde{g}^{\,+0}&=\bar{g}^{\,+0}-f_1(0)+4M^2f_1(0)[F_2^{+\prime}(0)+'F_2^+(0)]\,.\end{aligned}\tag{3.26}$$

We are now in the position to write down the threshold contribution for the physical process ($\delta=0$). This yields to first order in ε:

$$\mathcal{M}^{+0}=\frac{-ie}{2M}\varepsilon\epsilon^\mu\bar{u}(0)\gamma_5\gamma_\mu\{\bar{c}^{\,+0}+C_{10}^{+0}(0,0)+\tfrac{1}{2}[\tilde{g}^{\,+0}+G_{00}^{+0}(0,0)]\}u(0)+O(\varepsilon^2)\,. \tag{3.27}$$

On the other hand, the soft pion limit ($\varepsilon=0$) yields to first order in δ:

$$\mathcal{M}^{+0}=\frac{-ie}{2M}\delta\epsilon^\mu\bar{u}(0)\gamma_5\gamma_\mu\{\bar{c}^{\,+0}+C_{10}^{+0}(0,0)+\tfrac{1}{2}[\bar{g}^{\,+0}+G_{00}^{+0}(0,0)]\}u(0)+O(\delta^2)\,. \tag{3.28}$$

The PCAC hypothesis, Eq. (2.37), constrains the last amplitude to be zero, which means that the expression between the curly brackets must vanish. This enables us to determine precisely the combination of the unknown residual amplitudes which enters the physical production process, Eq. (3.27):

$$\begin{aligned}\tfrac{1}{2}G_{00}^{+0}(0,0)+C_{10}^{+0}(0,0)&=-(\tfrac{1}{2}\bar{g}^{\,+0}+\bar{c}^{\,+0})=-\{M^2[f_1'(0)+'f_1(0)-f_2'(0)-'f_3(0)]\\&\qquad-[f_2(0)+f_3(0)][F_2^+(0)-F_2^-(0)]\\&\qquad-f_1(0)F_2^-(0)+F_2^+(0)[f_2(0)+f_3(0)]\}\\&=-\{M^2[f_1'(0)+'f_1(0)-f_2'(0)-'f_3(0)]+F_2^-(0)[f_2(0)+f_3(0)-f_1(0)]\}\,.\end{aligned}\tag{3.29}$$

We repeat here that the isospin superscripts $+0$ in F_2^- are suppressed to avoid confusion.

This relationship has two important consequences. First, in our very general derivation we get back the result of de Baenst.[11] Substituting Eq. (3.29) into the matrix element of the physical process, Eq. (3.27), yields to first order in ε:

$$\begin{aligned}\mathcal{M}^{+0}&=\frac{-ie}{2M}\varepsilon\epsilon^\mu\bar{u}(0)\gamma_5\gamma_\mu\\&\quad\times[\bar{c}^{\,+0}-\bar{c}^{\,+0}+\tfrac{1}{2}(\tilde{g}^{\,+0}-\bar{g}^{\,+0})]u(0)+O(\varepsilon^2)\\&=\frac{ie}{2M}\varepsilon\epsilon^\mu\bar{u}(0)\gamma_5\gamma_\mu\tfrac{1}{2}f_1(0)u(0)+O(\varepsilon^2)\\&=\frac{iegm_\pi}{4M^2}\epsilon^\mu\bar{u}(0)\gamma_5\gamma_\mu u(0)+O(\varepsilon^2)\,.\end{aligned}\tag{3.30}$$

This is the same as the term linear in m_π/M in the low energy theorem of de Baenst, which yields the proton and neutron π^0 photoproduction amplitudes according to Eq. (A2). While de Baenst assumed pseudoscalar coupling, our general derivation immediately shows that the result is also the same for pseudovector coupling [see Eqs. (2.27)–(2.30)]. Without the PCAC constraint, pseudoscalar and pseudovector coupling yield different threshold amplitudes. The resulting amplitude is zero for the neutron in this order. The second consequence is that Eq. (3.29) relates the lowest nonvanishing contribution of the undetermined class B diagrams, $\mathcal{M}^R$ in Eq. (3.13), to the "known" elements of class A. This relation is the low energy limit of the gauge conditions, Eq. (2.26) for the proton and the analogous one for the neutron, extended to include PCAC. It provides a consistency condition for a microscopic description of photoproduction (beyond a phenomenological Lagrangian or "Born term" approach) and constrains the way form factors can go off-shell at threshold. For example, Ohta,[37] by using minimal substitution, has generated a class B contribution for a general πNN vertex that guarantees gauge invariance. In order to fulfill PCAC, which was not considered by Ohta, the constraint given in Eq. (3.29) must be satisfied. As another example of a possible application of Eq. (3.29), we mention the work of Gross and Riska.[43] These authors give a recipe for including phenomenological form factors in strong and electromagnetic vertices in exchange current contributions that is consistent with current conservation. Again Eq. (3.29) yields a consistency condition for the corresponding class R terms if one attempts to extend their prescription to the closely related process of pion photoproduction and PCAC is imposed.

In the derivation of de Baenst,[11] the terms were split up into Born terms and "rest." Pseudoscalar pion nucleon coupling was assumed for the Born terms. We can reproduce his results in our general framework as follows. Using the prescription as given in Eqs. (2.27) and (2.29) to obtain the pseudoscalar Born terms as class A, the condition, Eq. (3.29), then becomes much simpler

$$\tfrac{1}{2}G_{00}^{+0}(0,0)+C_{10}^{+0}(0,0)=\kappa^{+0}g\,. \tag{3.31}$$

This is precisely the result of de Baenst. For neutral pion production no class B terms are needed in his approach to fulfill "strong" gauge invariance, because the right-hand side of Eq. (2.26) vanishes for pseudoscalar coupling. Had we chosen pseudovector coupling Born terms[12] as class A in our derivation, according to the prescription given in Eqs. (2.27), (2.29), and (2.30), the PCAC constraint would therefore not require contributions from the "rest" in the physical amplitude in this order, i.e., the right-hand side of Eq. (3.29) vanishes. This is a consequence of the well-known fact that pseudovector pion nucleon coupling Born terms yield a chirally invariant theory.

Following de Baenst we can also obtain the second-order contribution to the photoproduction amplitude, i.e., expand the matrix element in Eq. (3.17) to order ε^2. For zero mass difference crossing symmetry, Eq. (3.4), holds. (Note that we do not have to keep the mass difference in this order any more after the PCAC constraint has been imposed.) Using crossing symmetry in the expansions for the invariant amplitudes of the residual matrix element, Eq. (3.18), we obtain:

$$C_{01}^{+0}=G_{10}^{+0}=C_{20}^{+0}=0 \ , \tag{3.32}$$

and therefore the second-order term of the residual amplitude R is zero. This means that we can obtain the second-order term by expanding the "known" amplitudes C_T and G_T which are the ones that contribute at threshold. As is clear from Eq. (3.17), C_T has to be expanded to second order, while G_T only has to be expanded to first order. This straightforward but rather tedious calculation (see Appendix C) results in the following *second-order* contribution:

$$C_T^{+0}+\tfrac{1}{2}\varepsilon G_T^{+0}=\tfrac{1}{4}f_1[1+2F_2^{+}(0)]=\tfrac{1}{4}g(1+2\kappa^{+0}) \ . \tag{3.33}$$

Again, the spinor expansion yields no contribution (see Appendix B). This final result is exactly the same as the low energy theorem of de Baenst,[11] and can again be converted to proton and neutron π^0 photoproduction amplitudes by using Eq. (A2). For the neutron, this is the first nonvanishing term in the low energy limit.

In order to obtain this result we used invariance under space and time reversal which yields relations between various form factors at the on-shell point (see Appendix C). Furthermore crossing symmetry is crucial to have direct and crossed terms cancel in this order. The quoted result holds only for threshold kinematics, because in this limit only the C and G invariant amplitudes contribute to the matrix element.

We would like to conclude this section with several caveats. We have assumed that a power-series expansion of class B terms exists. As was shown by Li and Pagels,[44] this might not be justified if a Goldstone symmetry of the Hamiltonian, realized by massless bosons, gets broken. Furthermore, if isospin breaking is taken into account and the charged and neutral pion masses become different, a cusp occurs at the charged pion threshold, which has to be treated carefully. Finally, we mention that expansion methods as used above are not *a priori* useful in nuclear pion production; this is discussed in Ref. 45.

IV. SUMMARY AND CONCLUSIONS

Recently, there has been a lot of interest in the electromagnetic interaction of an off-shell nucleon. This was triggered by recent experiments investigating the knock out of nucleons in quasifree electron scattering from nuclei. Electromagnetic form factors of an off-shell nucleon already play a role in simpler situations, namely, two-step processes on a free nucleon, such as pion photoproduction. Of course, in that case one also deals with the electromagnetic pion form factor and the strong pion-nucleon vertex in situations where the intermediate particles are off their mass shell. However, it has been shown that at low energies, the leading terms of an expansion of the physical amplitudes can be expressed in terms of a few global on-shell properties of the hadrons. One of these low energy theorems, which made a prediction for the photoproduction of a π^0 on a proton has come under close scrutiny after recent experiments apparently yielded incompatible results.

In the derivation of the low energy theorem for Compton scattering by Gell-Mann and Goldberger[13] the general structure of the off-shell vertices was explicitly taken into account to obtain the famous result in terms of on-shell properties: mass, charge, and magnetic moment. We have in this paper gone through an analogous derivation of the low energy theorem for π^0 production where the off-shell behavior was explicitly considered. While in Compton scattering one deals with two conserved currents, photoproduction only involves one and additional information is needed to go beyond the lowest order Kroll-Ruderman[10] result. This additional constraint is provided by the PCAC hypothesis. In the derivation of the threshold amplitude by de Baenst,[11] only Born terms are explicitly treated and the remainder, which includes the off-shell parts, is constrained in a general fashion by PCAC, crossing symmetry and conservation of the electromagnetic current. Similarly, in current algebra approaches, off-shell terms do not show up as intermediate on-shell single nucleon states are inserted in the current commutators.

Just as in the work of de Baenst[11] and, e.g., Vainshtein and Zakharov,[12] we neglect the mass difference between charged and neutral pions and, furthermore, we assume that a power-series expansion of the nonpole terms exists. Both assumptions exclude the cusp that occurs when the production channel of the charged meson opens above the π^0 threshold. The approach chosen in this paper differs from the derivation of de Baenst[11] in several respects. One important difference is that instead of Born terms we split off amplitudes which contain an isolated nucleon or pion pole. These "class A" terms have a more general structure and include off-shell vertices and propagators. Another major difference is that we incorporate gauge invariance on the operator level, rather than only requiring that the total amplitude is conserved. In our approach, just as in de Baenst's,[11] "rescattering" of the pion (Fig. 2) on the nucleon is included and therefore needs not to be taken into account explicitly.

In our general framework, we end up obtaining the well-known low energy theorem for π^0 photoproduction on a nucleon. Just as in earlier derivations, we have not taken into account isospin breaking, i.e., different pion masses which might be necessary to directly compare to the measurements. Therefore, it is clear from this work that additional rescattering corrections, which have been proposed to explain the threshold data, should disappear in the limit of equal masses. Our approach includes pure pseudoscalar πNN coupling, as assumed by de Baenst,[11] and pure pseudovector coupling, assumed by Vainshtein and Zakharov,[12] as special cases. We have explicitly demonstrated how the off-shell behavior of the different vertices enters and disappears in the final answer. In doing so, we obtain a new relation between the off-shell behavior of strong and electromagnetic form factors and the lowest-order nontrivial class B contribution. We have shown that this relation provides a useful consistency check for microscopic approaches to photopion production.

ACKNOWLEDGMENTS

The authors would like to thank J. W. Bos, R. Davidson, T. W. Donnelly, and J. A. Tjon for stimulating discussions. J. H. K. thanks the Theoretical Physics Group of the Universita di Trento for its hospitality. This work was made possible by financial support from the Foundation for Fundamental Research on Matter (FOM), the Netherlands' Organization for Scientific Research (NWO), and by a NATO Collaborative Research Grant. The work of J. L. F. was performed under the auspices of the U.S. Department of Energy.

APPENDIX A: ISOSPIN

The invariant amplitudes have the following structure in isospin space

$$X=\chi'^{\dagger}(X^{+}\delta_{\beta 3}+X^{-}\tfrac{1}{2}[\tau_\beta,\tau_3]+X^{0}\tau_\beta)\chi \ , \tag{A1}$$

where χ' and χ are the isospinor of the initial and final nucleons and β denotes the isospin of the outgoing pion. The four physical processes are given by the following combinations:

$$\begin{aligned}\mathcal{M}(\pi^+)&=\sqrt{2}(\mathcal{M}^0+\mathcal{M}^-)\ ,\\ \mathcal{M}(\pi^-)&=\sqrt{2}(\mathcal{M}^0-\mathcal{M}^-)\ ,\\ \mathcal{M}(p\pi^0)&=\mathcal{M}^++\mathcal{M}^0\ ,\\ \mathcal{M}(n\pi^0)&=\mathcal{M}^+-\mathcal{M}^0\ .\end{aligned} \tag{A2}$$

APPENDIX B: SPINOR EXPANSION

If one expands the total matrix element of the pion photoproduction process one has also to expand the spinors.[36] The spinor of the initial state can be expanded at threshold in the CM frame as

$$u(\mathbf{p})=u(\mathbf{0})-\frac{\mathbf{k}\cdot\boldsymbol{\gamma}}{2M}u(\mathbf{0})+O(k_0^2)\ . \tag{B1}$$

Let us assume that the photon momentum defines the j direction ($j=1,2,3$, corresponding with x,y,z, respectively), and rewrite Eq. (B1) in terms of the expansion parameters ε and δ. This yields

$$u(\mathbf{p})=u(\mathbf{0})-\tfrac{1}{2}(\varepsilon+\delta)\gamma_j u(\mathbf{0})+O(\varepsilon^2,\varepsilon\delta,\delta^2)\ . \tag{B2}$$

For the final spinor, of course, no expansion is necessary

$$\bar{u}(\mathbf{p}')=\bar{u}(\mathbf{0})\ . \tag{B3}$$

It is clear from Eqs. (3.17), (3.20), and (3.22) that the first-order matrix element, Eq. (3.30) does not change because a possible correction due to the expansion (B2) is of higher order. It could contribute to the second-order matrix element; however this term, denoted with τ, actually yields zero. We obtain

$$\tau\sim\bar{u}(\mathbf{0})\gamma_5\gamma_i\gamma_j u(\mathbf{0})\ , \tag{B4}$$

where $i\neq j$, because of the transversality of the real photon. The product of the γ matrices can be worked out easily

$$\begin{aligned}\gamma_5\gamma_i\gamma_j&=\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{bmatrix}\\ &=\begin{bmatrix}0 & -\sigma_i\sigma_j\\ -\sigma_i\sigma_j & 0\end{bmatrix}\ .\end{aligned} \tag{B5}$$

Since this matrix is off diagonal and because the spinor of a particle with zero three-momentum has only upper components, we indeed obtain $\tau=0$.

APPENDIX C: SECOND-ORDER TERMS

We give explicit formulas for the intermediate results of the second-order terms. First we give the second-order contribution of C:

$$\begin{aligned}C_{T,2}^{+0}=&-\tfrac{1}{2}M^2f_2'(0)-M^4f_2''(0)+\tfrac{1}{2}M^2\,{}'f_3(0)+M^4\,{}''f_3(0)-[\tfrac{1}{2}f_2(0)+2M^2f_2'(0)][F_2^+(0)-F_2^-(0)]\\ &-2M^2f_2(0)[F_2^{+\prime}(0)-F_2^{-\prime}(0)]-Q_1+Q_2\\ &+[\tfrac{1}{2}f_3(0)+2M^2\,{}'f_3(0)][F_2^+(0)-F_2^-(0)]+2M^2f_3(0)[\,{}'F_2^+(0)-{}'F_2^-(0)]\ ,\end{aligned} \tag{C1}$$

with

$$Q_1=M^2f_1(0)[F_2^{+\prime}(0)+2M^2F_2^{+\prime\prime}(0)]+4M^4f_1'(0)F_2^{+\prime}(0)+M^2F_2^+(0)[2M^2f_1''(0)+f_1'(0)+f_{1,q}(0)]\ , \tag{C2}$$

and

$$Q_2=M^2 f_1(0)['F_2^+(0)+2M^2\,''F_2^+(0)]+4M^4\,'f_1(0)\,'F_2^+(0)+M^2F_2^+(0)[2M^2\,''f_1(0)+'f_1(0)+f_{1,q}(0)]\ , \tag{C3}$$

where

$$f_{1,q}(0)=\left.\left(\frac{\partial f_1}{\partial q^2}\right)\right|_{q^2=0,p^2=p'^2=M^2}\ . \tag{C4}$$

The first-order term in G (which gives a second-order contribution to the matrix element because it gets an factor $\varepsilon/2$) is

$$\begin{aligned}G_{T,1}^{+0}=&\tfrac{1}{4}f_1(0)+2M^4f_1''(0)+M^2f_{1,q}(0)+2Q_1+\tfrac{1}{2}f_1(0)F_2^+(0)-2M^2[f_1'(0)F_2^+(0)+f_1(0)F_2^{+'}(0)]\\&+2M^2f_1(0)[F_2^{+'}(0)-F_2^{-'}(0)]+2M^2f_1'(0)[F_2^+(0)-F_2^-(0)]\\&+4M^2[f_2'(0)F_2^+(0)+f_2(0)F_2^{+'}(0)]+\tfrac{1}{4}f_1(0)-2M^4\,''f_1(0)-M^2f_{1,q}(0)\\&-2Q_2+\tfrac{1}{2}f_1(0)F_2^+(0)+2M^2['f_1(0)F_2^+(0)+f_1(0)'F_2^+(0)]\\&-2M^2f_1(0)['F_2^+(0)-'F_2^-(0)]-2M^2\,'f_1(0)[F_2^+(0)-F_2^-(0)]-4M^2['f_3(0)F_2^+(0)+f_3(0)'F_2^+(0)]\ .\end{aligned} \tag{C5}$$

The total contribution is in second order given by:

$$\begin{aligned}C_{T,2}^{+0}+\tfrac{1}{2}\varepsilon G_{T,1}^{+0}=&\tfrac{1}{4}f_1+\tfrac{1}{2}M^2('f_3-f_2')+M^4(''f_3-f_2''+''f_1-f_1'')\\&+[F_2^+(0)-F_2^-(0)][\tfrac{1}{2}(f_3-f_2)+2M^2('f_3-f_2')+M^2('f_1-f_1')]\\&+M^2[F_2^{+'}(0)-F_2^{-'}(0)](f_1-2f_2)-M^2['F_2^+(0)-'F_2^-(0)](f_1-2f_3)\\&+M^2F_2^+(0)('f_1-f_1'+2f_2'-2'f_3)+\tfrac{1}{2}f_1F_2^+(0)+2M^2[f_2F_2^{+'}(0)-f_3'F_2^+(0)]-M^2f_1[F_2^{+'}(0)-F_2^{-'}(0)].\end{aligned} \tag{C6}$$

Note that we used

$$f_i(0)=f_i+O(\varepsilon^2)\ . \tag{C7}$$

Proper behavior under space and time reversal[28] yields the following relations for the form factors at the on-shell points:

$$f_2=f_3,\quad f_2'='f_3,\quad f_2''=''f_3,\quad f_1'='f_1,\quad f_1''=''f_1,\quad F_2^{+'}(0)='F_2^+(0),\quad F_2^{-'}(0)='F_2^-(0)\ . \tag{C8}$$

If we insert these relations in Eq. (C6) then we find the result quoted in Sec. III because all model-dependent terms cancel. Alternatively one can immediately use these relations in the expressions for C and G separately. This yields a vanishing C amplitude while the G amplitude gives the final result. Note that this amplitude is the only one which is *not* constrained by weak gauge invariance.

---

[1]E. Mazzucato *et al.*, Phys. Rev. Lett. **57**, 3144 (1986).
[2]R. Beck, University of Mainz Dissertation, 1989; B. Schoch, Proceedings of the Elba Workshop, 1987 (to be published).
[3]D. Drechsel, in *Proceedings of the XXIII Yamada Conference on Nuclear Weak Processes and Nuclear Structure,* Osaka, Japan, 1989 (to be published).
[4]L. Tiator and D. Drechsel, in Proceedings of the 12th International Conference on the Few Body Problem, Vancouver, Canada, 1989 (to be published).
[5]R. Dahm and S. Scherer, in *Proceedings of the International Workshop of Nuclear Physics, XII,* Erice, Italy, 1989 (to be published).
[6]S. N. Yang, contributed paper G23 at the Proceedings of the 12th International Conference on the Few Body Problem (Ref. 4).
[7]L. M. Nath and S. K. Singh, Phys. Rev. C **39**, 1207 (1989).
[8]S. Nozawa, T.-S. H. Lee, and B. Blankleider, TRIUMF report, 1989 (unpublished).
[9]M. Araki, Phys. Lett. B **219**, 135 (1989).
[10]N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1953).
[11]P. de Baenst, Nucl. Phys. **B24**, 633 (1970).
[12]A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. **B36**, 589 (1972).
[13]M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).
[14]Y. Takahashi, Nuovo Cimento **6**, 371 (1957).
[15]M. S. Bhatia and P. Narayanaswamy, Phys. Rev. **172**, 1742 (1968).
[16]S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40**, 1171 (1965).
[17]G. M. Radutskii, Yad. Fiz. **41**, 995 (1985) [Sov. J. Nucl. Phys. **41**, 639 (1985)].
[18]R. D. Peccei, Phys. Rev. **181**, 1902 (1969).
[19]J. F. Mathiot, Phys. Rep. **173**, 63 (1989).
[20]G. Furlan, N. Paver, and C. Verzegnassi, Nuovo Cimento, **20**, 295 (1974).
[21]J. T. MacMullen and M. D. Scadron, Phys. Rev. D **20**, 1069 (1971); *ibid.* **20**, 1081 (1971).
[22]D. Drechsel and L. Tiator, Phys. Lett. **148B**, 413 (1984).
[23]S. Scherer, D. Drechsel, and L. Tiator, Phys. Lett. B **193**, 1

(1987).
[24]J. F. Donoghue and B. R. Holstein, Phys. Rev. D **40**, 2378 (1989).
[25]A. A. Chumbalov, S. S. Kamalov, and T. V. Tetereva, Phys. Lett. B **213**, 255 (1988).
[26]H. Primakoff, Phys. Rev. **81**, 899 (1951).
[27]M. G. Olsson and E. T. Osypowski, Phys. Rev. D **17**, 174 (1978).
[28]A. M. Bincer, Phys. Rev. **118**, 855 (1960).
[29]S. D. Drell and H. R. Pagels, Phys. Rev. **140**, B397 (1965).
[30]H. W. L. Naus and J. H. Koch, Phys. Rev. C **36**, 2459 (1987).
[31]E. Kazes, Nuovo Cimento, **13**, 1226 (1959).
[32]G. Barton, *Dispersion Techniques in Field Theory* (Benjamin, New York, 1965), Chap. 8.
[33]R. D. Peccei, Phys. Rev. **176**, 1812 (1968).
[34]G. M. Radutsky and V. A. Serdjutsky, Nucl. Phys. **B54**, 302 (1973).
[35]L. M. Nath and B. K. Bhattacharyya, Z. Phys. C **5**, 9 (1980).
[36]H. W. L. Naus and J. H. Koch, Phys. Rev. C **39**, 1907 (1989).
[37]K. Ohta, Phys. Rev. C **40**, 1335 (1989).
[38]G. W. Gaffney, Phys. Rev. **161**, 1599 (1967).
[39]S. L. Adler, Phys. Rev. **139**, B1638 (1965).
[40]M. Ericson and M. Rho, Phys. Rep. **5C**, 57 (1972).
[41]G. S. Ball, Phys. Rev. **124**, 2014 (1961).
[42]G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).
[43]F. Gross and D. O. Riska, Phys. Rev. C **36**, 1928 (1987).
[44]L. F. Li and H. Pagels, Phys. Rev. Lett. **26**, 1204 (1971).
[45]J. L. Friar and E. L. Tomusiak, Phys. Lett. **122B**, 11 (1983).