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Predictions of the Paris N-N potential for three-nucleon continuum observables: Comparison of two approaches

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Two different approaches for solving three-nucleon scattering are compared. The one relies on separable subsystem interactions, while the other one can use any two-nucleon potential and solves the three-body integral equations directly. Both approaches are tested in the case of a finite-rank expansion of the Paris N-N potential (the so-called PEST) and the results are then compared to the predictions calculated straight from the original interaction. By considering cross sections and polarization observables of elastic n-d scattering at $E_n = 10$ MeV, it is found that both methods lead to compatible results. Where appropriate a comparison is also made to experimental data showing a remarkably good agreement between theory and experiment.

I. INTRODUCTION

Over many years the theoretical treatment of the three-nucleon ($3N$) continuum with realistic N-N forces has been a great challenge. In addition to the ^{3}H and ^{3}He bound states, $3N$ scattering processes provide a wealth of further observables making it possible to study many more aspects of the $3N$ problem. Above all there is the primary question of the performance of present-day N-N interactions in the $3N$ system. In particular, one is most interested in whether or not the meson-exchange theory of nuclear forces, which in the $2N$ system is capable of reproducing the numerous experimental data, can also fully describe the more complex $3N$ system. Especially one would also like to know, if $3N$ forces play a sensible role. Clearly all these questions can only be answered along with an exact solution of the three-body equations employing the most-advanced N-N potentials.

While $3N$ bound-state observables could be calculated to great accuracy for quite some time,[1] the $3N$ continuum problem witnessed a major advance only recently. The solution of the $3N$ scattering with realistic forces has become possible by following two different approaches.

(i) The first one (we call it *approach A*) consists in designing a separable representation of a given N-N interaction V via an on-shell and off-shell-equivalent finite-rank expansion;[2] this is subsequently used as input for the three-body Faddeev equations, which then result in a coupled system of one-dimensional integral equations to be solved in one of the established ways.[3]

(ii) The second one (*approach B*) consists in applying the given N-N potential V directly in the three-body Faddeev equations, what amounts to numerically solving a coupled system of two-dimensional integral equations without resort to any approximations.[4] Clearly it is now most interesting to estimate the reliability of these two approaches. The purpose of the present paper is to provide a quantitative comparison between them by means of the Paris potential predictions for elastic N-d scattering.

For our comparison we proceed in two steps. In the following Sec. II we first confront formalism and technique of approach A to the one of approach B by keeping the N-N interaction in both calculations strictly the same (namely, a separable representation of the Paris potential); this sheds light on the exactness of the two methods. Next, we compare the results of Sec. II to the predictions of the original Paris potential obtained from approach B; thereby we get insight into the reliability of the finite-rank expansions used (Sec. III). It stands to reason that ideally all these results should agree to a satisfactory accuracy. Only then both approaches A and B can be considered as reliable and lead to the true predictions of the underlying N-N interaction. We discuss this in Sec. IV.

II. APPROACH A VS APPROACH B UNDER THE SAME DYNAMICAL INPUT

We now want to compare the results from approaches A and B for the very same N-N interaction. For this purpose we must evidently apply the separable representation obtained for the Paris potential.[5] Following the work of Haidenbauer and Plessas[6] we make use of the Ernst-Shakin-Thaler (EST) method.[7] In the present investigation we in particular employ the separable representation specified by the interpolation energies as given in Table I. This defines the ranks of the separable expansion in each channel; they are, in fact, the same as in Ref. 8. Throughout all partial waves the *a priori* numerical EST form factors are expressed by the analytical forms as already introduced before,[8] i.e.,

TABLE I. Finite-rank approximation of the Paris potential (PEST) used.

Partial wave	Rank	Selected interpolation energies E_i (MeV) or ensembles $\alpha_i=\{E_i l_i\}$; see also Refs. 6 and 8.
${}^1s_0$	3	$E_1=0,\ E_2=100,\ E_3=500$
${}^1p_1, {}^3p_0, {}^3p_1, {}^1d_2, {}^3d_2$	2	$E_1=10,\ E_2=-50$
${}^3s_1-{}^3d_1$	6	$\alpha_1=\{-2.2249,-\},\ \alpha_2=\{125,2\},\ \alpha_3=\{100,0\}$ $\alpha_4=\{425,2\},\ \alpha_5=\{-50,0\},\ \alpha_6=\{-50,2\}$
${}^3p_2-{}^3f_2$	3	$\alpha_1=\{75,1\},\ \alpha_2=\{175,3\},\ \alpha_3=\{300,1\}$

$$g_{li}(p)=\frac{p^l}{(p^2+\beta_{li}^2)^{l+\nu_{li}}}[C_{li}+p^2 y_{li}(p)]\ , \quad (i=1,\ldots,N)\ . \tag{1}$$

Here N is the rank of the approximation, and the parameters C, β, and ν are fixed so as to exactly reproduce the threshold behavior of the numerical form factor. For the remaining term in Eq. (1) the defect function $y_{li}(p)$ is most conveniently represented via a series of Gegenbauer polynomials. Such an expansion is feasible after performing the mapping

$$p^2=\gamma_{li}\frac{1+t}{1-t},\quad t\in[-1,1]\ . \tag{2}$$

The actual values of the parameters occurring in Eqs. (1) and (2) as well as a subroutine for calculating $g_{li}(p)$ can be obtained from the authors upon request.

We emphasize that the EST method involves no open parameters. Only the analytical representation of the pertinent form factors, which originally result in numerical form, require a suitable parametrization. Closed expressions for the separable form factors are needed in our calculation[3] along approach A to perform a contour deformation in solving the integral equations.

In approach A the so-obtained separable representation of the Paris potential (PEST) is used to rewrite the three-body Faddeev equations, in the Alt-Grassberger-Sandhas (AGS) form,[9] as a coupled system of one-dimensional integral equations. Approach B does not exploit this possibility but employs the separable interactions, just as any other arbitrary potential, in solving the two-dimensional integral equations.[10] Evidently the structure of the dynamical equations to be solved in the two approaches is totally different. In addition, the singularities arising from the free nucleon propagator are completely avoided in approach A by performing a contour rotation into the complex plane, while they are treated by a subtraction method and a spline interpolation in approach B. In both calculations all N-N partial waves up to $j\le 2$ are included. For the evaluation of the $3N$ scattering observables three-body amplitudes up to total angular momentum $J=17/2$ and $J=25/2$ have been used in approaches B and A, respectively; it was already shown in Ref. 4 that for the observables considered here a convergent result is practically achieved with $J=17/2$.

Table II gives the results of integrated n-d cross sections at $E_n=10$ MeV. At this instance we compare the figures in the first two columns (same interaction PEST). The agreement between the two approaches A and B is quite good, since the differences are at most 0.4%.

Table III shows the results for the differential cross section of elastic n-d scattering at $E_n=10$ MeV. Though at extreme forward and backward angles there are differences of about 1%, the agreement is again very satisfactory. The same information can also be read off from Fig. 1, where it is seen that the crosses (approach A) essentially coincide with the curve (approach B).

A similar good agreement shows up also in the various spin observables for elastic n-d scattering. As examples we show the three deuteron tensor polarizations T_{20}, T_{21}, and T_{22} in Figs. 2–4. Only in the particular case of T_{20} at very forward angles $\theta<40°$, where this spin observable takes rather small values, there remains a certain discrepancy which is probably due to an insufficient number of mesh points used in approach B.

From the preceding comparison we may thus conclude that the two approaches with their independently developed codes for solving the three body equations practically yield identical results (for one and the same N-N interaction), within an accuracy of 1–2 % (except in the above-mentioned case of T_{20} at small scattering angles). About 1% is also the uncertainty estimated for each of the two computer programs.

TABLE II. Integrated total, elastic, and reaction cross sections for n-d scattering at $E_n=10.0$ MeV as obtained from the different approaches.

	PEST Approach A	PEST Approach B	Paris potential Approach B	Experiment Ref. 11
σ_{tot}(mb)	1043.1	1043.1	1043.7	1055±10
σ_{el}(mb)	900.6	900.0	900.0	
σ_{react}(mb)	142.5	143.1	143.6	

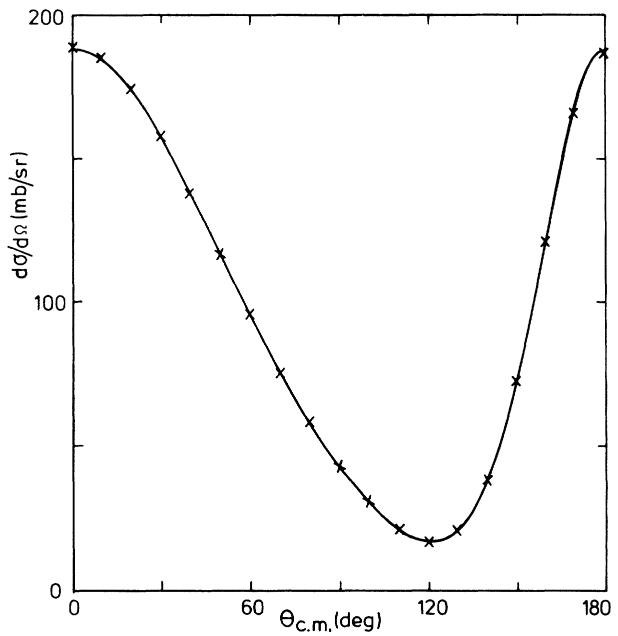

FIG. 1. Differential cross section for elastic n-d scattering at $E_n = 10$ MeV in case of the PEST interaction for approaches A (crosses) and B (solid line).

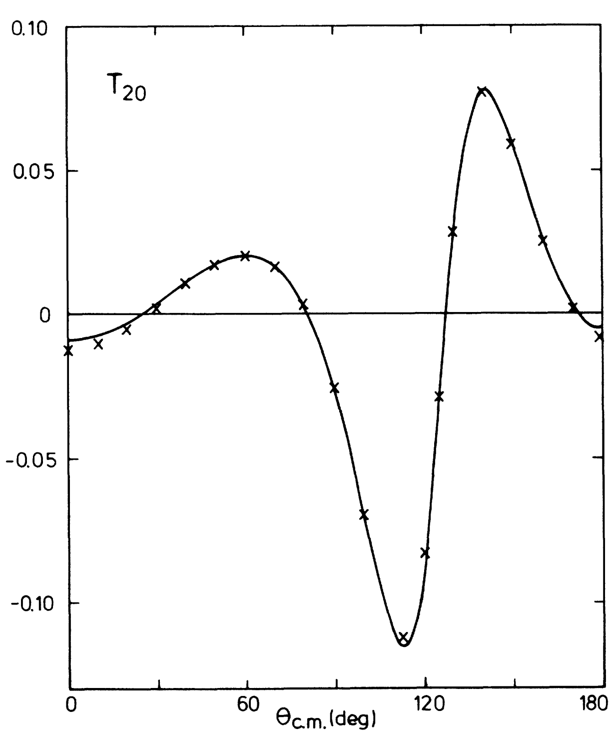

FIG. 2. Deuteron tensor polarization T_{20} for elastic n-d scattering at $E_n = 10$ MeV. Same description as in Fig. 1.

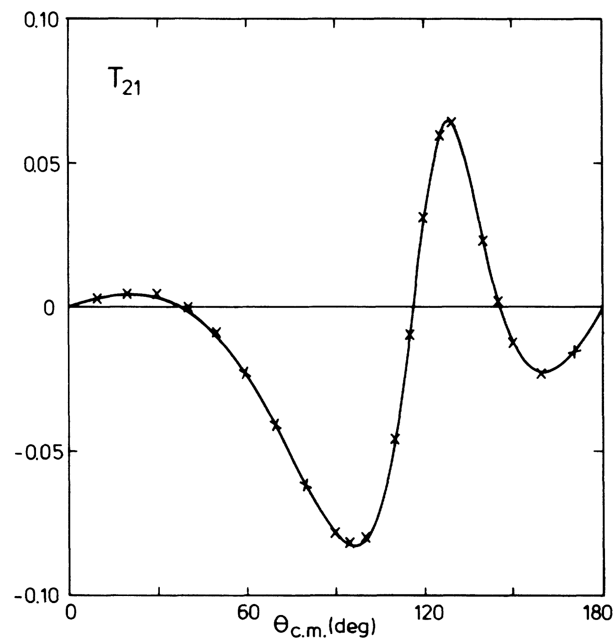

FIG. 3. Deuteron tensor polarization T_{21} for elastic n-d scattering at $E_n = 10$ MeV. Same description as in Fig. 1.

III. PREDICTIONS FROM PEST AND THE ORIGINAL PARIS POTENTIAL

In the next step we aim at examining the reliability of the separable-expansion method. Within approach B we compare the results obtained from PEST (in the previous section) with the ones from the original Paris potential.[5] As before, both interactions are taken to act in all N-N partial waves $j \leq 2$.

Again we first regard the integrated n-d cross sections in Table II. Comparing the figures in the second and third columns we notice that there is essentially no

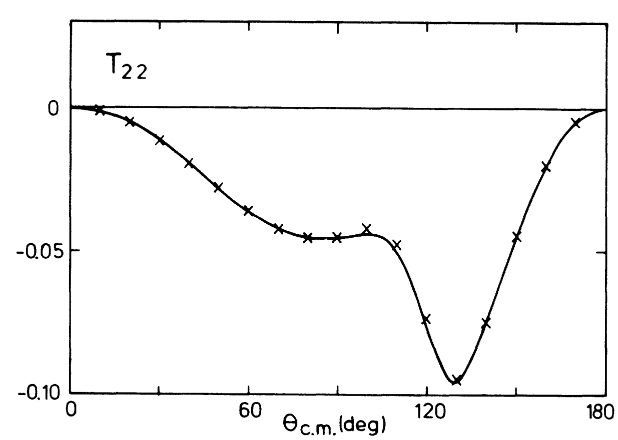

FIG. 4. Deuteron tensor polarization T_{22} for elastic n-d scattering at $E_n = 10$ MeV. Same description as in Fig. 1.

TABLE III. Differential cross section for n-d elastic scattering at $E_n = 10.0$ MeV as obtained from the different approaches.

	$d\sigma/d\Omega$(mb/sr)			
	PEST		Paris potential	
Θ_{cm}	Approach A	Approach B	Approach B	Experiment
0	189.32	188.69	188.24	
15	180.67	180.10	179.70	
30	157.59	157.18	156.92	
45	126.86	126.59	126.54	
60	95.24	95.00	95.11	
75	66.84	66.61	66.80	
90	43.27	43.03	43.22	
105	25.54	25.34	25.49	
120	17.22	17.17	17.24	
135	28.19	28.35	28.26	
150	72.84	73.74	73.44	
165	145.43	147.41	146.95	
170	166.53	168.68	168.19	170.3 ± 2.0[a]
172.5	174.71	176.91	176.41	174.7 ± 2.1[b]
180	185.89	188.15	187.65	187.2 ± 3.5[c]

[a]Measurement at $\Theta_{cm} = 170.5°$ by the Uppsala group (Ref. 12).
[b]Measurement at $\Theta_{cm} = 172.1°$ by the Uppsala group (Ref. 12).
[c]Extrapolated value (Ref. 12).

significant difference between the calculations with PEST and the original Paris potential (within approach B). Together with our comparison in the previous sections this means that with respect to integrated cross sections the PEST finite-rank expansions are completely reliable as are the solutions of the Faddeev equations within both approaches A and B. We may also note that the so obtained Paris potential predictions agree well with experiment (last column of Table II); cf. also the integrated total cross sections at other energies given in Ref. 8.

The same arguments hold for the differential cross sections of elastic n-d scattering given in Table III. Here too all results are in convincing agreement. Also the backward experimental data can be well reproduced by the Paris potential; thus the long-standing problem of theoretical results undershooting experiments appears to be resolved. In this connection we should, however, mention that the Paris potential predictions stay at variance with the measurement of the Karlsruhe group,[13] above all

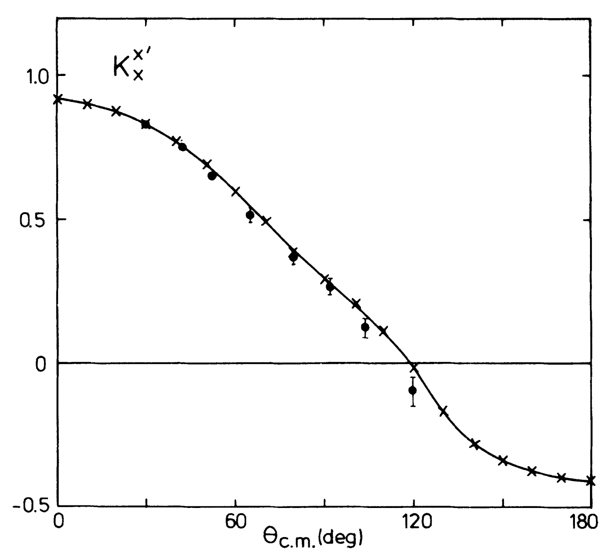

FIG. 5. Nucleon-to-nucleon spin-transfer coefficient $K_x^{x'}$ for elastic N-d scattering at $E_N = 10$ MeV. The theoretical results are calculated within approach B for the Paris (solid line) and PEST (crosses) interactions; they refer to n-d scattering, while the experimental data (Ref. 15) are for p-d.

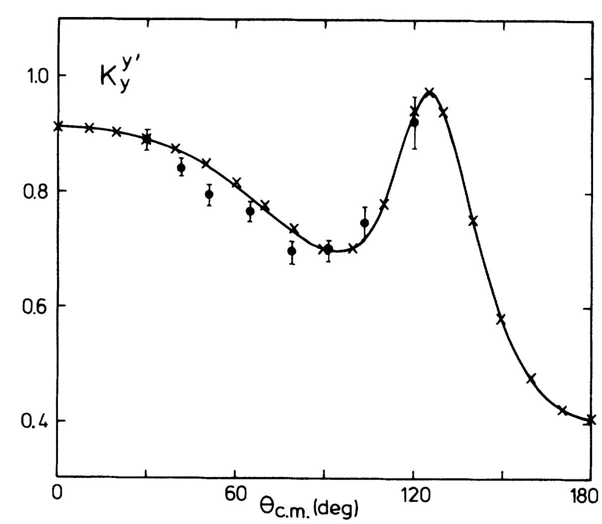

FIG. 6. Nucleon-to-nucleon spin-transfer coefficient $K_y^{y'}$ in elastic N-d scattering at $E_N = 10$ MeV. Same description as in Fig. 5.

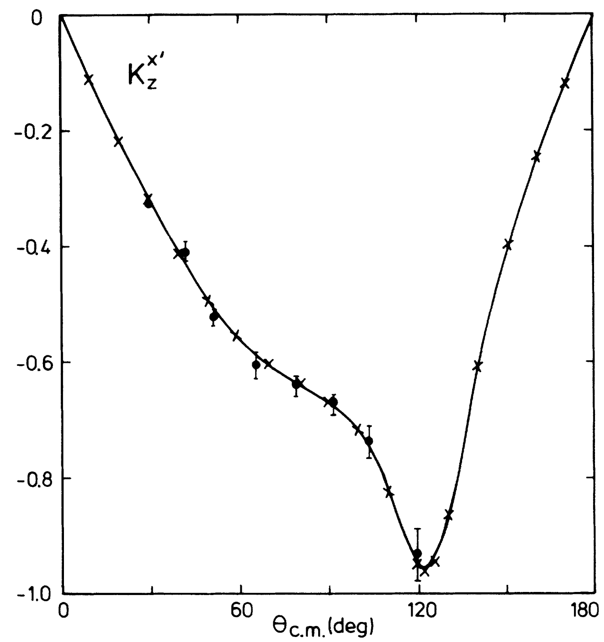

FIG. 7. Nucleon-to-nucleon spin-transfer coefficient $K_z^{x'}$ in elastic N-d scattering at $E_N=10$ MeV. Same description as in Fig. 5.

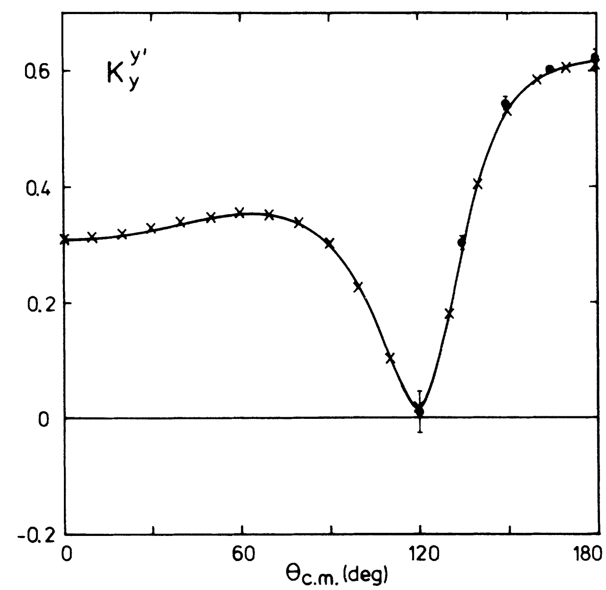

FIG. 9. Nucleon-to-deuteron spin-transfer coefficient $K_y^{y'}$ in elastic N-d scattering at $E_N=10$ MeV. Same description as in Fig. 8.

in the backward-scattering domain. This is also true at neighboring energies.[8] There was a subsequent repetition of the Karlsruhe experiment at higher energies,[14] and it turned out that at least at the energy of $E_n=20$ MeV the corresponding new data are in good agreement with the Paris potential result.[8] Consequently it seems that the former data set is not completely reliable and it would be desirable to have further measurements, especially in view of the obvious sensitivity of the backward-scattering regime to details of the N-N force, here the deuteron wave function (cf. the findings in Ref. 3).

Finally we regard for our comparison several spin observables, for which rather accurate data exist at $E_N=10$ MeV; these are the spin-transfer coefficients depicted in Figs. 5–10. Of course, the experimental data are for p-d scattering, while our calculations are for the uncharged case n-d, but it is expected that at our energy Coulomb effects are negligibly small in these particular observables.[17] The results given in Figs. 5–10 confirm the reliability of the Paris potential predictions obtained with approaches A and B. At the same time a convincing agreement with experiment is found.

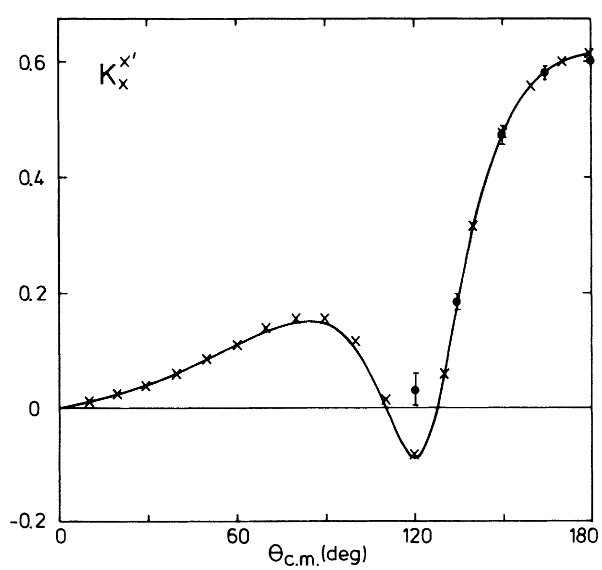

FIG. 8. Nucleon-to-deuteron spin-transfer coefficient $K_x^{x'}$ in elastic N-d scattering at $E_N=10$ MeV. Same description as in Fig. 5. Experimental data are from Ref. 16.

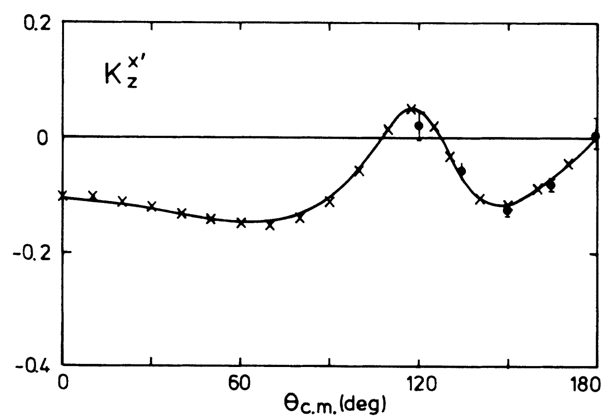

FIG. 10. Nucleon-to-deuteron spin-transfer coefficient $K_z^{x'}$ in elastic N-d scattering at $E_N=10$ MeV. Same description as in Fig. 8.

IV. CONCLUSIONS

We have compared two different approaches for calculating $3N$ scattering observables for present-day realistic N-N interactions. By the consideration of elastic n-d scattering at $E_n = 10$ MeV we have found that both approaches, though conceptually rather different, yield results that agree within less than 2%. In view of the complexity involved in the corresponding calculations and numerical algorithms this is a remarkable achievement.

In particular we have seen that the method of applying separable $2N$ t-matrices in the three-body Faddeev integral equations (approach A) yields the same results as the direct method of solving the two-dimensional integral equations right away (approach B). Either approach is thus suitable for calculating predictions of $3N$ (scattering) observables for any given interaction V. The first approach A requires to this end a correct separable representation of the corresponding off-shell N-N t-matrix. We have demonstrated that in the case of the Paris potential the separable PEST interactions fully meet this requirement. Their use allows to deduce in a reliable way predictions of $3N$ observables for the true Paris potential. The method of finite-rank expansion thus appears to be quite efficient.

In the present paper we have considered cross sections and several polarization observables for elastic n-d scattering and found good agreement in all places. Of course, it might happen that for even other observables, at other energies, or for breakup reactions larger differences between the two approaches can show up. If such a finding affected the separable PEST representations, they would have to be further refined to include additional interpolation energies (higher ranks).

With respect to the PEST representations we may add that already the result obtained with them for the ^3H binding energy[18] was in perfect agreement with the value calculated from the original Paris potential by the Hanover group.[1] In this light the above investigation of $3N$ scattering observables further supports the reliability of the PEST interactions.

Finally it should be noted that by using the PEST representations within approach A also the p-d elastic scattering problem with inclusion of the long-range Coulomb force can now practically be solved and, indeed, this was quite recently pushed into a promising stage. In particular the Graz group succeeded in calculating p-d (polarization) observables below breakup threshold from a simplified (rank-1) EST approximation of the Paris potential.[19] By taking into account in addition to $^1s_0$ and $^3s_1-{}^3d_1$ also all N-N p-waves a satisfactory description of all p-d data in the low-energy domain could be achieved.

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