Projection operator in the boson expansion techniques

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The projection operator onto the physical space is investigated in the boson expansion techniques. An explicit expression for the projection operator contains many body terms. From this expression, recurrent relations are obtained. Based on the properties of the projection operator, a new general form of boson mapping is derived.

I. INTRODUCTION

In recent years, considerable effort has been devoted to the study of methods that map many-fermion problems onto many-boson ones. First introduced in the theory of spin waves,[1,2] the concept of the boson representation of the bifermion operators has been extended to general fermion systems.[3–5] Several approaches have been suggested to develop boson representation techniques.[6–14]

To define a boson mapping, an ideal boson space is introduced. The physical boson states, i.e., those that are in one-to-one correspondence with the fermion states, span a subspace of the ideal boson space. Therefore, when working in the entire boson space instead of the physical subspace, the problem of determining and disentangling physical and nonphysical states should be solved. The nonphysical states cannot easily be identified and removed except in certain special cases. Large effort has been devoted to solving this problem.[7,11–13] Especially, questions arise if the original Hilbert space is truncated to a collective subspace. Moreover, consideration of the whole ideal boson space implies a diagonalization of matrices greater than needed for the physical problem actually.

In this respect, the projection operator onto the physical boson space is of considerable importance. The projection operator was written as the Taylor expansion in terms of the boson creation and annihilation operators.[5] However, it is rather prohibitive to apply this form to a general case.

In the present paper, we investigate the projection operator in more detail. Despite the fact that the obtained results are still rather formal and difficult to use in a general case, they could perhaps be useful in attempts at getting approximate forms of the projection operator. Using the properties of the projection operator, we derive a new general form of the boson mapping.

II. PROJECTION OPERATOR

Let us look for the boson operator $\exp(-\hat{A})$ (Ref. 15) obeying the relation

$$\exp(-\hat{A})b_{st}^{\dagger}=B_{st}^{\dagger}\exp(-\hat{A}) \ , \tag{2.1}$$

where $b_{st}^{\dagger}=-b_{ts}^{\dagger}$ is the ideal boson creation operator and

$$B_{st}^{\dagger}=b_{st}^{\dagger}-\sum_{uv}b_{su}^{\dagger}b_{tv}^{\dagger}b_{uv} \ .$$

One finds that the following identity holds

$$B_{st}^{\dagger}=b_{st}^{\dagger}-[\hat{F},b_{st}^{\dagger}] \tag{2.2}$$

with

$$\hat{F}^{\dagger}=\tfrac{1}{4}\sum_{stpq}b_{st}^{\dagger}b_{pq}^{\dagger}b_{sp}b_{tq} =\tfrac{1}{4}(2\hat{N}-\hat{N}^{2}+\hat{S}) \ .$$

In the above, $\hat{N}=\sum_{st}b_{st}^{\dagger}b_{st}$, and $\hat{S}$ is the operator defined by Park[13] as a combination of the linear and quadratic Casimir operators of the group $U(k)$ with the generators $\hat{\rho}_{st}=\sum_u b_{tu}^{\dagger}b_{su}$ (k being the dimension of the original fermion space).

From Eq. (2.1) one easily obtains

$$\exp(-\hat{A})\hat{N}=\sum_{st}B_{st}^{\dagger}\exp(-\hat{A})b_{st} \ . \tag{2.3}$$

Moreover, multiplying Eq. (2.1) from the left by b_{st}, commuting b_{st} and $B_{st}^{\dagger}$, and summing over s and t, we have

$$[k+\hat{N}(2k+\hat{N}-3)-\hat{S}]\exp(-\hat{A})=\sum_{st}b_{st}\exp(-\hat{A})b^{\dagger}{}_{st} \ . \tag{2.4}$$

Equation (2.1) determines the operator $\exp(-\hat{A})$ up to a normalization factor, which is fixed by

$$\langle 0|\exp(-\hat{A})|0\rangle=1 \ . \tag{2.5}$$

Let us calculate matrix elements of Eq. (2.3) in the ideal boson basis specified by the $U(k(k-1)/2)\supset U(k)$ reduction. Then, the states are characterized by the total boson number n, the Young partition $[f]$ of the group $U(k)$, and additional quantum numbers a. Of course, such states are eigenstates of the operators $\hat{N}$ and $\hat{F}$ with the eigenvalues $2n$ and $\frac{1}{4}[C_{U(k)}^{2}-2n(1-k)]$, respectively. A formula for the eigenvalues of the quadratic Casimir operator $C_{U(k)}^{2}$ can be found in Ref. 13. To further treatment, the following eigenvalues of $\hat{F}$ are of importance

$$F(n[1^{2n}])=n-n^2 \tag{2.6a}$$

and

$$F(n+1[2^2,1^{2n-2}])=n-n^2+1 . \tag{2.6b}$$

The physical boson space is characterized by the partition $[1^{2n}]$.

Inserting complete sets of intermediate states, we have from Eq. (2.3)

$$2(n+1)\langle n+1[f]a|\exp(-\hat{A})|n'+1[f']a'\rangle = \sum_{stf_1f_2a_1a_2} (1-F(n+1[f])+F(n[f_1]))\langle n[f_1]a_1|\exp(-\hat{A})|n'[f_2]a_2\rangle \times\langle n+1[f]a|b_{st}^{\dagger}|n[f_1]a_1\rangle\langle n'+1[f']a'|b_{st}^{\dagger}|n'[f_2]a_2\rangle \tag{2.7}$$

Equation (2.7) represents a recurrent formula from which the matrix elements of $\exp(-\hat{A})$ can be obtained. From Eq. (2.3), it follows that

$$2n\langle 0|\exp(-\hat{A})|n[f]a\rangle=0 .$$

One deduces immediately that the matrix elements $\langle n|\exp(-\hat{A})|n'\rangle$ are zero for $n'>n$. Requiring an additional condition of hermicity of the operator $\exp(-\hat{A})$, we find that this operator does not change the total number of bosons

$$\langle n|\exp(-\hat{A})|n'\rangle=0, \quad \text{for } n\neq n' . \tag{2.8}$$

Starting from Eq. (2.5), we obtain for one-boson states

$$\langle 1[1^2]a|\exp(-\hat{A})|1[1^2]c\rangle=\delta_{ac} .$$

From Eq. (2.7), the matrix elements for two-boson states are

$$\langle 2[1^4]a|\exp(-\hat{A})|2[1^4]c\rangle=3\delta_{ac} , \tag{2.9a}$$

$$\langle 2[2^2]a|\exp(-\hat{A})|2[2^2]c\rangle=0 , \tag{2.9b}$$

and

$$\langle 2[2^2]a|\exp(-\hat{A})|2[1^4]c\rangle =\langle 2[1^4]c|\exp(-\hat{A})|2[2^2]a\rangle=0 . \tag{2.9c}$$

Let us generalize Eqs. (2.9) and assume that all matrix elements of $\exp(-\hat{A})$ are zero except for the diagonal matrix elements between the physical states characterized by the partition $[1^{2n}]$. We infer

$$\langle n[f]a|\exp(-\hat{A})|n[f']c\rangle =(2n-1)!!\delta_{[f][f']}\delta_{[f][1^{2n}]}\delta_{ac} . \tag{2.10}$$

Proving the above relation by induction, one should remember that the operator $b_{st}^{\dagger}$ carries the symmetry $[1^2]$. In Eq. (2.7), only those symmetries of the intermediate states $[f_1]$ and $[f_2]$ are allowed for which the final and initial symmetries $[f]$ and $[f']$ are contained in the decomposition of $[f_1]\otimes[1]$ and $[f_2]\otimes[1]$, respectively. Using Eqs. (2.6), Eq. (2.10) is easily proved. Perhaps, the matrix element for the partition $[f]=[2^2,1^{2n-2}]$, which is connected by the operator $b_{st}^{\dagger}$ with the physical intermediate state $[f_1]=[1^{2n}]$, requires a special note. Then, the difference of $1-F(n+1[f])+F(n[f_1])=0$ appears in Eq. (2.7).

We have thus shown that the Hermitian operator $\exp(-\hat{A})$ obeying Eq. (2.3) is determined uniquely and that it has the matrix elements given by Eqs (2.8) and (2.10). We still need to prove that the operator $\exp(-\hat{A})$ really fulfills Eq. (2.1). Calculating matrix elements of Eq. (2.1), one gets

$$\langle n[f]a|b_{st}^{\dagger}|n'[f']a'\rangle\langle n[f]a|\exp(-\hat{A})|n[f]a\rangle =[1-F(n[f]a)+F(n'[f']a')]\langle n[f]a|b_{st}^{\dagger}|n'[f']a'\rangle\langle n'[f']a'|\exp(-\hat{A})|n'[f']a'\rangle ,$$

which is seen to be true for every s and t.

The operator $\exp(-\hat{A})$ has the property

$$\exp(-\hat{A})|n[f]a\rangle=\delta_{[f][1^{2n}]}(2n-1)!!|n[f]a\rangle .$$

The operator

$$\hat{P}=\frac{1}{(\hat{N}-1)!!}\exp(-\hat{A}) \tag{2.11}$$

is therefore the projection operator onto the physical space.

Multiplying Eq. (2.10) from the right by $1/(\hat{N}-1)!!$ and commuting the $\hat{N}$ dependent factors with the operators $\exp(-\hat{A})$ and $b_{st}^{\dagger}$ with help from $[\exp(-\hat{A}),\hat{N}]=0$ and $[b_{st}^{\dagger},\hat{N}]=-b_{st}^{\dagger}$, one gets

$$B_{st}^{\dagger}\hat{P}=\hat{P}b_{st}^{\dagger}(\hat{N}+1) . \tag{2.12}$$

From Eq. (2.3), we obtain

$$\sum_{st} B_{st}^{\dagger}\hat{P}b_{st}=\hat{P}\hat{N}(\hat{N}-1) . \tag{2.13}$$

In fact, Eq. (2.12) has been derived previously in Ref. 5. The above treatment thus proves again that the projection operator $\hat{P}$ obeys this equation. Moreover, we have shown that Eq. (2.12) and/or the consequent Eq. (2.13) determine the projection operator uniquely.

We decompose the projection operator $\hat{P}$ into components acting in the space with a given total boson number n $\hat{P}=\sum_{n=0}\hat{P}_n$, $\hat{P}_n=\hat{P}_n\hat{\mathcal{P}}_n$, where $\hat{\mathcal{P}}_n$ denotes the projection operator onto states with n bosons. It follows from Eq. (2.13) that

$$\hat{P}_n=\frac{1}{\hat{N}(\hat{N}-1)}\sum_{st}B^\dagger_{st}\hat{P}_{n-1}b_{st}\ . \tag{2.14}$$

Starting from $\hat{P}_0=1$ and $\hat{P}_1=1$, we get

$$\hat{P}_2=\frac{1}{\hat{N}(\hat{N}-1)}\sum_{st}B^\dagger_{st}b_{st}=1-\tfrac{1}{12}\hat{S}\ . \tag{2.15}$$

Equation (2.15) is easy to understand as there are only two possible $[f]$ partitions in the $n=2$ space, namely the physical one $[1^4]$ and the nonphysical one $[2^2]$ for which the eigenvalues of the operator $\hat{S}$ are 0 and 12, respectively. From the recurrence (2.14), one has generally

$$\hat{P}_n=\frac{2}{(2n)!}\sum_{\substack{s_1,\ldots,s_{n-1}\\ t_1,\ldots,t_{n-1}}}B^\dagger_{s_1t_1}\cdots B^\dagger_{s_{n-1}t_{n-1}}b_{s_1t_1}\cdots b_{s_{n-1}t_{n-1}}\ . \tag{2.16}$$

After rearranging Eq. (2.16) in the normal ordered form, the projection operator $\hat{P}_n$ onto the physical space with a given total boson number n is given as a combination of $(n-1)-$ and n-body operators.

In fact, Eq. (2.16) could also be derived from a representation of the projection operator given in Ref. 5:

$$\hat{P}=\sum_{n=0}\frac{1}{(2n)!}\sum_{\substack{s_1,\ldots,s_n\\ t_1,\ldots,t_n}}B^\dagger_{s_1t_1}\cdots B^\dagger_{s_nt_n}|0\rangle\langle 0|b_{s_1t_1}\cdots b_{s_nt_n}\ ,$$

from which

$$\hat{P}_n=\frac{1}{(2n)!}\sum_{\substack{s_1,\ldots,s_n\\ t_1,\ldots,t_n}}B^\dagger_{s_1t_1}\cdots B^\dagger_{s_nt_n}|0\rangle\langle 0|b_{s_1t_1}\cdots b_{s_nt_n}$$

$$=\frac{1}{(2n)!}\sum_{\substack{a\geq b\\ a_1\geq b_1}}\sum_{\substack{s_1,\ldots,s_n\\ t_1,\ldots,t_n}}B^\dagger_{s_1t_1}\cdots B^\dagger_{s_{n-1}t_{n-1}}|ab\rangle\langle ab|B^\dagger_{s_nt_n}b_{s_nt_n}|a_1b_1\rangle\langle a_1b_1|b_{s_1t_1}\cdots b_{s_{n-1}t_{n-1}}\ ,$$

with $|ab\rangle$ and $|a_1b_1\rangle$ being complete sets of the one-boson states. Using

$$\sum_{s_nt_n}\langle ab|B^\dagger_{s_nt_n}b_{s_nt_n}|a_1b_1\rangle=2\delta_{aa_1}\delta_{bb_1}$$

in the above, we obtain Eq. (2.16) again.

III. EXPONENTIAL FORM OF PROJECTION OPERATOR

As the operator $\exp(-\hat{A})$ is diagonal in the $U(k)$ basis, it should be expressed through the Casimir operators of the $U(k)$ group. Suppose that we have constructed Hermitian operators $\hat{G}_n$ with the properties

$$[\hat{G}_n,b^\dagger_{st}]=(b^\dagger\hat{\rho}^n)_{st}\ . \tag{3.1}$$

For instance, from Eq. (2.2) it follows that $\hat{G}_1=\hat{F}$. Using (3.1), we write

$$\exp\left[-\frac{1}{n}\hat{G}_n\right]b^\dagger_{st}\exp\left[\frac{1}{n}\hat{G}_n\right]=\left[b^\dagger\exp\left[-\frac{1}{n}\hat{\rho}^n\right]\right]_{st}$$

and

$$\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\hat{G}_n\right]b^\dagger_{st}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\hat{G}_n\right]=\left[b^\dagger\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\hat{\rho}^n\right]\right]_{st}$$
$$=\{b^\dagger\exp[\ln(1-\hat{\rho})]\}_{st}=[b^\dagger(1-\hat{\rho})]_{st}\ . \tag{3.2}$$

Moreover, we can prove that

$$\exp\left[-\frac{1}{n}\hat{G}_n\right]\hat{\rho}_{st}\exp\left[\frac{1}{n}\hat{G}_n\right]=\hat{\rho}_{st}\ . \tag{3.3}$$

Indeed, $[\hat{G}_n,\hat{\rho}]=0$. Comparing Eqs. (2.1) and (3.2), we find that

$$\hat{A}=\sum_{n=1}^{\infty}\frac{1}{n}\hat{G}_n\ . \tag{3.4}$$

Of course, in the above procedure, Eq. (3.2) should be understood in a formal manner. The inverse operator $\exp(\hat{A})$ to the operator $\exp(-\hat{A})$ is unbounded in the total boson space.

Let us try to construct the operators $\hat{G}_n$ obeying Eq. (3.1). First of all, we find the commutators

$$[\hat{\rho}_{st},b^\dagger_{uv}]=\delta_{su}b^\dagger_{tv}-\delta_{sv}b^\dagger_{tu} \quad (3.5)$$

and

$$[\hat{\rho}^2_{st},b^\dagger_{uv}]=2\delta_{st}b^\dagger_{uv}-\delta_{sv}b^\dagger_{ut}+\delta_{su}b^\dagger_{vt}+b^\dagger_{ut}\hat{\rho}_{sv} -b^\dagger_{vt}\hat{\rho}_{su}+(b^\dagger\hat{\rho})_{ut}\delta_{sv}-(b^\dagger\hat{\rho})_{vt}\delta_{su} \ . \quad (3.6)$$

Looking at Eqs. (3.5) and (3.6), we infer that in the general case

$$[\hat{\rho}^n_{st},b^\dagger_{uv}]=\sum_{m=0}^{n-2}\sum_{k=0}^{m}\beta^n_{mk}(b^\dagger\hat{\rho}^k)_{uv}\hat{\rho}^{m-k}_{st} +\sum_{m=0}^{n-1}\sum_{k=0}^{m}\alpha^n_{mk}[(b^\dagger\hat{\rho}^k)_{ut}\hat{\rho}^{m-k}_{sv} -(b^\dagger\hat{\rho}^k)_{vt}\hat{\rho}^{m-k}_{su}] \ . \quad (3.7)$$

Using Eq. (3.7) and the additional relations

$$(b^\dagger\hat{\rho}^n)_{st}=-(b^\dagger\hat{\rho}^n)_{ts} \ ,$$

$$[\hat{\rho}^n_{st},\hat{\rho}^m_{uv}]=\sum_{k=0}^{n-1}(\hat{\rho}^{m+n-k-1}_{ut}\hat{\rho}^k_{sv}-\hat{\rho}^k_{ut}\hat{\rho}^{m+n-k-1}_{sv}) \ ,$$

we evaluate the commutator $[\hat{\rho}^{n+1}_{st},b^\dagger_{uv}]$. Putting the result in the form (3.7), we obtain the recurrent relations for α^n_{mk} and β^n_{mk}

$$\beta^{n+1}_{mk}=(1-\delta_{mk})(1-\delta_{m0})\beta^n_{m-1,k}+2\delta_{mk}\sum_{l=0}^{m}\alpha^n_{ml} \ , \quad (3.8)$$

$$\alpha^{n+1}_{mk}=(1-\delta_{m0})(1-\delta_{k0})\alpha^n_{m-1,k-1} +(1-\delta_{nm})(1-\delta_{n-1,m})\beta^n_{mk} -(1-\delta_{nm})\alpha^n_{mk}+\delta_{k0}\delta_{nm} \ , \quad (3.9)$$

with $\alpha^1_{00}=1$.

It follows from Eq. (3.7) that

$$[\mathrm{Sp}(\hat{\rho}^n),b^\dagger_{st}]=\sum_{m=0}^{n-2}\sum_{k=0}^{m}\beta^n_{mk}(b^\dagger\hat{\rho}^k)_{st}\mathrm{Sp}(\hat{\rho}^{m-k}) +\sum_{m=0}^{n-1}2\gamma^n_m(b^\dagger\hat{\rho}^m)_{st} \ , \quad (3.10)$$

where $\gamma^n_m=\sum_{k=0}^{m}\alpha^n_{mk}$. From Eq. (3.8), we get

$$\beta^{n+1}_{mk}=2\gamma^{n-m+k}_k \ . \quad (3.11)$$

Solving Eq. (3.9), we find

$$\gamma^n_m=(-1)^{n-m+1}d_{n-m}\frac{n!}{m!(n-m)!} \ , \quad (3.12)$$

where the coefficients d_k are the solutions of the equations

$$d_k=d_{k-1}+2d_{k-2} \ ,$$

$$d_1=d_2=1 \ .$$

After a lengthy but, in principle, simple transformation, we put Eq. (3.10) into the more convenient form

$$[\hat{X}_n,b^\dagger_{st}]=\lambda^{n-1}_{st}+\sum_{l=1}^{n-2}\lambda^{n-l-l}_{st}\hat{X}_l \ , \quad (3.13)$$

where

$$\hat{X}_n=\sum_{l=1}^{n}(-k)^{n-l}\mathrm{Sp}(\hat{\rho}^l)$$

and

$$\lambda^{n-1}_{st}=\sum_{m=0}^{n-1}2\gamma^n_m(b^\dagger\hat{\rho}^m)_{st} \ .$$

For example, we find with the help of Eq. (3.13) explicitly

$$\hat{G}_1=\tfrac{1}{4}\hat{X}_2+\tfrac{1}{4}\hat{X}_1$$

$$\hat{G}_2=\tfrac{1}{6}\hat{X}_3+\tfrac{1}{4}\hat{X}_2-\tfrac{1}{12}\hat{X}_1-\tfrac{1}{12}\hat{X}^2_1 \ ,$$

$$\hat{G}_3=\tfrac{1}{8}\hat{X}_4+\tfrac{1}{4}\hat{X}_3-\tfrac{1}{8}\hat{X}_2-\tfrac{1}{4}\hat{X}_1-\tfrac{1}{8}\hat{X}^2_1-\tfrac{1}{8}\hat{X}_1\hat{X}_2 \ ,$$

$$\hat{G}_4=\tfrac{1}{10}\hat{X}_5+\tfrac{1}{4}\hat{X}_4-\tfrac{1}{6}\hat{X}_3-\tfrac{1}{2}\hat{X}_2+\tfrac{1}{20}\hat{X}_1+\tfrac{1}{12}\hat{X}^2_1 -\tfrac{1}{4}\hat{X}_1\hat{X}_2-\tfrac{1}{20}\hat{X}^2_2-\tfrac{1}{10}\hat{X}_1\hat{X}_3+\tfrac{1}{30}\hat{X}^3_1 \ .$$

IV. NEW FORM OF BOSON MAPPING

By analogy with Eq. (3.2), we obtain the following result:

$$\exp\left[-\alpha\sum_{n=1}^{\infty}\frac{1}{n}\hat{G}_n\right]b^\dagger_{st}\exp\left[\alpha\sum_{n=1}^{\infty}\frac{1}{n}\hat{G}_n\right] =\left[b^\dagger\exp\left[-\alpha\sum_{n=1}^{\infty}\frac{1}{n}\hat{\rho}^n\right]\right]_{st} =\{b^\dagger\exp[\alpha\ln(1-\hat{\rho})]\}_{st}=[b^\dagger(1-\hat{\rho})^\alpha]_{st} \ . \quad (4.1)$$

Realizing that $\hat{P}^\alpha=\hat{P}$ $(\alpha>0)$, we have immediately from Eq. (4.1)

$$[b^\dagger(1-\hat{\rho})^\alpha]_{st}\hat{P}=\hat{P}b^\dagger_{st}(\hat{N}+1)^\alpha \ , \quad (4.2)$$

or in a somewhat more general form

$$[b^\dagger(1-\hat{\rho})^\alpha]_{st}(\hat{N}+1)^\gamma\hat{P}=\hat{P}b^\dagger_{st}(\hat{N}+1)^{\gamma+\alpha} \ . \quad (4.3)$$

Similarly, we obtain

$$\hat{P}(\hat{N}+1)^{1-\alpha-\beta-\gamma}[(1-\hat{\rho})^\beta b]_{st}=(\hat{N}+1)^{1-\alpha-\gamma}b_{st}\hat{P} \ . \quad (4.4)$$

The commutation relations of the right-hand sides of (4.3) and (4.4), and the operator $\hat{\rho}_{st}\hat{P}$ reproduce the commutation relations of the original bifermion $SO(2k)$ algebra. They do that for the particular case of the Dyson boson mapping ($\alpha=1,\gamma=0$) as well as for the general α, β, and γ. Thus, we come to the general boson mapping of the bifermion operators

$$c^\dagger_s c^\dagger_t\to[b^\dagger(1-\hat{\rho})^\alpha]_{st}(\hat{N}+1)^\gamma\hat{P}=Pb^\dagger_{st}(\hat{N}+1)^{\alpha+\gamma} \ ,$$

$$c^\dagger_t c_s\to\hat{\rho}_{st}\hat{P} \ , \quad (4.5)$$

$$c_t c_s\to\hat{P}(\hat{N}+1)^{1-\alpha-\beta-\gamma}[(1-\hat{\rho})^\beta b]_{st} =(\hat{N}+1)^{1-\alpha-\gamma}b_{st}\hat{P} \ .$$

Mapping (4.5) generalizes and unifies the Dyson mapping[2,5] ($\alpha=1,\gamma=0$), the Holstein-Primakoff mapping[1,5] ($\alpha=\frac{1}{2},\beta=\frac{1}{2},\gamma=0$), and the Marumori mapping[4] ($\alpha=1$, $\beta=1,\gamma=-\frac{1}{2}$). If a general fermion Hamiltonian written in the pairing form is mapped according to the rightmost sides of relations (4.5), one gets a boson Hamiltonian in the one-body form discussed recently by Marshalek.[14]

V. CONCLUSIONS

In this paper, we have investigated some properties of the projection operator $\hat{P}$ onto the physical boson space. The projection operator $\hat{P}_n$ acting in the space with a given total boson number n has been shown to contain n-body and $(n-1)$-body operators. It is thus a quite difficult task to calculate the general projector from this expression.

However, a relation has been shown from which matrix elements of the projection operator could be recurrently calculated. Equation (2.14) provides matrix elements of $\hat{P}$ (the norm matrix) in the space of n bosons once the norm matrix in the space of $(n-1)$ bosons and the fractional parentage coefficients connecting n-boson space and $(n-1)$-boson space are known. We project onto the physical space by diagonalizing the norm matrix and restricting to the space with nonvanishing eigenvalues.

This recurrent procedure seems to be worthy of further study if a truncation to the collective subspace is performed. Restricting in Eq. (2.14) the sum of intermediate $(n-1)$-boson states to the collective subspace and knowing the norm matrix in the $(n-1)$-boson collective subspace, we obtain matrix elements of $\hat{P}$ in n-boson collective subspace immediately. This is, of course, an approximate procedure that becomes less exact with the increasing boson number and whose validity is dependent on the choice of a collective subspace. Further investigations in this direction are needed.

From the properties of the projection operator, we have derived a new form of boson mapping. From the point of view of practical applications, this new general form does not seem to bring any considerable simplification. However, the new form is interesting as it generalizes and unifies the previously known methods of boson mapping.

Finally, we note that one can proceed quite analogously with the above treatment in the case of boson mapping of boson systems introduced recently.[16] With that, the boson operators symmetrical in single boson indices are used $b_{ts}^\dagger=b_{st}^\dagger$ and the operator $B_{st}^\dagger$ has the form

$$B_{st}^\dagger=b_{st}^\dagger+\sum_{uv} b_{su}^\dagger b_{tv}^\dagger b_{uv} \ .$$

The physical states are characterized by the symmetrical $U(k)$ partitions $[2n]$. Equations (2.3), (2.11), (2.14), and (2.16) do not formally change. In Eq. (4.5), the factor $(1+\hat{\rho})$ instead of $(1-\hat{\rho})$ should be written.

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