Constraints on the $^{12}C(\alpha,\gamma)^{16}O$ astrophysical S factor from the beta-delayed α emission of ^{16}N

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The $E1$ astrophysical S factor for the $^{12}C(\alpha,\gamma)^{16}O$ reaction is studied using R-matrix theory. The constraints on this quantity from the α-particle spectrum following ^{16}N β decay are investigated with a set of pseudodata in an attempt to identify what additional information might be extracted from future measurements. The possibility of an experimental separation of the p- and f-wave α particles is discussed. Additional precise measurements of the β-delayed α particles at total center-of-mass energies near 1.0 MeV and the (α,γ) cross section at energies below and above previous measurements are particularly useful in constraining the S factor.

I. INTRODUCTION

The astrophysical S factor for the $^{12}C(\alpha,\gamma)^{16}O$ reaction has considerable importance in determining the evolution of massive stars.[1] In particular, it is the value of the S factor at 0.3 MeV [$S(0.3)$] that is most important for the astrophysical calculations, as this is the energy where most of the reactions take place during helium burning. (Although the S factor has contributions from both $E1$ and $E2$ processes, we will study only the $E1$ part in this paper). The $E2$ S factor is discussed in detail in other works.[4,7,11] In spite of long-term theoretical and experimental studies of this quantity, a large uncertainty still exists. The R-matrix and K-matrix analyses of the $E1$ S factor using the recently measured p-wave α-particle $+^{12}C$ scattering phase shifts[2] and the $E1$ $^{12}C(\alpha,\gamma)^{16}O$ cross sections[3] show that it has a value ranging from 0.0 to 0.18 MeV b,[3,4] which is very unsatisfactory in terms of the astrophysical calculations. Therefore it is important to consider other experimental information that might allow more precise determination of the S factor.

The α-particle spectrum from the β decay of the ^{16}N ground state offers a unique opportunity in this respect.[5] The R-matrix analysis of the α spectrum was performed in Ref. 6 and was reanalyzed recently in Ref. 7. Unfortunately, the observed α spectrum contains both the p- and f-wave outgoing α particles. A combined R-matrix analysis involving two channels is therefore necessary and parameters associated with the f wave R matrix, such as the energies, α widths, and β strengths of the R-matrix poles, must be introduced in the analysis. If these parameters are unconstrained, the χ^2 will have a shallow minimum as a function of $S(0.3)$ and the β-decay data will not be useful in determining the S factor. However, if they are constrained without a firm foundation, the χ^2 minimum could be strongly biased.

In this paper, we study what constraints the p-wave α particles from the ^{16}N β decay could impose on the S factor if an experimental separation of the p- and f-wave α particles were performed. Our analysis is performed with R-matrix theory and a model-dependent p-wave α-particle spectrum. We find that the β-decay data could provide a stringent constraint on $S(0.3)$ if the interference scheme in the (α,γ) reaction can be determined and/or if the background level can be constrained. We also discuss briefly the experimental questions associated with the β-spectrum separation.

II. R-MATRIX FORMULAS

The multilevel R-matrix theory for the α-^{12}C scattering phase shift, the $^{12}C(\alpha,\gamma)^{16}O$ reaction cross section, and the α spectrum from the ^{16}N β decay can be found in Refs. 6 and 7. For convenience, we summarize its main ingredients here. For the relative p-wave motion of an α particle and ^{12}C, the three-level approximation for the R matrix is appropriate.[7] The first level corresponds to the 7.12 MeV 1^- state of ^{16}O. This level is $\sim$45 keV below the α-^{12}C channel threshold and its γ width is well known, while its β amplitude has been taken according to Ref. 9. The second level corresponds to the broad 1^- resonance at 9.59 MeV in ^{16}O, while the third level has a large pole energy and represents the contribution from high-lying 1^- states. This latter state is sometimes called the "background" level. Since the energy and widths of the background level are highly correlated, we fix the former at 23 MeV above the α-^{12}C threshold in ^{16}O.

The α-^{12}C scattering phase shift in the 1^- channel at center-of-mass energy E is

$$\delta^{1^-}(E)=-\phi(E)+\arctan\left[\frac{P(E)}{R^{-1}(E)-\mathcal{S}(E)+B}\right], \quad (1)$$

where

$$R(E)=\sum_{i=1}^{3}\frac{\gamma_{i\alpha}^2}{E_i-E} \quad (2)$$

is the R matrix. Here, $\phi(E)$ is the hard-sphere scattering phase shift and $P(E)$ and $\mathcal{S}(E)$ are the energy-dependent penetration factor and shift factor, which we calculate with Coulomb functions at channel radius $a=5.5$ fm. The constant boundary condition parameter B is chosen to be $\mathcal{S}$ at a certain reference energy; and E_i is the pole energy and $\gamma_{i\alpha}^2$ the reduced α width of level i. The phase-shift data we choose to fit are the same as those used in Ref. 4.

The $E1$ cross section for the $^{12}C(\alpha,\gamma)^{16}O$ reaction can be parametrized in R-matrix theory as

$$\sigma^{(1^-)}(E)=\frac{6\pi}{k_\alpha^2}P(E)\left|\frac{\sum_{i=1}^{3}[\gamma_{i\alpha}\Gamma_{i\gamma}^{1/2}/(E_i-E)]}{1-[\mathcal{S}(E)-B+iP(E)]R(E)}\right|^2, \tag{3}$$

where $\Gamma_{i\gamma}$ is the γ width of level i. The bound-state formal γ width, $\Gamma_{1\gamma}$, is determined from the observed width through

$$\Gamma_{1\gamma}=\Gamma_{1\gamma}^{\mathrm{obs}}\left[1+\gamma_{1\alpha}^2\frac{d\mathcal{S}(E)}{dE}\bigg|_{E=E_1}\right]. \tag{4}$$

We take the observed γ width to be 55 meV in our fits.

The ground state of ^{16}N is 2^-, while the spin parities of both the α particle and ^{12}C are 0^+. In the allowed approximation, ^{16}N can then decay only to the relative 1^- and 3^- channels of the $\alpha+{}^{12}C$. The number of α particles per unit energy for a total center-of-mass energy E in the 1^- channel can be parametrized in R-matrix theory as[6]

$$W^{(1^-)}(E)=f_\beta(E)P(E)\left|\frac{\sum_{i=1}^{3}[A_i/(E_i-E)]}{1-[\mathcal{S}(E)-B+iP(E)]R(E)}\right|^2, \tag{5}$$

where $f_\beta(E)$ is the integrated Fermi function $F(W,Z)$ with $Z=8$ and $W=(3.768-E)/m_e$. The beta-decay amplitude (or feeding factor) for each level, A_i, is proportional to the hadronic matrix element of the Gamow-Teller operator between initial and final hadron states. For the bound state, A_1 can be calculated through[9]

$$A_1^2=\frac{NY(7.12)\gamma_{1\alpha}^2(1+\gamma_{1\alpha}^2 d\mathcal{S}/dE|_{E=E_1})}{\pi Y(9.59)F(7.46,8)}, \tag{6}$$

where N is total number of α particles and $Y(7.12)$ and $Y(9.59)$ are the experimental β-branching ratios to the respective levels.

All level parameters $(\gamma_{i\alpha},E_i)$ in the R matrix (2) depend on the value given to the boundary condition constant B. When the latter constant is modified, one can obtain, from algebraic equations,[8,10] the new parameters leaving unchanged the energy dependence of observable quantities such as the phase shift and the cross section. Following Barker,[6] we have applied the constraints on the parameters of a level E_i when the boundary condition constant B is such that the shift of that level, $\mathcal{S}(E_i)-B$, is zero (we apply this throughout the paper). However, our fits are carried out at the boundary condition $B=\mathcal{S}(0.3)=-3.504$. This allows us to use $S(0.3)$ as one of the input parameters, replacing $\gamma_{1\alpha}$, in order to more easily determine the error in $S(0.3)$ as was done in Ref. 4. The choice of the channel radius ($a=5.5$ fm) is based on the empirical nuclear radius formula. We did not study the sensitivity of our fits to different choices of a as was done in Refs. 6 and 7.

III. CONSTRAINTS IMPOSED BY THE p WAVE α SPECTRUM

To see the role of the p wave α spectrum from the ^{16}N decay in determining the S factor, we generated a set of pseudodata, which are obtained by subtracting the theoretical f-wave spectrum of Ref. 6 from the inclusive experimental α spectrum.[5] These pseudodata cannot be taken too seriously because of the theoretical bias involved in the f-wave spectrum. Nevertheless, we expect that they retain the main features of the real experimental p-wave spectrum so that the broad conclusions from our analysis remain. From the results of Ref. 6, the f-wave contribution to the total α spectrum from the ^{16}N decay is important only at small and large α-particle energies.

In Ref. 3, the χ^2 fit was done with the p wave $\alpha+{}^{12}C$ elastic-scattering phase shifts and the $E1$ $^{12}C(\alpha,\gamma)^{16}O$ reaction data. We reproduced the same fit here and the χ^2 is shown as a function of $S(0.3)$ in Fig. 1. The χ^2 minimum is located at a small S factor and the χ^2 grows slowly as $S(0.3)$ increases. The dashed and dotted lines show the separate contributions to the total χ^2 from the phase-shift data and the $E1$ cross sections, respectively. It is clear from the figure that the constraint on $S(0.3)$ is mainly from the (α,γ) data. If one allows a 30% increase in χ^2 as an acceptable fit, $S(0.3)$ has a range from 0.00 to 0.16 MeV b. A recent K-matrix analysis of the same data reached a similar conclusion.[4]

For comparison, we performed a fit to both the pseudo-β-decay data and the elastic phase shifts. In Fig. 2, we show the χ^2 as a function of $\gamma_{1\alpha}$ $[B=\mathcal{S}(-0.045)]$,

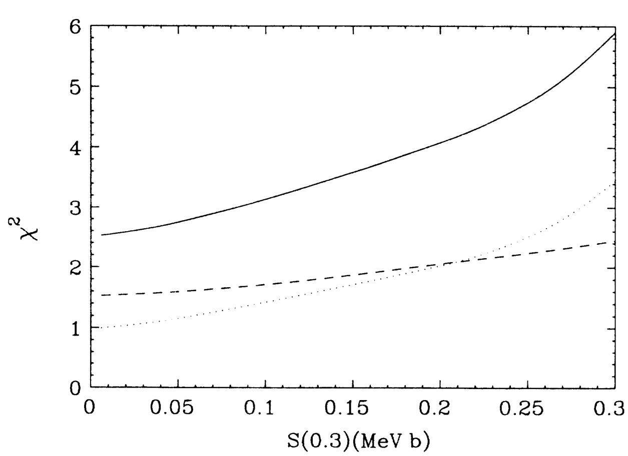

FIG. 1. χ^2 per number of data points obtained in the fit to the experimental $\alpha+{}^{12}C$ p-wave elastic-scattering phase shift and the $^{12}C(\alpha,\gamma)^{16}O$ cross section. The dashed line is the χ^2 for phase shift, the dotted line is that for (α,γ) cross section, and the solid line is the sum of the two.

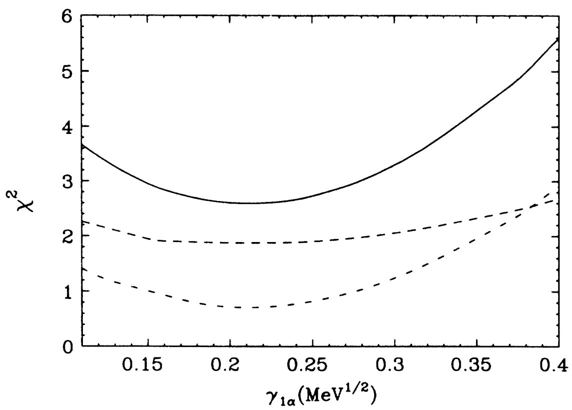

FIG. 2. χ^2 obtained in the fit to the experimental $\alpha+{}^{12}$C p-wave elastic-scattering phase shift and the pseudo$-{}^{16}$N β-decay α-particle spectrum. The dashed line is the χ^2 for phase shift, the dotted-dashed line is that for the β decay, and the solid line is the sum of the two.

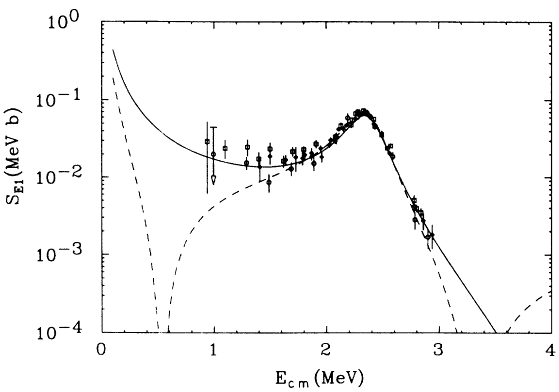

FIG. 4. The S-factor function for the ^{12}C$(\alpha,\gamma)^{16}$O reaction. The solid curve is calculated using the best-fit parameters with constructive interference, while the dashed curve is calculated using the best fit parameters with destructive interference. The circles represent the data from Ref. 3 which are used in the fit; the squares represent the data from Ref. 11; and the diamonds represent the data from Ref. 12.

the amplitude of the alpha-particle reduced width of the bound state. The minimum of χ^2 is located at 0.22 MeV$^{1/2}$ [corresponding to $S(0.3)=0.11$ MeV b for a reasonable choice of Γ_γ widths]. As was the case for the fits to the phase shift and the (α,γ) cross section, the χ^2 is not very steep and therefore a large range of $\gamma_{1\alpha}$, which in turn means a large range of $S(0.3)$, is permitted by the data. Again, the figure shows that the phase-shift data do not significantly constrain the S factor.

The three types of data can also be fitted simultaneously. Figure 3 shows the results of such a fit. Interestingly, there are two branches of χ^2, each with its own minimum. One branch, with a minimum at $S(0.3)=0.01$ MeV b, corresponds to destructive interference between

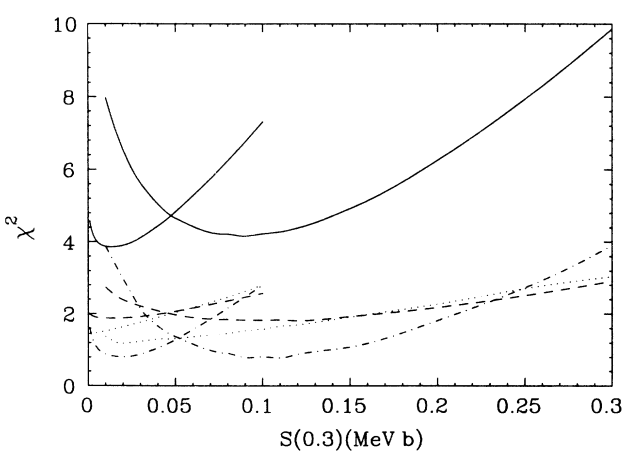

FIG. 3. χ^2 obtained in a simultaneous fit to the experimental $\alpha+{}^{12}$C p-wave elastic-scattering phase shift, the ^{12}C$(\alpha,\gamma)^{16}$O cross section, and the pseudo$-{}^{16}$N β-decay α-particle spectrum. The different curves have the same meaning as they have in Figs. 1 and 2.

the first and second levels in the $E1$ cross section. The other branch, with a minimum at $S(0.3)=0.09$ MeV b,corresponds to constructive interference. The two different cases result in very different S factors below 1.5 MeV and above 3 MeV, as is shown in Fig. 4. The experimental data from Refs. 11 and 12 are also shown in Fig. 4 for comparison. Clearly if the sign of the interference between the first two states can be determined, and if this corresponds to constructive interference, the beta-decay data can rule out a very small $S(0.3)$, and the range of acceptable fits would be 0.03–0.16 MeV b, a narrower range than that determined by the (α,γ) cross section alone. However, without definite information on the interference scheme, the beta-decay data does not constrain $S(0.3)$; the range of the S factor is similar to that deduced from Fig. 1. Consequently, new measurements of the $E1$ (α,γ) cross section below 1.5 MeV and above 3 MeV can be very useful for excluding small values of $S(0.3)$.

To obtain a better determination of $S(0.3)$, one must have better constraints on some of the parameters in the R-matrix theory. For instance, Barker suggested in Ref. 6 that the background level has a many-particle, many-hole structure and should make a very small contribution to the (α,γ) reaction and ^{16}N β decay. We can also investigate how a constraint on the background level will influence the fit. Figure 5 shows a fit similar to that of Fig. 1, except that the γ width of the background level is set to zero. The restriction on the background level shifts the χ^2 minimum to a larger $S(0.3)$, 0.125 MeV b. As expected, the χ^2 value at the new minimum is larger than that of the previous unconstrained background fit at its minimum. But, the main effect of the restriction is to worsen the fit for small $S(0.3)$. The range of $S(0.3)$ is still nearly as large as before. In contrast, a restriction on the beta-decay amplitude of the third level, i.e., $A_3=0$ in Eq. (5), did not shift the minimum significantly, as shown in Fig. 6, the analog of Fig. 2. The acceptable range for $\gamma_{1\alpha}$ is now much smaller, 0.24 to 0.29 MeV$^{1/2}$. In a combined fit of the three types of data, the χ^2 is a very sensi-

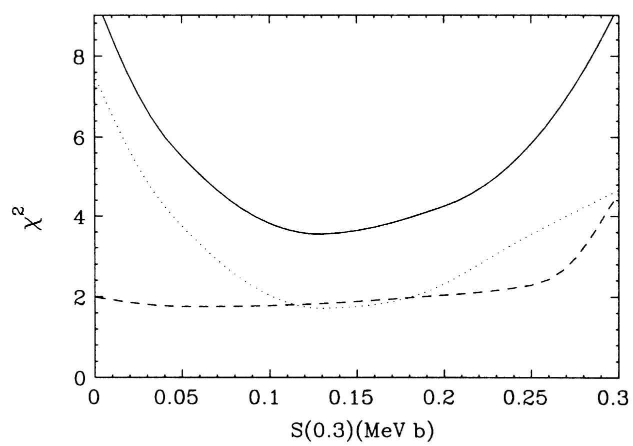

FIG. 5. Same as Fig. 1 except that the γ width for the background level is set to zero.

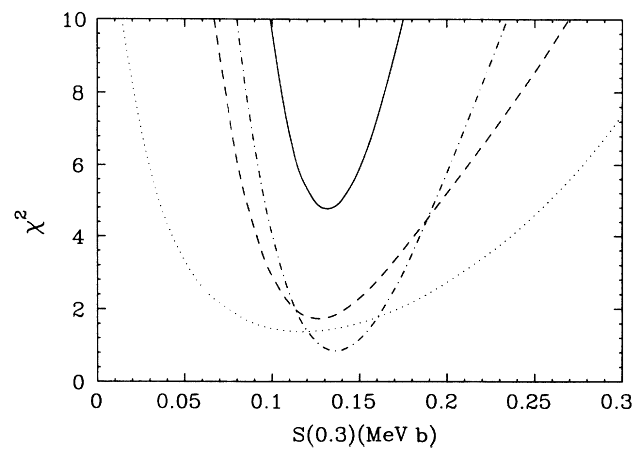

FIG. 7. Same as Fig. 3 except that the γ width and β amplitude for the background level are set to zero.

tive function of $S(0.3)$, as Fig. 7 shows. The best value of the $S(0.3)$ is now 0.13 MeV b with a range from 0.11 to 0.15 MeV b; the uncertainty is about 20%.

Thus it seems that $S(0.3)$ can be determined very well from the (α,γ) cross section, the (α,α) phase shift and the α spectrum from the ^{16}N β decay *if* some restrictions on the background level can be applied. The sensitivity of the analysis to the background strength can be estimated by varying this parameter. Taking the α spectrum as an example, we show in Fig. 8 the predicted spectrum for two different choices of the background strength A_3. The solid curve is calculated with $A_3=0$ and the dashed curve with $A_3=-7000$ MeV$^{1/2}$. Apart from the large peak corresponding to the 9.6-MeV state, there is a small peak generated by the interference between the first and second states, whose peak height depends upon A_3 (the counts per channel at the peak differs by 40% in this example). If one can measure the spectrum with an accuracy of 10% then one can easily distinguish $A_3=0$ from $A_3=-7000$, which corresponds to $S(0.3)=0.13$ MeV b and 0.09 MeV b, respectively.[13] Therefore a measurement of the α-particle spectrum for total center-of-mass energy in the 1.0 MeV region can provide important constraints on $S(0.3)$.

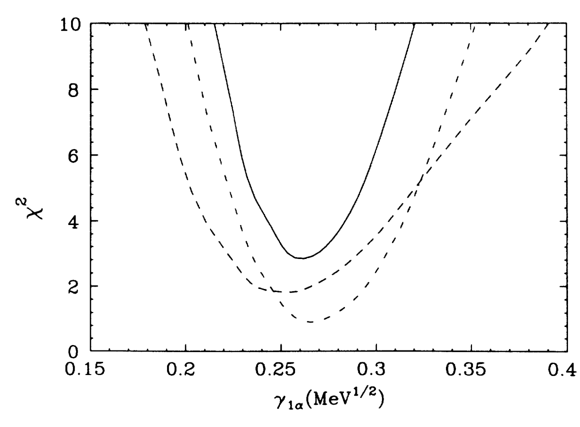

FIG. 6. Same as Fig. 2 except that the β amplitude for the background level is set to zero.

IV. EXPERIMENTAL SEPARATION OF THE p- AND f-WAVE α PARTICLES

The preceding analysis shows that the p-wave α-particle spectrum from ^{16}N β decay is a useful complement to the (α,γ) data. However, the experimental data available so far are only the total α spectrum, which consists of both p- and f-wave α particles. We now study the possibilities for separating these two components.

Using the Gamow-Teller part of the weak interaction Hamiltonian and ignoring the final-state Coulomb distortion of the electron for simplicity of presentation, we find the ^{16}N β-decay probability to be

$$dP=\frac{C_F^2}{(2\pi)^5}\delta\left[\frac{Q^2}{2M}+\frac{k^2}{2\mu}+E_e+E_\nu-\Delta E\right]\frac{1}{E_eE_\nu}$$

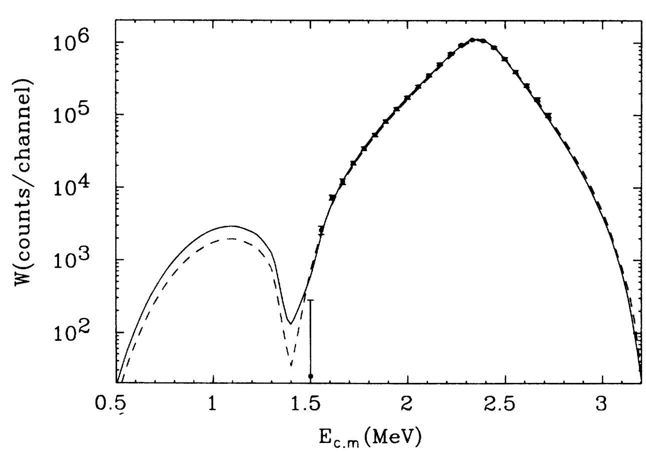

FIG. 8. The α-particle spectrum from ^{16}N β decay. The data is the experimental inclusive spectrum less the theoretical f-wave spectrum. The solid curve is calculated with the best-fit parameters and $A_3=0$ [$S(0.3)=0.13$ MeV b], and the dashed curve with $A_3=-7000$ MeV$^{1/2}$ [$S(0.3)=0.09$ MeV b].

$$\times \left\{ -\frac{12}{5\sqrt{4\pi}} \mathrm{Re}[e^{i\delta_1 - i\delta_3} \langle 1^- \| \Sigma \| 2^- \rangle \langle 2^- \| \Sigma \| 3^- \rangle][(\mathbf{p}_e \otimes \mathbf{p}_\nu)^{(2)} \otimes Y_2(\mathbf{k})]^{(0)} \right.$$
$$\left. + \frac{1}{4\pi} \sum_l^{1,3} |\langle l \| \Sigma \| 2 \rangle|^2 (E_e E_\nu - \tfrac{1}{3} \mathbf{p}_e \cdot \mathbf{p}_\nu) \right\} \delta(\mathbf{p}_e + \mathbf{p}_\nu + \mathbf{Q}) d\mathbf{p}_e d\mathbf{p}_\nu d\mathbf{Q}\, d\mathbf{k} \, , \tag{7}$$

where $|2^-\rangle$ is shorthand for the ^{16}N ground state and $|1^-\rangle$ and $|3^-\rangle$ are the α-^{12}C scattering states. The hadronic reduced matrix elements $\langle l \| \Sigma \| 2 \rangle$ contain the nuclear structure physics. $\mathbf{Q}$ and $\mathbf{k}$ are center-of-mass and relative momenta of the α and ^{12}C, M and μ are the total and reduced mass, δ_1 and δ_3 are the elastic-scattering phase shifts in the 1^- and 3^- channels, respectively, E_e, $\mathbf{p}_e$, E_ν and $\mathbf{p}_\nu$ are lepton energies and momenta and ΔE is the total energy release. The first term in the curly brackets is the interference term between p- and f-wave contributions, while the second is the sum of noninterfering terms that contribute to the total α-particle spectrum.

According to Eq. (7) there are two kinds of angular correlation experiments by which the p- and f-wave spectra could be separated. The first is to observe the angular correlation between the outgoing electron and α particle. After integrating out the neutrino and hadron recoil momenta, the decay probability is

$$dP = \frac{C_F^2}{2(2\pi)^2} \left\{ -\frac{4\sqrt{6}}{5} I_1 \mathrm{Re}[e^{i\delta_1 - i\delta_3} \langle 1^- \| \Sigma \| 2^- \rangle \langle 2^- \| \Sigma \| 3^- \rangle] P_2(\cos\theta_{e\alpha}) + \left[I_\infty - \frac{I_1}{3} \right] \sum_l^{1,3} |\langle l \| \Sigma \| 2 \rangle|^2 \right\} d\mathbf{p}_e d\mathbf{k} \, , \tag{8}$$

where I_1 and I_∞ are phase space integrals:

$$I_1 = \int \delta \left[\frac{Q^2}{2M} + \frac{k^2}{2\mu} + E_e + E_\nu - \Delta E \right] \frac{\mathbf{p}_e \cdot \mathbf{p}_\nu}{E_e E_\nu} d\mathbf{p}_\nu \, , \tag{9}$$

$$I_\infty = \int \delta \left[\frac{Q^2}{2M} + \frac{k^2}{2\mu} + E_e + E_\nu - \Delta E \right] d\mathbf{p}_\nu \, . \tag{10}$$

If there were no recoil term in the energy conserving delta function, the first term would vanish after integrating over the neutrino momentum. The size of the interference term is thus only of the order of the recoil term. Therefore one has to measure the coincidence spectrum very precisely to separate out the interference contribution.

The second type of experiment involves measurements of the momenta of the α particle and the ^{12}C recoil. After integrating out the lepton momenta, one has

$$dP = \frac{C_F^2}{2(2\pi)^2} \left\{ \frac{4\sqrt{6}}{5} (I_1' + I_2') \mathrm{Re}[e^{i\delta_1 - i\delta_3} \langle 1^- \| \Sigma \| 2^- \rangle \langle 2^- \| \Sigma \| 3^- \rangle] P_2(\cos\theta_{\mathbf{Qk}}) \right.$$
$$\left. + \left[I_\infty' + \frac{I_0' + I_1'}{3} \right] \sum_l^{1,3} |\langle l \| \Sigma \| 2 \rangle|^2 \right\} d\mathbf{Q}\, d\mathbf{k} \, , \tag{11}$$

where again I_0', I_1', I_2' and I_∞' are phase-space integrals:

$$\begin{aligned} I_0' &= \int \delta \left[\frac{Q^2}{2M} + \frac{k^2}{2\mu} + E_e + E_\nu - \Delta E \right] \frac{p_e^2}{E_e |\mathbf{p}_e + \mathbf{Q}|} d\mathbf{p}_e \, , \\ I_1' &= \int \delta \left[\frac{Q^2}{2M} + \frac{k^2}{2\mu} + E_e + E_\nu - \Delta E \right] \frac{\mathbf{p}_e \cdot \mathbf{Q} P_1(\cos\theta_{e\mathbf{Q}})}{E_e |\mathbf{p}_e + \mathbf{Q}|} d\mathbf{p}_e \, , \\ I_2' &= \int \delta \left[\frac{Q^2}{2M} + \frac{k^2}{2\mu} + E_e + E_\nu - \Delta E \right] \frac{p_e^2 P_2(\cos\theta_{e\mathbf{Q}})}{E_e |\mathbf{p}_e + \mathbf{Q}|} d\mathbf{p}_e \, , \\ I_\infty' &= \int \delta \left[\frac{Q^2}{2M} + \frac{k^2}{2\mu} + E_e + E_\nu - \Delta E \right] d\mathbf{p}_e \, . \end{aligned} \tag{12}$$

The interference term now has a size comparable to the total spectrum, but the measurement of the ^{12}C recoil is in general difficult. Furthermore, one has to measure both the α particle and ^{12}C momenta very precisely so that the resulting center-of-mass and relative momenta can be extracted with sufficient precision.

In conclusion, the p-wave α-particle spectrum from the ^{16}N β decay can be very useful for constraining the ^{12}C$(\alpha,\gamma)^{16}$O S factor. If the (α,γ) cross section can be determined to have constructive interference between the first and second levels, then small values of $S(0.3)$ can be excluded by the α spectrum data. In addition, constraints on the background contributions can be provided by measurements of the ^{16}N β decay α spectrum at energies around 1.0 MeV. Experimental separation of the p-wave α-particle spectrum is in principle possible, but is likely very difficult in practice.

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