PHYSICAL REVIEW C VOLUME 41, NUMBER 3 MARCH 1990

Supersymmetric quantum mechanics, the Pauli principle, and nucleon-alpha scattering

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(Received 21 August 1989)

Supersymmetric quantum mechanics provides a simple way to understand the scattering of a particle from a potential that contains a bound state that is Pauli blocked to the projectile. The singularity of the resulting potential found empirically in studies of nucleon-alpha scattering emerges naturally in supersymmetric quantum mechanics as does a direct and unique way to calculate the complete potential.

If one tries to describe nucleon-alpha scattering with a local potential, the Pauli forbidden $1S$ bound state causes difficulty. It can be shown, empirically, that a singular repulsive potential can reproduce the nucleon-alpha scattering data and still have no $1S$ bound state, but the origin of the repulsion, and the credentials of the new potential are uncertain.[1] A valid nucleon-alpha potential is important if one is to use it in three-body nucleon-nucleon-alpha calculations.[2] Supersymmetric quantum mechanics[3] (SSQM) provides a simple and elegant solution to all these difficulties including providing the precise local form of the singular potential that describes nucleon-alpha scattering but has no $1S$ bound state.

Empirically nucleon-alpha scattering is quite well described by a folding-type attractive potential. The s-wave phase shift begins at π at zero energy and falls (ignoring inelasticity) to zero at high energy, just as it should for one bound state.[1] Indeed the folding attractive potential has a $1S$ bound state. But there is no such bound state in nature due to the Pauli principle. Thus the problem is to construct a potential that has exactly the scattering of the attractive folding potential, but does not have the bound state. This is just what supersymmetric quantum mechanics applied to scattering can do, as was first emphasized by Baye.[4] Starting with a potential with one bound state and making two supersymmetric transformations, one arrives at a potential with exactly the same scattering (phase equivalence) but without the bound state. This local potential is *unique* (neglecting inelasticity). Inverse scattering theory guarantees that there is a unique local potential that gives the scattering and has no bound states. It is just that potential that the supersymmetric transformation gives. Since by construction the new scattering phase still obeys a bound state like Levinson's theorem, but there is no bound state, the new potential must be singular.[5] This is what has been discovered empirically.

Supersymmetric quantum mechanics begins by noting that a partial wave Schrödinger Hamiltonian can be written in factored form[6] (we take $\hbar=2m=1$)

$$H_1=-\frac{d^2}{dr^2}+V(r)=A_1^+A_1^- -B\ , \tag{1}$$

where B is the binding energy ($B>0$) of the lowest bound state of H_1, $H_1\Phi_{10}=-B\Phi_{10}$, and where

$$A_1^{\pm}=\pm\frac{d}{dr}+W_1\ . \tag{2}$$

The "superpotential" W_1 is given in terms of the bound state wave function by

$$W_1=\frac{\frac{d}{dr}\Phi_{10}}{\Phi_{10}}\ . \tag{3}$$

In the lth partial wave, the bound state wave function goes like r^{l+1} for small r and like e^{-Kr} for large r, where $K^2=B$. Hence for small r, W_1 goes like $(l+1)/r$, and is $-K$ for large r. There is a partner Hamiltonian to H_1, H_2 given by

$$H_2=A_1^-A_1^+-B\ . \tag{4}$$

It has the same spectrum (same bound states and continuum) as H_1 except for the state at $-B$ that is present in H_1 but absent in H_2. The eigenfunctions of H_2 corresponding to energy E are given in terms of those of H_1 at that energy by

$$\Psi_2(E)=A_1^-\Psi_1(E)\ . \tag{5}$$

We have shown[6] from this that the S matrix of H_2, S_2, is related to that of H_1, S_1, at scattering energy E ($E=k^2$) by

$$S_2=\frac{k-iK}{-k-iK}S_1 \ . \tag{6}$$

From the form of $A_1^{\pm}$ it is easy to show that H_2 is related to H_1 by

$$H_2=H_1-2\frac{dW_1}{dr} \ . \tag{7}$$

Because of the small r behavior of W_1 this means that H_2 will have an effective centrifugal potential of $(l+1)(l+2)/r^2$. At first glance this looks like we have promoted the angular momentum by one, but we must keep in mind that it is the partial wave problem we are solving, and the angular momentum has already been separated. We are still in the lth partial wave. But the potential is singular at the origin. That singularity is seen in the fact that S_2 goes to -1 at large k, if S_1 goes to 1.

The Hamiltonian H_2 has the same spectrum as H_1 except that the lowest bound state is removed. However, we see from (6) that H_2 is not phase equivalent (leads for positive energy to the same S matrix) as H_1. To achieve phase equivalence we need to make one more supersymmetric transformation.[4] We write

$$H_2=A_2^+A_2^- -B \tag{8}$$

with A_2 given by

$$A_2^{\pm}=\pm\frac{d}{dr}+W_2 \ , \tag{9}$$

and the "superpotential" W_2 is given by

$$W_2=\frac{\frac{d}{dr}\Phi_{20}}{\Phi_{20}} \tag{10}$$

in terms of the solution of

$$H_2\Phi_{20}=-B\Phi_{20} \ . \tag{11}$$

It might be objected that H_2 is constructed to have no normalizable solution at $E=-B$, but we do not need a normalizable solution to carry out this construction, only one that has no zeros. Because $-B$ is below the bound state spectrum of H_2, Φ_{20} will have no zeros, except at the origin where (recall the new "centrifugal" barrier) it goes like r^{l+2}. At large r, Φ_{20} goes like e^{Kr}. Hence W_2 goes like $(l+2)/r^2$ for small r and becomes $+K$ at large r. Now we construct the supersymmetric partner H_3 of H_2.

$$H_3=A_2^-A_2^+ -B \ . \tag{12}$$

H_3 has the same spectrum as H_2 with no missing states, but also with no extra states. In particular H_3 has no bound state at $-B$. The eigenfunctions of H_3 are given in terms of those of H_2 by

$$\Psi_3(E)=A_3^-\Psi_2(E) \ , \tag{13}$$

and from this we have that the S matrix of H_3, S_3, at energy $E=k^2$ is given in terms of that of H_2 by the analog of (6) with $K\rightarrow -K$, or

$$S_3=\frac{k+iK}{-k+iK}S_2 \ , \tag{14}$$

but now from (6) we get

$$S_3=\frac{(k+iK)}{(-k+iK)}\frac{(k-iK)}{(-k-iK)}S_1=S_1 \ , \tag{15}$$

which is the phase equivalence of H_3 and H_1.

Thus H_3 is the Hamiltonian that is phase equivalent to H_1, that is, it has the same scattering, but that does not have the bound state at $E=-B$. As before we can relate H_3 to H_2 and then to H_1. We have

$$H_3=H_1-2\frac{dW_2}{dr}-2\frac{dW_1}{dr}=H_1+V_3 \ . \tag{16}$$

Because of the small r behavior of W_2, we have an added centrifugal potential in H_3 of the form $(l+2)(l+3)/r^2$, corresponding to two steps in l. Note that W_1 can be calculated knowing the original potential V from its bound state solution and (3), and that W_2 can be calculated from the solution of (11). Hence the added potential in going from H_1 to H_3, V_3, is completely known. We remark that V_3 has short-range behavior,[7] in agreement with the notion that Pauli effects for composite systems are repulsive in the region of wave function overlap.

There have been a number of other studies of scattering with Pauli-blocked bound states in compound systems. Many of these have been based on imposing an orthogonality constraint.[8] It would be interesting to set straight the connection of the underlying many-body problem and these effective potential approaches. Supersymmetric quantum mechanics gives the promise of providing new insight into this problem. We emphasize again that considered as a problem in potential scattering, that is, neglecting inelasticity, supersymmetric quantum mechanics provides a unique construction for the local potential that both gives the scattering and respects the Pauli principle.

We now focus on the nucleon-alpha system where we are in $l=0$. The additional potential, V_3, goes like $6/r^2$ for small r corresponding to a d-wave-like potential, but of course still in s wave, and hence singular. This behavior has been found before from an empirical study of the nucleon-alpha scattering[5] and from a study of resonating group calculations[9] and Levinson's theorem. These studies were not able to give a detailed form for what we are calling V_3. We see that supersymmetric quantum mechanics not only provides a simple explanation of the connection between the empirical phase shift and the Pauli principle, but also gives a constructive way to calculate V_3 given the original Hamiltonian without the Pauli principle. This is particularly important in few-body calculations where the nucleon-alpha potential is the input to, for example, a deuteron-alpha calculation.[2]

Similar ideas may be useful in atomic physics. For example, in the interaction of an electron with the Li^+ ion, at long distances one has a simple Coulomb force, but the $1S$ state is in fact occupied. If one wants to describe the interaction with a local potential, that potential must be phase equivalent to the Coulomb potential, but have the 1S state removed. (One of us has shown explicitly how such a potential is constructed.[10]) This potential would be useful, for example, in developing the wave function of the outgoing electron in photodetachment of Li.

In summary we have seen, as first suggested by Baye,[4] that supersymmetric quantum mechanics provides a simple and elegant way to describe the scattering of complex systems with local potentials but still accounting for states forbidden by the Pauli principle. In applying these ideas to the nucleon-alpha system, we have emphasized how the 1S state is removed and how the unique, local, singular potential that removes it without changing the phase shift can be constructed explicitly, as is needed in few-body calculations.[2,9]

ACKNOWLEDGMENTS

We are grateful to the Paul Scherrer Institute for bringing us together. One of us (R.D.A.) thanks the U.S. National Science Foundation for support. The Division de Physique Theorique is a Research Unit of the Universities of Paris 11 and 6 associated with the Centre National de la Recherche Scientifique.

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