Quark tunneling in a one-dimensional nuclear model

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A general result for the eigenenergies of the one-dimensional Dirac equation for an arbitrary array of delta-function potentials of either Lorentz scalar or vector type is developed. This result is then used to discuss a finite one-dimensional model of the nucleus from which the energy bands of the constituent quarks may be obtained. The resulting energy eigenvalues are compared with those obtained previously via a similar model based on a Klein-Gordon equation. In contrast to this alternative treatment a system consisting of massless quarks confined to a regular periodic one-dimensional nucleus is found to be unbound. In the near ultra-relativistic limit binding to first order in the mass is found, thus conforming unexpectedly with the nonrelativistic limit. The possible surface states are examined and it is found that the so-called Dirac surface states are not expected for potentials of either Lorentz character. The existence and effect of quark surface states is also discussed in terms of our nuclear model. Finally the scattering coefficients for a relativistic particle in an arbitrary array of delta-function potentials are determined and their implications with regard to quark tunneling in nuclei are discussed.

I. INTRODUCTION

Since the Stanford Linear Accelerator Center (SLAC) deep inelastic electron scattering experiments twenty years ago the traditional view of the nucleus, as consisting of point nucleons bound by meson exchange, has had to change to accommodate the known quark substructure of nuclear matter. This change has been very slow in coming to be accepted, but in the last few years has been seen as a necessary development in nuclear physics. However, the opposite extreme, a model of the nucleus based solely on a quark shell model must be reconciled with the knowledge that baryons form color singlet states comprising a triplet of quarks. As a consequence, various models of the nucleus have been developed which occupy the region between these two extremes, bag[1] (usually containing at least six quarks) and potential[2] models being two such examples.

In this paper a one-dimensional potential model of the nucleus in which the effective gluon field manifests itself as a one-dimensional lattice of confining potentials will be employed. Any realistic nuclear model must include the fact that the spatial extent of a nucleon in a nucleus is comparable to, or possibly greater than,[3] that of a free nucleon. This implies that at nuclear matter densities, significant overlap of the nucleons wave functions must occur. We incorporate this feature in our model by allowing quark tunneling between adjacent nucleons via a finite internucleon potential. In our model this condition will be fashioned by the introduction of an effective delta function at the nucleon boundary. The confinement potential for our nucleus will be of a Lorentz scalar type[4] as strong Lorentz vector type potentials may be shown to suffer from Klein paradox problems.[4,5]

Similar potential models of the nucleus have appeared previously in the literature. Goldman and Stephenson[6] have shown that quark tunneling in such a system results in a significant increase in the binding energy of the nucleus due to the inherent decrease in quark localization. The quark energies for their finite three-dimensional model were obtained via an effective Klein-Gordon equation incorporating a Lorentz scalar confining potential.

The advantage of the Klein-Gordon model is that it permitted analytic solution in three dimensions. However, it ignored the Dirac nature of the quarks. To explore the latter, McKellar and Stephenson[7] analyzed a one-dimensional Dirac-Kronig-Penney model in which periodic boundary conditions are imposed and the energy bands for relativistic quarks, in the presence of both Lorentz scalar and vector potentials, are obtained. As our paper also utilizes the Dirac equation, we momentarily digress to discuss the use of delta-function potentials in this equation. As Woods and Callaway[8] first noted, a discrepancy exists between results obtained by solving the Dirac equation for a delta-function potential as compared to solving the Dirac equation for a square well and then taking the delta-function limit. Many authors[9] have since chosen to integrate the Dirac equation over the delta function, a procedure which McKellar and Stephenson[7] and Calkin *et al.*[10] have now shown to be incorrect. As they demonstrated, the correct result emerges by considering the delta function as the limit of a square well and in Sec. II we use this prescription to derive a general solution of the Dirac equation for an arbitrary array of delta-function potentials of both Lorentz scalar and vector types.

With this formalism as a basis in Sec. III, a determination of the energies of a quark confined to a one-dimensional box of finite length and impenetrable walls, our model of the nucleus, is made. The implications of these results for both small systems and for a finite periodic model are discussed. These results are then used to determine binding energies for a system of massless quarks. It is found that for periodic potentials, the sys-

tem is unbound. This feature is discussed and compared with the contrasting results of the Goldman-Stephenson and McKellar-Stephenson analyses.

Section IV introduces the relativistic Tamm model which is studied with regard to determining the surface states of the system and in resolving the problems associated with previous treatments of this model.[11] Both the Lorentz scalar and vector cases are discussed. Our treatment confirms the Subramanian and Bhagwat[12] calculation which showed that states which violate the Tamm existence condition[13] in the nonrelativistic limit, the Dirac surface states of Steslicka and Davison,[11] do not exist for the vector potential. Moreover, no abnormal surface states are found where the Tamm potential enters the Dirac equation as a Lorentz scalar. The energies of the quark surface states in our nuclear model are then obtained. The energy region occupied by these localized states is shown to occur at a much greater energy than that of the bulk bands and as such their relevance with regard to the ground state of the nucleus is quite limited.

In Sec. V we derive the scattering coefficients of a relativistic particle in an arbitrary array of delta-function potentials of either Lorentz type. The formalism of Sec. II is used to avoid the irregularities involving the delta function encountered by previous treatments of this problem.[14] In the infinite strength limit a comparison is made between the suitability of vector and scalar potentials in providing confinement. Transmission coefficients for several simple systems are explicitly calculated and discussed in terms of quarks tunneling in nuclei. Our conclusions are outlined in Sec. VI.

II. GENERAL FORMALISM

The one-dimensional time-independent Dirac equation for a particle of rest mass m in a potential $U(x)$ is given by

$$\left[i\alpha_x\frac{d}{dx}+E-\beta m-U(x)\right]\Psi(x)=0\ , \tag{2.1}$$

where we have taken $\hbar=c=1$. As the potential $U(x)$ may consist of both Lorentz vector and scalar components, we write it explicitly as

$$U(x)=V(x)+\beta S(x)\ . \tag{2.2}$$

In one dimension we are able to represent the Dirac matrices α_x and β by 2×2 matrices. Initially it is not necessary to choose a specific representation for the Dirac matrices, instead we proceed simply by introducing a general algebra of 2×2 matrices, choosing as a basis σ_μ ($\mu=0,1,2,3$), defined in terms of α_x and β by

$$\sigma_0=I,\quad \sigma_1=\alpha_x,\quad \sigma_2=i\alpha_x\beta,\quad \sigma_3=\beta\ , \tag{2.3}$$

which satisfy

$$\sigma_0\sigma_\mu=\sigma_\mu,\quad \sigma_i\sigma_j=\delta_{ij}+i\varepsilon_{ijk}\sigma_k,\quad \mathrm{Tr}\sigma_\mu=2\delta_{\mu0}\ . \tag{2.4}$$

Clearly one possible representation is to take the σ_i as the usual Pauli matrices. This particular representation of the algebra is not necessary in much of our analysis. Utilizing this notation we may rewrite (2.1) as

$$\left[i\sigma_1\frac{d}{dx}+(E-V)-\sigma_3(m+S)\right]\Psi(x)=0\ , \tag{2.5}$$

where $\Psi(x)$ is a two-component spinor. Rearranging (2.5) we obtain

$$\frac{d}{dx}\Psi(x)=G(x)\Psi(x)\ , \tag{2.6}$$

where $G(x)=i(\eta_1 k-\sigma_1 V+i\sigma_2 S)$ and $\eta_1=(1/k)(\sigma_1 E+i\sigma_2 m)$. The 2×2 matrix η_1 belongs to a similarity class of the σ_μ defined by $\eta_\mu=C^{-1}\sigma_\mu C$, where

$$C=(\tfrac{1}{2})[(r+1)\sigma_0+(r-1)\sigma_3]$$

and $r=k/(E+m)$. As a consequence, the η_μ also obey the algebra (2.4), and in particular $\eta_\mu^2=1$.

In analogy with the standard technique for solving the one-dimensional time-dependent Schrödinger equation, we obtain an iterative solution of (2.6)

$$\Psi(x)=P_x\left[1+\int_{x_0}^{x}dx'G(x')+\int_{x_0}^{x}dx'G(x')\int_{x_0}^{x'}dx''G(x'')+\cdots\right]\Psi(x_0)$$

which can be rewritten more concisely as

$$\Psi(x)=T(x_0,x)\Psi(x_0)\ , \tag{2.7}$$

where

$$T(x_0,x)=P_x\exp\left[\int_{x_0}^{x}G(x')dx'\right]$$

and P_x is the spatial ordering operator. Continuing the analogy with the Schrödinger equation we may interpret $T(x_0,x)$ as a "spatial" evolution operator as it describes the development of the wave function with displacement from the point x_0 to the point x in the presence of a potential $U(x)$. Intuitively we would expect $T(x_0,x)$ to be related to the transfer matrix.[15] In Sec. V this relationship will be explicitly determined.

For certain potentials, namely square barriers or delta functions of either pure Lorentz scalar or vector type, $T(x_0,x)$ assumes a particularly simple form. Since this paper will primarily be based on this class of potentials we include a brief discussion of them. We choose to analyze the scalar case in detail while simply quoting the vector results which may be developed in an analogous fashion.

A. Square barrier

For a potential consisting of a series of square barriers of width a_i and height H_i separated by field-free regions of length (l_i-a_{i-1}) we have

$$S(x)=\begin{cases}0 & \text{if } x_0<x<x_1\\ H_i & \text{if } x_i<x<x_i+a_i\\ 0 & \text{if } x_i+a_i<x<x_{i+1}\end{cases} \tag{2.8}$$

for $i=1,2,3,\ldots,N-1$ as in Fig. 1. The total length of the region of interest is $L=x_N-x_0=\sum_{i=1}^{N}l_i$.

By substituting (2.8) into (2.7) and utilizing Feynman's notation for spatially ordered operators,[16] we find that $T(x_0,x)$ assumes the form

$$T(x_0,x_N)=\exp[i\eta_1 k(l_N-a_{N-1})]\exp[ia_{N-1}(\eta_1 k+i\sigma_2 H_{N-1})]$$
$$\times\exp[i\eta_1 k(l_{N-1}-a_{N-2})]\cdots\exp[i\eta_1 k(l_2-a_1)]\exp[ia_1(\eta_1 k+i\sigma_2 H_1)]\exp(i\eta_1 k l_1)\ , \tag{2.9a}$$

where $l_i=x_i-x_{i-1}$.

For the vector equivalent potential of (2.8) we obtain

$$T(x_0,x_N)=\exp[i\eta_1 k(l_N-a_{N-1})]\exp[ia_{N-1}(\eta_1 k-\sigma_1 H_{N-1})]$$
$$\times\exp[i\eta_1 k(l_{N-1}-a_{N-2})]\cdots\exp[i\eta_1 k(l_2-a_1)]\exp[ia_1(\eta_1 k-\sigma_1 H_1)]\exp(i\eta_1 k l_1)\ . \tag{2.9b}$$

This form results because the elements of the exponent commute at spatially separated points for regions where the potential remains constant.

It is interesting to note that for a massless particle in a vector potential $\eta_1=\sigma_1$ and the exponential elements of (2.9b) commute allowing us to write

$$T(x_0,x_N)=\exp\left[i\sigma_1\left[kL-\sum_{i=1}^{N-1}H_i a_i\right]\right]\ . \tag{2.10}$$

As $T(x_0,x_N)$ is merely a phase, we see from (2.7) that the amplitude of the wave function of a massless particle is not diminished by the presence of the vector potential. The vector potential is transparent for a massless particle. This phenomenon arises because in the massless limit a fermion has a definite chirality which is unaffected by the vector interaction. Reflection of the particle by a barrier, which implies a reversal of momentum, must therefore be accompanied by a change in the spin of the particle in order for chirality to be conserved. However, this would violate spin conservation and therefore the only physically consistent scenario is for complete transmission to occur.

For a massless particle in a scalar field, $T(x_0,x_N)$ does not reduce to a single exponential analogous to (2.10). On a purely mathematical level we can see that this occurs because of the anticommutivity of σ_1 and σ_2. From a more physical standpoint we note that the scalar potential enters the Dirac equation with the mass term and thus the particle obtains an effective mass

$$M(x)=m+S(x)\ , \tag{2.11}$$

which is nonzero even for the massless particle. Consequently, the phenomenon of barrier transparency is peculiar to the vector interaction as the scalar field does not preserve the chirality of the massless fermion. This property of the scalar potential makes it suitable for confinement of massless quarks.

B. Delta-function limit

We may simplify the square barriers to delta functions by taking the barrier width to zero and the barrier height to infinity while allowing the product to remain finite:

$$H_i\to\infty,\quad a_i\to 0\quad\text{but } H_i a_i=\lambda_i\ . \tag{2.12}$$

The potential described by (2.8) now becomes

$$S(x)=\sum_{i=1}^{N-1}\lambda_i\delta(x-x_i)\ , \tag{2.13}$$

and for the scalar potential $T(x_0,x_N)$ is then given by

$$T(x_0,x_N)=\exp(i\eta_1 k l_N)\exp(-\sigma_2\lambda_{N-1})\exp(i\eta_1 k l_{N-1})\cdots\exp(i\eta_1 k l_2)\exp(-\sigma_2\lambda_1)\exp(i\eta_1 k l_1)\ . \tag{2.14a}$$

Similarly for the vector potential we obtain

$$T(x_0,x_N)=\exp(i\eta_1 k l_N)\exp(-i\sigma_1\lambda_{N-1})\exp(i\eta_1 k l_{N-1})\cdots\exp(i\eta_1 k l_2)\exp(-i\sigma_1\lambda_1)\exp(i\eta_1 k l_1)\ . \tag{2.14b}$$

The discontinuity in the wave function due to the delta function is in agreement with the results of McKellar and Stephenson[7] and Calkin *et al.*[10]

C. General properties of $T(x_0,x)$

The general structure of $T(x_0,x)$ may be deduced by considering the effect of an infinitesimal translation. For Δx small, to first order in Δx, using (2.6) and (2.7) we find that

$$T(x_0,x_0+\Delta x)=I+\Delta x G(x_0)\ . \tag{2.15}$$

Since $T(x_0,x)$ is a linear operator we may utilize the group property to write

$$T(x_0,x_0+\Delta x_1+\Delta x_2)$$
$$=T(x_0+\Delta x_1,x_0+\Delta x_1+\Delta x_2)T(x_0,x_0+\Delta x_1)\ , \tag{2.16}$$

for translations of Δx_1 and then Δx_2, respectively. Writ-

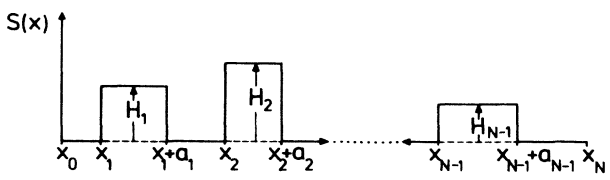

FIG. 1. An array of square barriers of height H_i and width a_i.

ing $G(x)=i[\sigma_1 E(x)+i\sigma_2 M(x)]$, where $E(x)=E-V(x)$ and $M(x)=m+S(x)$ are real quantities, and substituting this into (2.16) shows that $T(x_0,x)$ has the form

$$T(x_0,x)=\rho_0(x)\sigma_0+i\rho_1(x)\sigma_1+\rho_2(x)\sigma_2+\rho_3(x)\sigma_3 , \tag{2.17}$$

where the $\rho_\mu(x)$ are real functions of E and M. This particular representation of $T(x_0,x)$ will prove most useful in Sec. III.

This particular form of $T(x_0,x)$ is intimately related to current conservation in the interval $[x_0,x]$. For a stationary state the continuity equation reduces in one dimension to the current conservation condition $(d/dx)j_x=0$. In our notation this implies $j_x=\Psi^+(x)\sigma_1\Psi(x)=$const. Applying this condition at the points x_0, and some distant point $x>x_0$, we find from (2.7) that the function $T(x_0,x)$ must satisfy the constraint

$$T^+(x_0,x)\sigma_1 T(x_0,x)=\sigma_1 . \tag{2.18}$$

If we multiply both sides of (2.18) by each of the four σ_μ, take the trace and then invoke (2.4), then a system of four complex equations will result, the solution of which is consistent with the representation of $T(x_0,x)$ as given in (2.17). Furthermore, we find that $T(x_0,x)$ must satisfy $\det[T(x_0,x)]=1$, a feature of $T(x_0,x)$ that may previously have been deduced from (2.7). In terms of the parameters ρ_μ, the condition $\det[T(x_0,x)]=1$ implies $\rho_0^2+\rho_1^2-\rho_2^2-\rho_3^2=1$.

D. Unit cell

In most of this paper we will use potentials of the form shown in (2.13). As such it is convenient to define a unit cell as in Fig. 2.

The delta function of strength λ_j is situated at the point $x_j-\varepsilon$. If we define $\Theta^j(\lambda_j)$ as the operator connecting the wave function at x_j to the wave function at x_{j-1}, then for the scalar potential we have (for $\varepsilon\to 0$)

$$\Theta^j(\lambda_j)_s=\exp(-\sigma_2\lambda_j)\exp(i\eta_1 k l_j) . \tag{2.19a}$$

For the vector potential we obtain

$$\Theta^j(\lambda_j)_v=\exp(-i\sigma_1\lambda_j)\exp(i\eta_1 k l_j) , \tag{2.19b}$$

where $l_j=x_j-x_{j-1}$ and η_1 is as given in (2.6).

$T(x_0,x)$, for the potential as given by (2.13), may then be written as

$$T(x_0,x_N)=\exp(i\eta_1 k l_N)\Theta^{N-1}(\lambda_{N-1})\cdots\Theta^1(\lambda_1) , \tag{2.20}$$

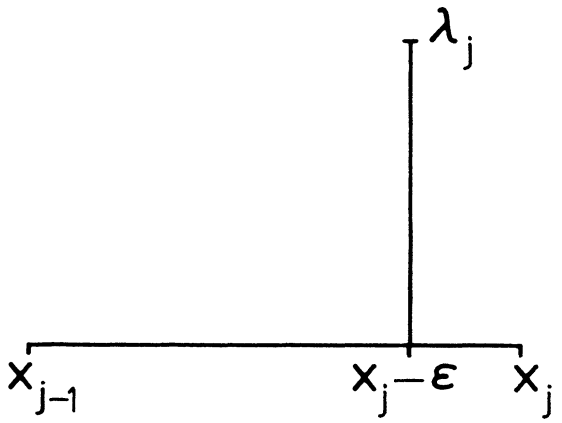

FIG. 2. The unit cell.

where $\Theta^j(\lambda_j)$ is given by either (2.19a) or (2.19b), depending on the Lorentz character of the potential. For future reference we note that $\det[\Theta^j(\lambda_j)]=1$, as $\mathrm{Tr}\sigma_1=\mathrm{Tr}\sigma_2=\mathrm{Tr}\eta_1=0$.

E. The periodic system

The simplest structure is the periodic system where a particular unit cell structure is repeated along the entire length of the interval. This potential is normally referred to as being of Kronig-Penney type.[17] If the cell width and delta-function strength are given by l and λ, respectively, then we can express Θ in a form analogous to (2.17), as

$$\Theta(\lambda)=\alpha_0\sigma_0+i\alpha_1\sigma_1+\alpha_2\sigma_2+\alpha_3\sigma_3 , \tag{2.21}$$

where $\alpha_\mu=(\frac{1}{2})\mathrm{Tr}[\Theta(\lambda)\sigma_\mu]$ may be determined from (2.19a) or (2.19b), and are given explicitly in Eq. (3.4). It is now a simple matter to determine $[\Theta(\lambda)]^n$ for any integer n as this will prove useful in our later work. Two cases exist. (i) $|\alpha_0|\le 1$: If we let $\alpha_0=\cos\omega$, for some real ω, then we can show by induction that

$$[\Theta(\lambda)]^n=\sigma_0\cos n\omega + (i\alpha_1\sigma_1+\alpha_2\sigma_2+\alpha_3\sigma_3)\mathrm{cosec}\,\omega\sin n\omega . \tag{2.22}$$

(ii) $|\alpha_0|>1$: If we let $\alpha_0=(-1)^q\cosh n\nu$ for some $\nu>0$ and $q=0$ or 1 then we also find

$$[\Theta(\lambda)]^n=(-1)^{nq}[\sigma_0\cosh n\nu+(-1)^q(i\alpha_1\sigma_1+\alpha_2\sigma_2+\alpha_3\sigma_3)\mathrm{cosech}\,\nu\sinh n\nu] . \tag{2.23}$$

III. THE ONE-DIMENSIONAL BOX

In our model of the nucleus the constituent quarks, while being confined in nucleon-like structures, are permitted to tunnel into adjacent nucleons. This tunneling lifts the degeneracy of quark energies and results in the formation of energy bands. Using a similar model based on the Klein-Gordon equation, Goldman and Stephenson[6] have shown that an appreciable contribution to the binding energy of the nucleus arises due to the splitting of the degenerate states, even where the barrier to tunneling is large and the corresponding bandwidth is small.

For our model we introduce the internucleon tunneling along with the long-range confinement property of the nucleus via a composite scalar potential of the form

$$S(x)=S_c(x)+S_{\text{in}}(x) , \tag{3.1}$$

where $S_c(x)$ represents the confining potential and $S_{\text{in}}(x)$ is the internal potential representing the barrier to quark tunneling. In contrast to the Goldman and Stephenson formulation, which used the Klein-Gordon equation, our treatment is based on the Dirac equation. Regrettably this, with the desire to obtain insight from an analytic model, forces us to limit our consideration to one dimension. The potential chosen is assumed to be of the Lorentz scalar type for reasons discussed above and in Sec. V.

For a scalar internucleon potential given explicitly by (2.13) we may relate the wave function at the point x_0 to the wave function at some distant point x_j (immediately after the jth delta function) via (2.7) as

$$\Psi(x_j)=T(x_0,x_j)\Psi(x_0) , \tag{3.2}$$

where $T(x_0,x_j)$ is of the form given by (2.20) for a scalar potential. As such we may write $\Theta^j(\lambda_j)$ for the jth cell in a form analogous to (2.21) as

$$\Theta^j(\lambda_j)=\alpha_0^j\sigma_0+i\alpha_1^j\sigma_1+\alpha_2^j\sigma_2+\alpha_3^j\sigma_3 , \tag{3.3}$$

where the α_μ^j are given by

$$A_j^{\pm}=\alpha_0^j\pm\alpha_2^j=\left[c_j\mp\frac{m}{k}s_j\right]\exp\{\mp\lambda_j\} , \tag{3.4a}$$

$$B_j^{\pm}=\alpha_3^j\pm\alpha_1^j=\pm\frac{E}{k}s_j\exp\{\mp\lambda_j\} , \tag{3.4b}$$

for $c_j=\cos kl_j$ and $s_j=\sin kl_j$.

Confinement is modeled by introducing delta-function potentials of infinite strength at the extremes of the system. As is shown in Sec. V, this is sufficient to provide zero transmission of the wave function when the potential enters the Dirac equation as a scalar. In our model the confinement potential is given explicity by

$$S_c(x)=\lim_{\lambda\to\infty}\lambda[\delta(x-x_0)+\delta(x-x_N)] . \tag{3.5}$$

The presence of an infinite strength delta function places constraints on the wave function in its immediate neighborhood. To exhibit this we recall from (2.14a) that the discontinuity in the wave function due to a scalar delta function of strength λ at some point x_j is given by

$$\Psi(x_j\pm\varepsilon)=\exp(\mp\sigma_2\lambda)\Psi(x_j\mp\varepsilon) , \tag{3.6}$$

where ε represents an infinitesimally small positive length.

Writing $\xi=\exp(\lambda)$ allows this to be rewritten as

$$\Psi(x_j\pm\varepsilon)=\tfrac{1}{2}[\xi(\sigma_0\mp\sigma_2)+(1/\xi)(\sigma_0\pm\sigma_2)]\Psi(x_j\mp\varepsilon) . \tag{3.7}$$

By multiplying both sides of (3.7) by $(\sigma_0\pm\sigma_2)$ and utilizing the algebra of the σ_μ given in (2.4) we obtain

$$(\sigma_0\pm\sigma_2)\Psi(x_j\pm\varepsilon)=(1/\xi)(\sigma_0\pm\sigma_2)\Psi(x_j\mp\varepsilon) . \tag{3.8}$$

For an infinite strength, delta function $\lambda\to\infty$ and, hence, $1/\xi\to 0$. This implies that the wave function adjacent to an infinite strength delta function must satisfy the constraint

$$(\sigma_0\pm\sigma_2)\Psi(x_j\pm\varepsilon)=0 . \tag{3.9}$$

The presence of an infinite strength scalar delta-function potential therefore has the same effect as imposing the fixed boundary condition $\sigma_2\Psi=\pm\Psi$. In the Bag model[18] confinement is imposed by demanding that the wave function at the boundary of the cavity satisfies $\bar{\Psi}\Psi=0$. It is a simple problem, using the anticommutivity of the σ_μ, to show that the constraint (3.9) necessarily implies that the wave function on either side of an infinite strength delta function also satisfies $\bar{\Psi}\Psi=0$. (This result is an example of the continuity of $\bar{\Psi}\Psi$ over a scalar delta function of arbitrary strength, a fact noted by McKellar and Stephenson.[7]) The resulting eigenenergies of our model will coincide with those of a one-dimensional bag with internal potentials.

For our confinement potential the constraint (3.9) implies that the wave function at the points $x_0+\varepsilon$ and $x_N-\varepsilon$ must satisfy the boundary conditions

$$(\sigma_0+\sigma_2)\Psi(x_0+\varepsilon)=0 \tag{3.10a}$$

and

$$(\sigma_0-\sigma_2)\Psi(x_N-\varepsilon)=0 , \tag{3.10b}$$

respectively.

On relating the wave function at the two extremes of the system via (2.7) we obtain

$$\Psi(x_N-\varepsilon)=T(x_0,x_N)\Psi(x_0+\varepsilon) , \tag{3.11}$$

where $T(x_0,x_N)$ is given explicitly by (2.20) for a scalar potential.

In order to derive the eigenvalue condition it is not necessary to resort to this explicit representation. Rather we may substitute the general form of $T(x_0,x_N)$ from (2.17) into (3.11). By applying (3.10b) to (3.11) and utilizing (3.10a) we are able to show that the following condition must hold for general $T(x_0,x_N)$

$$[\rho_0(x_N)-\rho_2(x_N)]\Psi(x_0+\varepsilon)=0 . \tag{3.12}$$

Since $\Psi(x_0+\varepsilon)\neq 0$, the quantized energy levels of the system must be given by the solutions of the linear equation

$$\rho_0(x_N)-\rho_2(x_N)=0 . \tag{3.13}$$

To solve this equation for a general system of N cells, it is convenient to define the following functions:

$$F_j=\rho_0(x_j)-\rho_2(x_j)=\tfrac{1}{2}\{\text{Tr}[\sigma_0T(x_0,x_j)]-\text{Tr}[\sigma_2T(x_0,x_j)]\} , \tag{3.14a}$$

$$G_j=\rho_3(x_j)+\rho_1(x_j)=\tfrac{1}{2}\{\text{Tr}[\sigma_3T(x_0,x_j)]-i\,\text{Tr}[\sigma_1T(x_0,x_j)]\} . \tag{3.14b}$$

In this notation the energy eigenvalue equation (3.13) obtains the simple form

$$F_N=0\ . \tag{3.15}$$

Since for arbitrary j we may write

$$T(x_{j-1},x_{j+1})=\Theta^{j+1}(\lambda_{j+1})T(x_{j-1},x_j)\ , \tag{3.16}$$

we are able to use (3.3) and (3.4) to derive the relationships

$$F_{j+1}=A^-_{j+1}F_j+B^-_{j+1}G_j=\{[c_{j+1}+(m/k)s_{j+1}]F_j -(E/k)s_{j+1}G_j\}\exp(\lambda_{j+1})\ , \tag{3.17a}$$

$$G_{j+1}=B^+_{j+1}F_j+A^+_{j+1}G_j=\{[c_{j+1}-(m/k)s_{j+1}]G_j +(E/k)s_{j+1}F_j\}\exp(-\lambda_{j+1})\ . \tag{3.17b}$$

Eliminating G from (3.17) produces a recursion relationship for the F_j

$$F_{j+1}-D_jF_j+C_jF_{j-1}=0\ . \tag{3.18}$$

where $C_j=(B^-_{j+1}/B^-_j)$ and $D_j=A^-_{j+1}+C_jA^+_j$.

Since for the $N=0$ and $N=1$ systems

$$F_0=1 \quad \text{and} \quad F_1=A^-_1=\alpha^1_0-\alpha^1_2\ , \tag{3.19}$$

respectively, we are able to use (3.18) iteratively to derive F_j for any integer j. Once F_N is determined, putting $F_N=0$ will give the energy eigenvalues. For general N we can rewrite the condition $F_N=0$ simply in terms of the α^j_μ as the continued fraction

$$A^-_1+\mathop{\mathsf{K}}_{j=1}^{N-1}\left[-\frac{C_j}{D_j}\right]=A^-_1-\cfrac{C_1}{D_1-\cfrac{C_2}{D_2-\cfrac{C_3}{D_3-\ddots-\cfrac{C_{N-1}}{D_{N-1}}}}}=0\ , \tag{3.20}$$

where K is a continued fraction in the notation of Jones and Thron.[19]

Equation (3.20) is the key result of this paper, it gives the energy eigenvalues for an arbitrary array of delta-function potentials confined in a finite box. We now apply it to some special cases and investigate some of the more simple systems.

A. $N=1$

The most simple system is that consisting of a single cell or nucleon. The $N=1$ result is the relativistic analogue of a particle in an infinite box.[20] Putting $F_1=0$ gives the quark energies as solutions of

$$\tan kl_1+k/m=0\ , \tag{3.21}$$

which is precisely the energy of a one-dimensional bag. Note that the allowed wave numbers k depend explicitly on the quark mass. For a system of N such nucleons, widely separated, each energy level would be N-fold degenerate. To see how this degeneracy is split when tunneling is allowed we look at the $N=2$ case.

B. $N=2$

Putting $F_2=0$ into (3.18) for $j=1$ results in the condition

$$[\tan kl_1+(k/m)][\tan kl_2+(k/m)]-(E^2/m^2)\exp(-2\lambda_1)\tan kl_1\tan kl_2=0\ . \tag{3.22}$$

The quark energies no longer correspond to those given by (3.21) except in the limit that λ_1 becomes infinite, in which case there is no tunneling. For finite λ_1, the second term in (3.22) remains nonzero, thus splitting the degenerate energy levels into doublets. As Kronig and Penney showed in the nonrelativistic system, the band structure is therefore a natural consequence of quark tunneling.

In principle, we could continue this analysis to larger N systems, but it is simpler to consider the periodic arrangement of cells (or nucleons) as a general formula for the allowed energies may then be obtained.

C. The periodic system

For a regular array of delta-function potentials with separation and internucleon potential barrier strength l and λ, respectively, we find that since $C_j=1$ and $D_j=2\alpha_0$ we are able to rewrite (3.20) as

$$\alpha_0-\alpha_2+\mathop{\mathsf{K}}^{N-1}\left[-\frac{1}{2\alpha_0}\right]=0\ . \tag{3.23}$$

This form of the eigenvalue equation allows us to calculate the energy eigenvalues for any N. However, for

large N, the extraction of solutions is quite difficult and cumbersome. Therefore we choose to derive an alternate form for (3.23) in which the properties of the solutions for general N become more transparent. In Appendix A we show that only solutions satisfying $|\alpha_0| \leq 1$, the energies of the relativistic Kronig-Penney (RKP) band,[21] are admissible as solutions of (3.23), thus implying that we need only consider (2.22) to determine the allowed energies. For $F_N = 0$ and

$$T(x_0, x_N) = \exp(i\eta_1 kl)[\Theta(\lambda)]^{N-1} ,$$

the required energies are found to be the solutions of

$$\omega = \frac{n\pi}{N} + \frac{1}{N}\tan^{-1}\left[\frac{\alpha_0}{\alpha_2}\tan\omega\right], \quad n = 0,1,2,\ldots,N-1 \tag{3.24}$$

where $\cos\omega = \alpha_0 = \cosh\lambda\cos kl + (m/k)\sinh\lambda\sin kl$ and the principal value of the inverse tangent is implied. The N solutions of (3.24) thus represent the band of energy levels into which the N-fold degenerate ground state has split.

Two interesting limiting cases of (3.24) occur for λ large and for $\lambda = 0$. For $\lambda > 1$ we find a good approximation

$$\omega \approx n\pi/N + 1 , \tag{3.25}$$

while for $\lambda = 0$ and $m = 0$ we find

$$\omega = kl = (\pi/N)(n - \tfrac{1}{2}) . \tag{3.26}$$

Furthermore, in the limit that the number of nucleons becomes infinite we obtain

$$\omega \to n\pi/N , \tag{3.27}$$

which is equivalent to the energies of a quark in a nucleus with the periodic boundary conditions $\Psi(x_0) = \Psi(x_0 + Nl)$. This suggests that for large nuclei the band levels are essentially independent of the boundary conditions and therefore, as one may have expected by analogy with the nonrelativistic result, the quark energies are independent of the mechanism of quark confinement to the nucleus.

We may determine the nonrelativistic limit of (3.24) by retaining only first-order terms in k/m and λ to obtain

$$\cos(n\pi/N) = \cos kl + (m\lambda/k)\sin kl , \tag{3.28}$$

a result that we have verified by a direct nonrelativistic calculation with the Schrödinger equation.

Previously, Goldman and Stephenson[6] have made a study of a similar model using the Klein-Gordon equation. Their replacement of the Dirac equation by a Klein-Gordon type equation permits a separation of variables in Cartesian coordinates, thus allowing a three-dimensional model of cubical nucleons to be developed. As in our model, delta-function potentials are used to represent the internucleon barriers while confinement is imposed via infinite strength scalar delta functions at the extremes of the system.[22] For massless quarks their one-dimensional energy eigenvalue equation is given by

$$\cos(n\pi/N) = \cos kl + (v/2k)\sin kl , \tag{3.29}$$

where l is the cell width and v is the internucleon delta-function strength (we have replaced $2b$ in Ref. 6 by l).

As the internucleon strength is reasonably large (so that quark barrier penetration is small) the N-plet of allowed states in the first band are confined to the small region of energies

$$k\varepsilon\left[\frac{2}{l}\tan^{-1}\left[\frac{v}{2k}\right], \frac{\pi}{l}\right] . \tag{3.30}$$

As Goldman and Stephenson have shown, such a small decrease in quark localization may still result in quite significant binding energies.

In contrast, our calculation with the Dirac equation shows that for massless quarks the quark energies are confined to the region

$$k\varepsilon\left[\frac{2}{l}\tan^{-1}\left[\tanh\frac{\lambda}{2}\right], \frac{2}{l}\tan^{-1}\left[\coth\frac{\lambda}{2}\right]\right] . \tag{3.31}$$

The fact that differences exist between the two treatments is quite evident. The reasons for this are varied and will now be discussed in terms of their effects on the respective binding energies of the systems.

D. Binding energies

Using results developed in Sec. III C, it is now possible to study the problem of determining the binding energy associated with the formation of a nucleus from its constituent nucleons in our one-dimensional model. Indeed the first question we may ask is: which configuration of nucleons possesses the lowest ground state energy? Possibly it is the 'discrete' system [Fig. 3(a)] representing N widely separated nucleons each of uniform width l with each of the quarks in the N-fold degenerate ground state, or it is the 'composite' system [Fig. 3(b)], the nucleus of our model, comprising a single box of width Nl in which the quarks occupy the N lowest energy states (which are singly filled due to the absence of the spin degree of freedom in one dimension).

In the Goldman and Stephenson formulation for massless quarks it is found that the composite system is the energetically favored configuration if the internucleon potential remains finite. (In the infinite limit the discrete and composite systems are identical and therefore energetically degenerate.) This feature also persists if the quarks are given masses. Naively, we would expect this behavior because of the extended nature of the quark wave functions in the composite system compared to the well-localized quark wave functions of the discrete system. Furthermore, it was found that, even where the internucleon potential is large, significant binding energies are still found because of the large quark momenta involved. In contrast, in our model we find that for massless quarks not only are the energies of the composite and discrete systems equivalent, but that this feature is independent of the internucleon potential in the composite system. There is no binding associated with the composite system for any delta-function strength $\lambda\varepsilon[0, \infty)$.

This result has a precedent. In fact, McKellar and Stephenson[7] have found that for a similar model of the nucleus with periodic boundary conditions, the system is unbound if the width of their internucleon square barriers tends to zero, as is the case in the delta-function limit. To obtain binding they had to use finite width square barriers.

In order to elucidate this remarkable result it is convenient to consider a specific example, the $N=2$ nucleus is the obvious choice as many features relevant for general N are also exhibited in this analytically simple system.

The ground state and first excited state energies for a massless particle confined to a one-dimensional box of width l are in general given by

$$k = g\pi/l$$

and (3.32)

$$k = (g+1)\pi/l \ ,$$

respectively, where $g\varepsilon(0,1]$ is a model-dependent parameter. For the discrete $N=2$ system the total energy E_d is then given by $E_d = 2g\pi/l$. For the composite system the total energy E_c will obviously be a function of the appropriate internucleon potential strength, say, λ. The effect of a nonzero delta-function potential is to either raise or lower the energy of a particular eigenstate, the choice being dependent upon the parity of the corresponding wave function. However, the movement of these eigenstates is such that for increasing λ, the sum of the individual eigenenergies E_c remains a nondecreasing function. As such, a minimum of E_c will occur for $\lambda=0$, which is simply given by

$$E_c(\lambda=0) = g\pi/(2l) + (g+1)\pi/(2l) = (g+\tfrac{1}{2})\pi/l \ .$$

The maximum binding energy of the composite system is then

$$E_b(\lambda=0) = E_d - E_c(\lambda=0) = (g-\tfrac{1}{2})\pi/l \ , \qquad (3.33)$$

thus implying that the composite system will be bound if and only if $g > \frac{1}{2}$.

In the Goldman and Stephenson model the ground state energy is $k=\pi/l$ and hence $g=1$. As such, the composite system is bound by an amount $E_b(v=0) = \pi/2l$. If $v\neq 0$ the binding energy is found to decrease because the ground state energy is shifted upward to $k = \pi/2l + (1/l)\tan^{-1}(v/2k)$ while the first excited state remains unaffected [Fig. 4(a)]. This lack of symmetry in the $v\neq 0$ case is directly related to the different parities of the ground state (even) and first excited state (odd) wave functions. At the position of the delta function, the wave function is zero for the odd parity solution and the energy remains unaffected. The even parity solution, however, is nonzero at this point and hence the energy of the state is a function of the delta-function potential strength. However, for v finite the binding energy will always remain nonzero:

$$E_b = (1/l)[(\pi/2) - \tan^{-1}(v/2k)] \ . \qquad (3.34)$$

When using the Dirac equation for massless particles the ground state energy is found to occur at $k=\pi/(2l)$, thus implying $g=\frac{1}{2}$. Therefore the maximum binding energy of the composite system is $E_b(\lambda=0)=0$; the discrete and composite systems are degenerate. Since this result remains in the limit $\lambda\to\infty$ (when the composite system reverts to the discrete form) it is perhaps not surprising to find that this feature also persists for any $\lambda\varepsilon[0,\infty)$. To see how this arises we must look at the wave function of

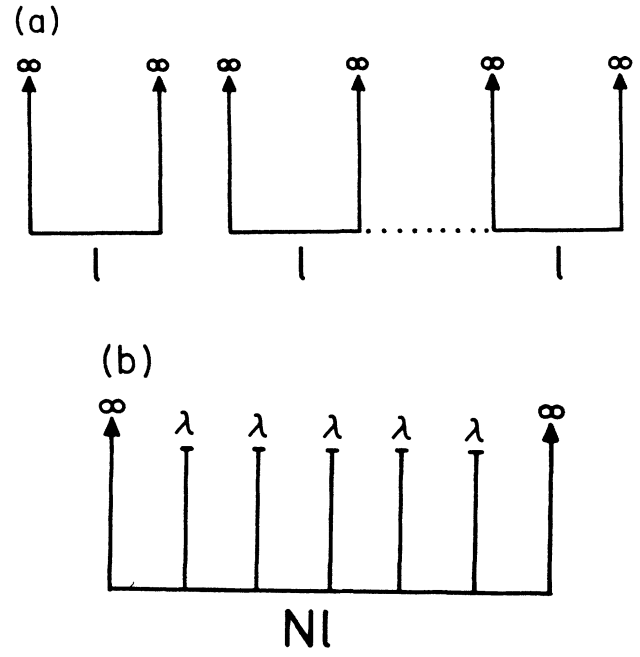

FIG. 3. (a) The discrete system consisting of N widely separated nucleons each of width l. (b) The composite system of width Nl.

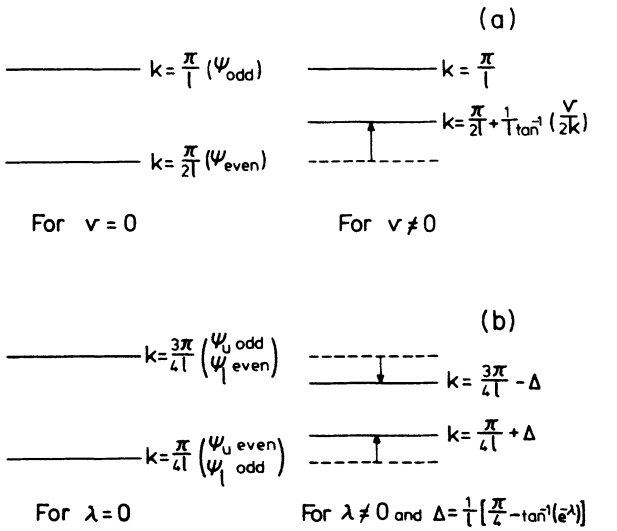

FIG. 4. First band energy spectrum for the massless $N=2$ composite system for both the zero and nonzero internucleon potential strength cases. The parities of the appropriate wave functions are also given. (a) Klein-Gordon analysis. (b) Dirac analysis.

the confined particle explicitly.

In Sec. II in order to describe the particle correctly in one dimension, a two-component spinor wave function was found to be necessary. The relative magnitude of the lower component is of order $r=k/(E+m)$, smaller than the upper component. In the nonrelativistic limit $r\rightarrow 0$ and the lower components become negligible; however, in the ultra-relativistic limit (i.e., when $m\rightarrow 0$) they assume a magnitude equal to that of the upper components of the wave function. Since the mass of the particle determines the magnitude of the lower components of the wave packet, it would seem reasonable to expect the particles eigenenergies to be explicit functions of the mass, as is the case in (3.21) and (3.22).

In the massless case the upper and lower components of the wave packet have equal magnitudes and therefore the ground and first excited state energies for $N=2$ and $\lambda=0$ are, from (3.26), $k=\pi/4l$ and $k=3\pi/4l$, respectively [Fig. 4(b)]. This ground state energy corresponds to a wave function having an even parity upper component and therefore the lower component is necessarily of odd parity.[23] For nonzero λ we would expect only the even part of the wave function, the upper components, to be affected. In analogy with the Klein-Gordon analysis we would expect the energy of the ground state to rise, and indeed we find it to be given by

$$k=(1/l)(\pi/4+\{\pi/4-\tan^{-1}[\exp(-\lambda)]\}) .$$

For the first excited state the wave function now has an upper component of odd parity and a lower component of even parity. The lower, even component can be expected to feel the delta-function potential so, for nonzero λ, it seems reasonable to expect that this would result in an analogous lowering of the first excited state energy. Indeed not only is the energy of the state lowered but, due to the inability of the scalar potential to differentiate between positive and negative energy states, it is found that this reduction is equal in magnitude to the increase in energy of the ground state. For nonzero λ the first excited state is given by

$$k=(1/l)(3\pi/4-\{\pi/4-\tan^{-1}[\exp(-\lambda)]\}) .$$

This symmetry persists only in the massless system. As the mass increases the lower component of the wave function decreases so that the decrease in energy of the first excited state becomes smaller than the increase in energy of the ground state. This loss of symmetry is most evident in the infinite mass limit where the lower component of the wave function becomes negligible and the wave numbers of the allowed states coincide with those of the Klein-Gordon or Schrödinger formulations. This result is not surprising as the Klein-Gordon equation of the Goldman and Stephenson theory was derived from the Dirac equation by ignoring the connection between the two energy states of the wave functions. Another way of looking at this involves recognizing that the Klein-Gordon equation is a second order differential equation while the Dirac equation is only of first order. Therefore, in order to completely specify the development of the Klein-Gordon wave function, we must specify the first derivative of the wave function in addition to the wave function itself. This procedure is equivalent to choosing the sign of the energy.

As the Dirac system is bound for an infinite quark mass we may now look to see if this property also persists in the finite mass domain. For small m (i.e., to first order in m/k) the single nucleon energy is, from (3.21), approximately given by $k\approx\pi/2l+2m/\pi$ and therefore we find a binding energy of order $E_b(\lambda=0)\approx 2m/\pi$ associated with the composite system. For small nonzero λ this will be sufficient to overcome the small rise in energy due to the loss of symmetry between the positive and negative states and to effectively bind the system.

In the nonrelativistic limit (i.e., to first order in k/m) the binding for λ small as obtained from (3.28) is of order $E_b\approx 2m\lambda/\pi$. Consequently, both the near relativistic and nonrelativistic limits produce binding to first order in the mass, a rather surprising result given their remarkably diverse origins.

The nonexistence of binding in the composite system, whilst demonstrated for the massless $N=2$ system, persists for any $N\geq 1$ as we shall now show using (3.23). For a given k the transformation $k\rightarrow\pi/l-k$ implies $\alpha_0\rightarrow-\alpha_0$, $\alpha_2\rightarrow-\alpha_2$, and $\mathrm{K}(-1/2\alpha_0)\rightarrow-\mathrm{K}(-1/2\alpha_0)$. Therefore if k is a solution of (3.23), then $(\pi/l-k)$ is as well. For even N this results in $N/2$ pairs of solutions of the form k and $(\pi/l-k)$, while for N odd there are $(N-1)/2$ pairs of solutions of this form as well as the single solution $k=\pi/2l$. In either case we find that $E_c=E_d=N\pi/2l$ and hence, $E_b=0$ for any $\lambda\varepsilon[0,\infty)$.

This general feature of the model is unfortunate. Our aim in this instance was to develop a simple one-dimensional model of the nucleus from which nominal binding energies could be determined with a minimum of parameters. However, as we have shown, our massless one-dimensional nucleus with delta-function internucleon barriers is insufficient in this respect as the system remains unbound independent of the internucleon delta-function strength.

This property of the system remains unchanged even upon the introduction of an additional quark degree of freedom, spin and/or color being the relevant choices. As stated previously, in one dimension spin is not a well defined concept whilst color, while succeeding in tripling the occupancy of the energy eigenstates, ultimately forces the baryons to form color singlets, with zero binding being the net result (color magnetic spin splitting would be expected to act in a repulsive fashion[2] and so it would not contribute to binding).

There are several ways of rectifying this difficulty. These include replacing the delta-function potentials with more realistic square barriers (as the increase in the potential strength as the quark moves away from the center of the confining potential could then be modeled more effectively), attributing a mass to the quarks, or introducing some random fluctuation of the internucleon barrier position or height. We may also replace the pure scalar potential with a combination scalar-vector potential as the presence of the vector potential component would destroy the symmetry between the two energy states of the particle without compromising the confinement prop-

erties of the scalar component, as long as $S(x)-V(x)$ is positive at the position of the delta function (as we will show in Sec. V). As our primary purpose is insight rather than numbers we will continue to study the possibilities exhibited by the one-dimensional system, leaving these developments for possible future work.

IV. THE TAMM MODEL

The formalism of Sec. II may also be utilized to analyze the problem of energy states localized about a particular surface. The potential chosen for this model is the relativistic analogue of the Tamm potential[13] comprising a semi-infinite array of periodic delta functions. Previous analyses have considered only a vector Tamm potential. However, as surface states arising from a scalar potential are of obvious significance with regard to our nuclear model of Sec. III, we will derive conditions for the existence of surface states for both potential types.

The Tamm potential for a scalar potential is given by (Fig. 5)

$$S(x)=S_0\theta(x_0-x)+\lambda\sum_{i=1}^{\infty}\delta(x-x_i),\quad S_0>0\ . \tag{4.1}$$

In order to solve the Dirac equation for this potential we are forced to specify a particular representation for the σ_μ. Without any loss of generality we choose the Dirac representation where

$$\sigma_0=I,\quad \sigma_1=\sigma_x,\quad \sigma_2=\sigma_y,\quad \sigma_3=\sigma_z\ , \tag{4.2}$$

with σ_x, σ_y, and σ_z the usual Pauli matrices.

Solving the Dirac equation in this representation in the region $(-\infty,x_0)$ we obtain the general form

$$\Psi(x)=\begin{bmatrix}1\\R_s\end{bmatrix}A\exp(iK_sx)+\begin{bmatrix}1\\-R_s\end{bmatrix}B\exp(-iK_sx)\ , \tag{4.3}$$

where $R_s=K_s/(E+S_0+m)$ and $K_s^2=E^2-(m+S_0)^2$.

A surface state by definition is an energy state corresponding to a particle localized about a surface. As a consequence, the wave function of such a particle must diminish as its displacement from the surface increases. In the interval $(-\infty,x_0)$ the wave function given by (4.3) will be oscillatory for energies $E>m+S_0$. Therefore we may confine our analysis to those energies which produce a wave function of exponential nature, namely $m<E<m+S_0$. As the quarks of our model satisfy this constraint, the possibility of quark surface state arises. We shall comment on this prospect after the equation for the existence of the surface states has been developed.

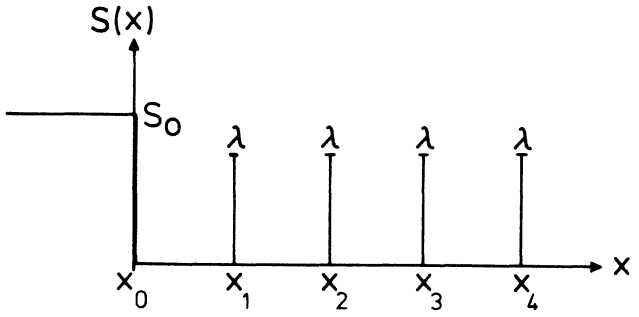

FIG. 5. The Tamm potential comprising a semi-infinite array of delta-function potentials of strength λ.

Similarly, with a vector potential the wave function is exponential for $m<E<m+V_0$ for $V_0<m$, and $V_0-m<E<V_0+m$ for $V_0>2m$. As the wave function must be bounded in the region $(-\infty,x_0)$, we choose the exponentially decaying part of (4.3) as

$$\Psi(x)=\begin{bmatrix}1\\-i|R|\end{bmatrix}B\exp(|K|x)\ , \tag{4.4}$$

where we have used $R=i|R|$ and $K=i|K|$.

For the region (x_0,∞) we may relate the wave function at the beginning of the first cell to the wave function at the beginning of the $(N+1)^{\text{st}}$ cell via

$$\Psi(x_N)=[\Theta(\lambda)]^N\Psi(x_0)\ . \tag{4.5}$$

Since the wave function is continuous at the point $x=x_0$ we find from (4.4) and (4.5) that

$$\Psi(x_N)=[\Theta(\lambda)]^N\begin{bmatrix}1\\-i|R|\end{bmatrix}B\exp[|K|x_0]\ . \tag{4.6}$$

The allowed energies are those for which $\Psi(x_N)$ is bounded in the limit $N\to\infty$. The two distinct classes of energies satisfying this criterion form the energy bands and the surface states.

A. Energy bands

For the case where $|\alpha_0|\le 1$ we have for $n=N$ in (2.22)

$$[\Theta(\lambda)]^N=\sigma_0\cos N\omega+(i\alpha_1\sigma_x+\alpha_2\sigma_y+\alpha_3\sigma_z)\operatorname{cosec}\omega\sin N\omega\ , \tag{4.7}$$

which is obviously bounded as $N\to\infty$ for all values of ω except perhaps where $\omega=j\pi$ ($|\alpha_0|=1$, j an integer). That this divergence does not eventuate for $|\alpha_0|=1$ becomes evident by noting that, for $\omega=j\pi$, (4.6) becomes

$$\Psi(x_N)=(-1)^{Nj}\begin{bmatrix}1+N(-1)^j[\alpha_3+|R|(\alpha_1-\alpha_2)]\\|R|\{-1+N(-1)^j[\alpha_3+(1/|R|)(\alpha_1+\alpha_2)]\}\end{bmatrix}B\exp[|K|x_0]\ .$$

For $\Psi(x_N)$ to remain finite in the limit $N\to\infty$ the coefficients of N in the spinor part of the wave function must both be zero. This in turn implies that the α_μ must satisfy the condition $\alpha_1^2-\alpha_2^2-\alpha_3^2=0$. Since $\det[\theta(\lambda)]=\alpha_0^2+\alpha_1^2-\alpha_2^2-\alpha_3^2=1$ we find that for $|\alpha_0|=1$ this condition is obeyed and thus $\Psi(x_N)$ remains finite.

The energy bands for a scalar potential therefore occur where

$$|\cos kl\cosh\lambda+(m/k)\sin kl\sinh\lambda|\le 1\ , \tag{4.8a}$$

while for a vector potential the energy bands are given by

$$|\cos kl\cos\lambda+(E/k)\sin kl\sin\lambda|\le 1\ . \tag{4.8b}$$

As mentioned previously these solutions result because of the geometry of the potential in the region (x_0,∞) (they form the RKP band[21]) and as such they are independent of the boundary conditions at x_0. The surface states, however, critically depend on the boundary conditions at the interface as we shall see in Sec. IV B.

B. Surface states

For the case where $|\alpha_0|>1$ we put $n=N$ into (2.23) to obtain

$$[\Theta(\lambda)]^N=(-1)^{Nq}[\Xi(\lambda)\sinh N\nu+I\exp(-N\nu)]\ , \tag{4.9}$$

where $\Xi(\lambda)=I+\xi(i\alpha_1\sigma_x+\alpha_2\sigma_y+\alpha_3\sigma_z)$ and $\xi^{-1}=(-1)^q\sinh\nu$ with ν defined as in (2.23).

For that range of energies satisfying

$$\Xi(\lambda)\Psi(x_0)=0\ , \tag{4.10}$$

the wave function (4.6) becomes

$$\Psi(x_N)=(-1)^{Nq}\exp(-N\nu)\Psi(x_0)\ ,$$

where $\Psi(x_0)$ is given by (4.4). This wave function decays exponentially as N increases, thus implying that states which satisfy (4.10) are localized about $x=x_0$, i.e., they represent surface states. From the two-component equation (4.10) we may derive the following two conditions for the development of surface states

$$-\alpha_3+|R|(\alpha_2-\alpha_1)=1/\xi\ , \tag{4.11a}$$

$$\alpha_3|R|+(\alpha_2+\alpha_1)=|R|/\xi\ , \tag{4.11b}$$

which, upon eliminating ξ, results in the condition

$$\alpha_3-\tfrac{1}{2}(|R|-|R|^{-1})\alpha_2+\tfrac{1}{2}(|R|+|R|^{-1})\alpha_1=0\ , \tag{4.12}$$

which must be satisfied by any surface state.

For the scalar potential (4.12) becomes

$$k\cot kl=S_0\coth\lambda-|K_s|\ , \tag{4.13a}$$

while for the vector potential we have

$$k\cot kl=V_0\cot\lambda-|K_v|\ , \tag{4.13b}$$

where $K_v^2=(E-V_0)^2-m^2$.

All surface states are solutions of (4.12) [and consequently (4.13)], however, due to this method of derivation, solutions not representing surface states may also occur. To extract the true surface states we must develop an existence condition which will exclude these extraneous solutions. The derivation of this condition is performed in Appendix B with the results given by

$$(m+S_0)-|K_s|\coth\lambda>0\ , \tag{4.14a}$$

for the scalar potential; while for the vector potential we find

$$(E-V_0)-|K_v|\cot\lambda>0\ , \tag{4.14b}$$

for energies satisfying $|\alpha_0^v|>1$ in agreement with the Subramanian and Bhagwat result.

In the nonrelativistic limit if we take $E=m+\varepsilon$ (where $\varepsilon\ll m$) then both (4.13a) and (4.13b) reduce to (where $|V_0|=|S_0|=U_0\ll 1$)

$$k\cot kl=(U_0/\lambda)-|K_{\rm nr}|\ , \tag{4.15}$$

where $|K_{\rm nr}|=2m(U_0-\varepsilon)$. The existence conditions also become indistinguishable in the nonrelativistic limit

$$U_0-\varepsilon<\tfrac{1}{2}m\lambda^2\ , \tag{4.16}$$

for energies satisfying

$$\left|\cos kl+\frac{m\lambda}{k}\sin kl\right|>1\ , \tag{4.17}$$

in accordance with the Tamm results. Steslicka and Davison[11] have contended that relativistic surface states having no nonrelativistic counterparts—the Dirac surface states (DSS)—exist for the vector Tamm potential. Our results concur with those of Subramanian and Bhagwat in finding that these DSS do not exist. We have further shown that DSS do not exist in the scalar case either.

For massless quarks with $S_0\gg E(=k)$ we find that surface states can exist for large λ for k satisfying [from (4.14a)]

$$k>S_0\,\mathrm{sech}\lambda\ , \tag{4.18}$$

as well as (4.13a) and $|\alpha_0^s|>1$.

Since in our model S_0 is extremely large, the lower band edge for the surface states will occur at large k even when the internucleon tunneling is not large (λ is large). The presence of these higher energy surface eigenstates in the quark energy spectrum is therefore of minimal importance in our model as the lowest energy states will be preferentially filled. Admittedly, the potential at the surface of our model is quite idealistic, a potential rising linearly with displacement from the boundary of the nucleus would be a more realistic representation of the actual physical situation;[24] however, the point remains that large potential gradients at the surface of the nucleus will force the surface eigenstates into energy domains which would be inaccessible to the constituent quarks of a ground state nucleus.

V. TRANSMISSION

We now investigate the phenomena of transmission for a relativistic particle in both Lorentz vector and scalar potentials, respectively. Recently, Gumbs[14] determined

the relativistic scattering states of a particle in a one-dimensional system consisting of a finite array of delta-function potentials of arbitrary position and strength. The Greens function method chosen required the value of the wave function at the position of each delta-function potential be known. To solve this problem Gumbs chose to represent the wave function by an average of the wave function before and after the delta function. However, the net effect is equivalent to integrating the Dirac equation over a delta function and, whereas, this procedure is permissible in the nonrelativistic treatment[25] its use in the relativistic case is, as discussed in Ref. 7, incorrect. As the formalism of Sec. II resolves this problem we are now able to determine the transmission coefficients of the relativistic system in such a way as to avoid this anomaly.

The scattering coefficients are usually obtained from the transfer matrix,[15] $M(x_0,x_N)$, a procedure we will also adopt. In Appendix C the relationship between $T(x_0,x_N)$ and $M(x_0,x_N)$ is found to be given by

$$T(x_0,x_N)=QM(x_0,x_N)Q^{-1} , \tag{5.1}$$

where

$$Q=\begin{bmatrix}1 & 1\\ r & -r\end{bmatrix}$$

and $r=k/(E+m)$.

The transmission ($\mathcal{T}$) and reflection ($\mathcal{R}$) coefficients may be expressed in terms of the elements of $M(x_0,x_N)$, i.e.,

$$\mathcal{T}=|M_{22}|^{-2} \text{ and } \mathcal{R}=|M_{21}/M_{22}|^2 .$$

In terms of the function $T(x_0,x_N)$ we obtain

$$\mathcal{T}=4\{[\mathrm{Tr}T(x_0,x_N)]^2-[\mathrm{Tr}\eta_x T(x_0,x_N)]^2\}^{-1} , \tag{5.2a}$$

$$\mathcal{R}=\frac{[\mathrm{Tr}\eta_z T(x_0,x_N)]^2+[\mathrm{Tr}\eta_y T(x_0,x_N)]^2}{[\mathrm{Tr}T(x_0,x_N)]^2-[\mathrm{Tr}\eta_x T(x_0,x_N)]^2} , \tag{5.2b}$$

where the η_μ are defined as in (2.6).

Using the general form of $T(x_0,x_N)$ as given by (2.17) along with the constraint $\det[T(x_0,x_N)]=1$, we can also show that $\mathcal{T}+\mathcal{R}=1$ as expected.

We may use (5.2) to determine the scattering coefficients for any finite array of delta functions of either scalar or vector origin. We begin by discussing some simple cases.

A. Single delta function

As is well known, for a single delta-function barrier of strength λ, the transmission coefficients are given by

$$\mathcal{T}_s^{-1}=1+(E^2/k^2)\sinh^2\lambda , \tag{5.3a}$$

and

$$\mathcal{T}_v^{-1}=1+(m^2/k^2)\sin^2\lambda , \tag{5.3b}$$

for the scalar and vector cases, respectively. It is evident from (5.3b) that the transmission coefficient for the vector potential is bounded to the region $[(1+m^2/k^2)^{-1},1]$ irrespective of the strength of the potential. This manifestation of the Klein paradox is most clearly demonstrated in the massless limit where the barrier becomes transparent to the nonrelativistic particle, as noted previously in Sec. II.

In contrast, from (5.3a) we find that the transmission coefficient becomes zero in the presence of an infinite scalar field, irrespective of the mass of the particle. Indeed for the massless case we find that

$$\mathcal{T}_s=\mathrm{sech}^2\lambda , \tag{5.4}$$

where $\mathcal{T}_s<1$ for $\lambda\neq 0$.

For a combination vector and scalar delta-function potential of strengths λ_v and λ_s, respectively, it is a simple problem, using the formalism developed above, to show that the transmission coefficient is given by

$$\mathcal{T}_{s/v}^{-1}=1+\frac{(E+m\gamma)^2}{k^2(1-\gamma^2)}\sinh^2\Lambda , \tag{5.5}$$

where $\Lambda^2=\lambda_s^2-\lambda_v^2$ and $\gamma=\lambda_v/\lambda_s$.

Using the appropriate limits in (5.5) allows the retrieval of the scalar and vector results given in (5.3); however, for the case where both λ_s and λ_v are nonzero, the implications are not as apparent.

If $\lambda_v>\lambda_s$ then, as Λ is necessarily complex, we must have $\mathcal{T}>0$, even in the limit $\lambda_v\to\infty$. Physically, this result implies that there will always be leakage through the barrier where the vector potential dominates the scalar potential.

If $\lambda_v=\lambda_s=\lambda$ then $\Lambda=0$ and $\gamma=1$, allowing (5.5) to be rewritten as

$$\mathcal{T}_{s=v}^{-1}=1+\frac{(E+m)^2\lambda^2}{k^2} . \tag{5.6}$$

In the limit that λ becomes infinite there will be zero transmission through the barrier. If $\lambda_v<\lambda_s$ then Λ is necessarily real such that in the limit $\lambda_s\to\infty$ ($\Lambda\to\infty$), there will be no transmission through the barrier.

Accordingly, we have shown that an infinite strength scalar delta-function potential provides confinement irrespective of the strength of the vector potential. This result agrees with that of Fishbane *et al.*[24] who found that the Klein paradox problems associated with a strong vector field may be avoided with a combination vector and scalar field if the scalar component of the field is considerably larger than the vector component. It was on the basis of this result, and others from the literature,[4,5] that an infinite strength scalar potential was used in Sec. III to provide confinement in our one-dimensional model of the nucleus.

B. Periodic delta functions

Again we consider the special case of a periodic system of N delta-functions of strength λ and separation l. We find the following relationship results for the scalar potential:

$$\mathcal{T}_s^{-1}=1+(E^2/k^2)\sinh^2\lambda g_N^2(\alpha_0^s) , \tag{5.7a}$$

while for the vector potential

$$\mathcal{T}_v^{-1}=1+(m^2/k^2)\sin^2\lambda g_N^2(\alpha_0^v)\ , \tag{5.7b}$$

where

$$g_N(\alpha_0^p)=(2\alpha_0^p)^{N-1}-\binom{N-2}{1}(2\alpha_0^p)^{N-3}+\binom{N-3}{2}(2\alpha_0^p)^{N-5}+\cdots$$

and p denotes whether the scalar or vector potential should be used.

While exhibiting the same gross features as the single delta-function case we also find that for energies satisfying

$$g_N(\alpha_0^p)=0,\quad N>1 \tag{5.8}$$

the transmission coefficient of (5.7) becomes unity for both Lorentz potentials irrespective of the strength of the potential (except for $\lambda\to\infty$ in the scalar case). This phenomena, which occurs in any symmetric many-barrier system, is known as resonant transmission. In this instance the reflection coefficient becomes zero due to destructive interference between multiple reflected waves.

In the nonrelativistic system, it is well known that the periodic barrier system becomes transparent for at least one energy from each allowed band.[26] From (5.8) we see that this feature persists in the relativistic domain as $\alpha_0^p=0$ is a root of $g_N(\alpha_0^p)$ independent of the potential strength. That this energy belongs to an allowed band is necessary because for $|\alpha_0^p|>1$ we find

$$g_N(\alpha_0^p)=\frac{\sinh N\nu}{\sinh\nu}\neq 0,\quad \text{for } N>0\ , \tag{5.9}$$

and therefore no roots of $g_N(\alpha_0^p)$ can exist for $|\alpha_0^p|>1$. Additionally we find that as $N\to\infty$ then $g_N(\alpha_0^p)\to\infty$, thus indicating that the transmission coefficient for each energy from a forbidden energy region must vanish in the large barrier limit. These results suggest that many of the general properties of nonrelativistic periodic systems, as expound by Kowalski and Fry,[26] may also persevere in the relativistic domain.

VI. CONCLUSION

The prime motivation of this paper was to examine the prominent features of a simple one-dimensional nuclear model comprising a lattice of confining potentials and incorporating quarks as its fundamental constituents. By keeping the model in such an elementary form it was hoped that insight into the confinement problem in a many nucleon system could be obtained. Some of the result thus far obtained have been quite unexpected, showing that the new insights provided by this model are valuable.

In light of previous work it had seemed reasonable to expect that the composite nucleus of our model would prove to be bound for massless quarks due to the inherent decrease in localization in the composite system. Surprisingly, this is not the case in our most basic model and therefore we must focus our attention elsewhere. Either we upgrade our simple model by introducing further mechanisms, such as nonzero quark masses, random fluctuations in delta-function position and strength, composite vector and scalar delta-function potentials or replacing the delta-function potentials with square barriers, or else we dispense with the one-dimensional system altogether and move onto the three-dimensional problem, or attempt both generalizations at once.

The main benefit of remaining in one dimension is that we are able to investigate specific features of the nucleus, as was the case with surface states, without compromising the analyticity of our formulation or the ease with which solutions can be obtained. With most of the relevant features studied in this fashion we would then possess a physical intuition from which an informed assault on the three-dimensional system could be made.

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APPENDIX A

For a periodic system (3.18) becomes, for $j=N-1$,

$$F_N-2\alpha_0F_{N-1}+F_{N-2}=0\ . \tag{A1}$$

From this recursion relationship we may write

$$X_N=\frac{F_{N-1}}{F_N}=(2\alpha_0-X_{N-1})^{-1}=\cfrac{1}{2\alpha_0-\cfrac{1}{2\alpha_0-\cfrac{1}{2\alpha_0-\cdots}}}\ . \tag{A2}$$

X_N is now a continued fraction of unit periodicity.[19] From (A1) we note that, for $F_N=0$, if either of F_{N-1} or F_{N-2} is zero then both must be zero. This would imply that $F_j=0$ for j any integer less than N. Since $F_0=1$ this is not true. Therefore if we put $F_N=0$ into (A2) then $F_{N-1}\neq 0$ implies that

$$X_N=\frac{F_{N-1}}{F_N}\to\infty\ ,$$

or that X_N must diverge. The behavior of X_N as $N\to\infty$ is determined by considering the fixed points of the transformation (A2):

$$X^*=\alpha_0\pm[(\alpha_0)^2-1]^{1/2}\ .$$

As long as X^* is real, $X_N\to X^*$ as $N\to\infty$. If, however, X^* is complex, then $X_N\to\infty$ as $N\to\infty$. The condition $F_N=0$ but $F_{N-1}\neq 0$ thus requires complex X^*. Hence, $|\alpha_0|\leq 1$ is the condition that defines the band energies.

APPENDIX B

A prescription for deriving an existence condition for surface states has been exhibited previously by Subramanian and Bhagwat,[12] at least for the vector potential case. However, for the scalar potential case we

must use a different technique in order to derive the appropriate necessary and sufficient condition for the existence of surface states. We being by reintroducing the definition of α_0 in terms of ν as given in (2.23) along with a rearrangement of (4.11a) to reveal the two relationships

$$(-1)^q\cosh\nu=\alpha_0 \tag{B1}$$

and

$$(-1)^q\sinh\nu=-\alpha_3+|R|(\alpha_2-\alpha_1)\ . \tag{B2}$$

From (4.13a) we have

$$\cos kl=(1/k)(S_0\coth\lambda-|K_s|)\sin kl\ , \tag{B3}$$

which, upon substitution into both (B1) and (B2), gives the two conditions

$$(-1)^q\sinh\nu=(1/k)\sin kl\sinh\lambda[(m+S_0)-|K_s|\coth\lambda]\ , \tag{B4}$$

$$(-1)^q\cosh\nu=(1/k)\sin kl\sinh\lambda[(m+S_0)-|K_s|\coth\lambda+S_0\operatorname{cosech}^2\lambda]\ . \tag{B5}$$

We may construct $\tanh\nu$ by dividing (B4) by (B5):

$$\tanh\nu=\frac{(m+S_0)-|K_s|\coth\lambda}{(m+S_0)-|K_s|\coth\lambda+S_0\operatorname{cosech}^2\lambda}\ .$$

As $0<\tanh\nu\le 1$ for $\nu>0$ then, since $S_0\operatorname{cosech}^2\lambda>0$, we must have

$$(m+S_0)-|K_s|\coth\lambda>0\ . \tag{B6}$$

This condition is necessarily obeyed by all surface states for a scalar potential. To show that it is sufficient we note that any solution of both (B3) and (B6) must also be a solution of both (B1) and (B2) as well. Therefore any solution of (B3) which satisfies (B6) and $|\alpha_0^s|>1$, represents a surface state for the scalar potential.

We may derive a necessary and sufficient condition for the existence of surface states in a vector potential in an analogous fashion. The desired result is given explicitly by

$$(E-V_0)-|K_v|\cot\lambda>0\ . \tag{B7}$$

APPENDIX C

We begin by solving the Dirac equation in the Dirac representation [see (4.3)] for a single, arbitrary cell, such as the jth cell of Fig. 2 comprising the interval (x_{j-1},x_j). The solution is given by

$$\Psi(x)=\begin{bmatrix}1\\ r\end{bmatrix}A_j\exp(ikx)+\begin{bmatrix}1\\ -r\end{bmatrix}B_j\exp(-ikx)\ . \tag{C1}$$

We may then introduce the function $\Phi(x)$, via the relationship

$$\Phi(x)=Q^{-1}\Psi(x)=\begin{bmatrix}A_j\exp(ikx)\\ B_j\exp(-ikx)\end{bmatrix}\ , \tag{C2}$$

where

$$Q=\begin{bmatrix}1 & 1\\ r & -r\end{bmatrix}\ .$$

In analogy with the definition of $\Theta^j(\lambda_j)$, which relates the wave function $\Psi(x)$ at the two extremes of the cell, we may define a similar function $\Omega^j(\lambda_j)$ via

$$\Phi(x_j)=\Omega^j(\lambda_j)\Phi(x_{j-1})\ , \tag{C3}$$

where relates the coefficients of a general solution of the Dirac equation, such as the A_j and B_j of (C1) above, with the appropriate coefficients of the wave function at some distant point. The function $\Omega^j(\lambda_j)$ therefore represents the transfer matrix[15] across the jth cell. For more complicated systems we may denote the transfer matrix by $M(x_0,x_N)$, as in the relationship

$$\Phi(x_N)=M(x_0,x_N)\Phi(x_0)\ ,$$

where

$$M(x_0,x_N)=\Omega^N(\lambda_N)\Omega^{N-1}(\lambda_{N-1})\cdots\Omega^2(\lambda_2)\Omega^1(\lambda_1)\ .$$

Returning to the single cell we find from (C2) and (C3) that the following relationship must exist:

$$\Theta^j(\lambda_j)=Q\Omega^j(\lambda_j)Q^{-1}\ . \tag{C4}$$

Generalizing this relationship to any number of cells we obtain the relation between $T(x_0,x_N)$ and the transfer matrix $M(x_0,x_N)$:

$$T(x_0,x_N)=QM(x_0,x_N)Q^{-1}\ . \tag{C5}$$

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