Possible Bose-Fermi symmetries in the giant resonance fragmentation of deformed odd-even nuclei

Giuseppe Maino
Comitato Nazionale per l'Energia Nucleare e le Energie Alternative, 40138 Bologna, Italy
(Received 18 January 1989)

A group-theoretical method for the description of the giant dipole resonance in odd-mass nuclei is described and analytical expressions for the strength splitting are derived when nuclei exhibit axially symmetric deformation, corresponding to the SU(3)⊗U(2) limit of the interacting boson-fermion model.

Recently, renewed interest has grown in developing group-theoretical techniques to deal with coupling between nuclear low- and high-energy (i.e., giant resonances) degrees of freedom.[1–4] Because of its feasible structure and the remarkable results achieved in reproducing low-lying collective spectra, the interacting boson model (IBA) (Ref. 5) turned out to be particularly suitable to this aim. Therefore, a further degree of freedom, namely one $J^{\pi}=1^{-}$ boson (p boson),[1–3] has to be introduced into the IBA model in addition to the usual s and d bosons which represent pairs of valence nucleons beyond shell closures. It is worth recalling that the microscopic structure of the p boson differs from that of s and d bosons, since it rather mimics 1p-1h collective excitations across a major shell, analogously to random-phase approximation (RPA) bosons.

By coupling s, d, and p bosons, it is then possible to describe the fragmentation of the giant dipole resonance (GDR) for a large class of even-even nuclei far from closed shells. The relevant IBA Hamiltonian has the following general form:

$$\hat{H}=\hat{H}(s,d)+\varepsilon_p\hat{n}_p+\alpha_0(d^{\dagger}\times\tilde{d})^{(0)}\cdot(p^{\dagger}\times\tilde{p})^{(0)}$$
$$+\alpha_1(d^{\dagger}\times\tilde{d})^{(1)}\cdot(p^{\dagger}\times\tilde{p})^{(1)}$$
$$+\alpha_2[s^{\dagger}\times\tilde{d}+d^{\dagger}\times\tilde{s}+\chi(d^{\dagger}\times\tilde{d})]^{(2)}\cdot(p^{\dagger}\times\tilde{p})^{(2)}\;, \quad (1)$$

where $\hat{H}(s,d)$ is the usual IBA Hamiltonian,[5] $b^{\dagger}$ (b) creates (annihilates) a boson of kind "b," $\tilde{b}_{j,m}=(-1)^{j+m}b_{j,-m}$, and $\hat{n}_p$ is the p boson number operator, equal to 0 and 1 for low- and high-energy (GDR) states, respectively.

In Eq. (1) the dominant term, responsible for the GDR fragmentation due to coupling with nuclear deformations and surface oscillations, is the last quadrupole-quadrupole interaction, as discussed in Ref. 1, from a qualitative point of view (note that in Ref. 1 this term is defined as "quadrupole-dipole coupling") and confirmed by numerical calculations as those in Refs. 2, 7, and 8.

The transitions strengths from GDR states to low-lying levels are given by means of the following dipole operator:

$$\hat{D}^{(1)}=D_0(p^{\dagger}+p)\;. \quad (2)$$

With these ingredients it is then possible to evaluate photon absorption and scattering cross sections by nuclei resorting to standard formalisms.[6] Once a suitable parametrization[6] is taken into account for the intrinsic widths associated with each GDR component (e.g., spreading and escape widths), this extended version of the interacting boson model allows us to satisfactorily reproduce both GDR splittings in shape-transition regions, like those of Nd-Sm (Refs. 7 and 8) and Os-Pt (Refs. 9 and 10) isotopes, and elastic and inelastic photon scattering cross sections.[2,11–13]

Moreover, in the case of deformed axially symmetric nuclei which are concerned with the SU(3) IBA symmetry,[5] analytic formulas can be derived for both photon absorption and scattering reactions,[3,14] since the p boson transforms under SU(3) like the components of a first-rank tensor. Relevant results for actinide nuclei compare well with the corresponding experimental data.[10,14]

Further extensions of the IBA plus GDR model are easily achievable by exploiting group-theoretical techniques, for instance, adding isospin degrees of freedom to deal with nuclei in the s-d shell.[15] In this paper, we introduce fermion operators in addition to boson ones, in order to describe GDR fragmentation in odd-mass nuclei, where a further mode-mode coupling—with respect to neighboring even-even nuclei—is to be taken into account between single-particle and giant resonance degrees of freedom.

The interacting boson-fermion model (IBFA)[16] for odd-mass nuclei has gained success in reproducing low-energy properties such as level schemes, electromagnetic moments and transitions, and one-nucleon spectroscopic factors. Dynamic Bose-Fermi symmetries are of particular interest in the model and arise when a usual IBA symmetry[5] describing the even-even boson core and one subgroup of U(m), to which the single-particle states belong, with m dimension of the fermion space, are isomorphic and can be combined together in a common fermion-boson group chain. If the complete Hamiltonian of the system can be written in terms of linear and quadratic Casimir operators of these chain subgroups only, then a dynamical Bose-Fermi symmetry results.[17,18]

Since we are concerned with GDR splitting, we refer here to dynamic Bose-Fermi symmetries connected with the SU(3) IBA limit and generalize the results of Ref. 3 to odd-A deformed nuclei. In particular, it is possible to derive closed-form expressions for GDR energy fragmentation and dipole excitation strengths, to be used in photon absorption and scattering cross-section calculations, that will be presented in a forthcoming paper.

A few years ago, a method to obtain a new class of supersymmetric states in nuclei was proposed,[19] based on the splitting of the angular momentum, j, of an odd (unpaired) nucleon into a pseudo-orbital part, k, and a pseudospin part, s. For instance, when $j=\frac{1}{2}$ and $\frac{3}{2}$, $k=1$ sits in the three-dimensional representation, [1], of SU(3) and $s=\frac{1}{2}$. In the more complicated case $j=\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$, one is faced with two possible choices ($k=0,2$; $s=\frac{1}{2}$ and ($k=1$; $s=\frac{3}{2}$).[17,18] More generally, a SU(3) Bose-Fermi symmetry results[20] whenever the single-particle space is spanned by all levels in an oscillator shell with $j=\frac{1}{2},\frac{3}{2},\ldots,n-\frac{1}{2},n+\frac{1}{2}$. Then the pseudo-orbital part assumes the values $k=n,n-2,\ldots,1$, or 0, while the pseudospin is $s=\frac{1}{2}$.

The group structure of the pseudo-orbital part is given by that of the associated harmonic oscillator:[20]

$$\mathrm{U}(m)\supset \mathrm{U}_k(m/2)\otimes \mathrm{U}_s(2)\supset \mathrm{SU}_k(3)\otimes \mathrm{U}_s(2)\ , \tag{3}$$

where the subscripts k and s refer to the pseudo-orbital and pseudospin terms, respectively; $m=\sum_j(2j+1)=(n+1)\cdot(n+2)$ corresponds to the dimension of fermion space.

The fermion group chain (3) is then coupled to the boson group chain, which contains $\mathrm{SU}^B(3)$ (in the following, B and F indices label boson and fermion groups, respectively):

$$\mathrm{U}^B(6)\otimes \mathrm{U}^F(m)\supset \mathrm{SU}^B(3)\otimes \mathrm{SU}^F_K(3)\otimes \mathrm{U}^F_S(2)\supset \mathrm{SU}^{\mathrm{BF}}(3)\otimes \mathrm{U}^F_S(2)\supset \mathrm{SO}^{\mathrm{BF}}(3)\otimes \mathrm{SU}^F_S(2)\supset \mathrm{Spin}(3)\supset \mathrm{Spin}(2)\ . \tag{4}$$

Therefore, under the assumption that boson and fermion degrees of freedom couple at the $\mathrm{SU}^{\mathrm{BF}}(3)$ level, one has to deal with the $(\lambda_B,\mu_B)\otimes(\lambda_F,\mu_F)$ product representation in Elliott's notation.[21] In particular $(\lambda_B,\mu_B)=(2N,0)$ for the ground-state rotational band, where N is the effective boson number and $(\lambda_F,\mu_F)=(n,0)$ or $(0,n)$.

One or the other of these conjugate representations is chosen according to the particular kind of relevant coupling, particle or hole like, respectively. The two resulting coupling schemes are related together by a particle-hole transformation.[17]

The basis for high-energy GDR states is obtained by further coupling the $\mathrm{SU}^{BF}(3)$ irreducible representation (irrep) in Eq. (4) to the $(1,0)$ irrep which labels the corresponding $\mathrm{SU}^p(3)$ representation for the p boson. To sum up, the group decomposition chain to be investigated is the following:

$$\mathrm{SU}^{\mathrm{BF}}(3)\otimes \mathrm{U}^F_S(2)\otimes \mathrm{SU}^p(3)\supset \mathrm{SU}^{\mathrm{BF}p}(3)\otimes \mathrm{U}^F_S(2)\supset \mathrm{SO}^{\mathrm{BF}p}(3)\otimes \mathrm{SU}^F_S(2)\supset \mathrm{Spin}(3)\supset \mathrm{Spin}(2)\ . \tag{5}$$

In this note we confine ourselves to the simple case $n=1$, i.e., to a restricted fermion space with $j=\frac{1}{2}$ and $\frac{3}{2}$, since we intend to sketch the formalism only and underlying ideas. More realistic cases, spanning larger fermion spaces, will be presented afterwards. However, the GDR fragmentation pattern is not very sensitive to details of low-energy spectrum and, therefore, the restricted single-particle space does not imply too drastic assumptions.

When $j=\frac{1}{2}$ and $\frac{3}{2}$, the low-energy states of an odd-A nucleus are given by the decompositions

$$(2N,0)\otimes(1,0)=(2N+1,0)\oplus(2N-1,1)\ , \tag{6a}$$

$$(2N,0)\otimes(0,1)=(2N,1)\oplus(2N-1,0)\ , \tag{6b}$$

for particle and hole coupling, respectively. The ground-state rotational band belongs to the irrep $(2N+1,0)$—or $(2N,1)$—with pseudo-orbital values $L=0,2,\ldots$ and total angular momenta $J=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$, in the former case (6a), while $L=1,2,3,\ldots$, and $J=\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{5}{2},\ldots$, in the latter case (6b). Moreover, starting from the particle-coupling mode (6a), the following product representation and decomposition for GDR states hold [see Eq. (5)]:

$$(2N+1)\otimes(1,0)=(2N+2,0)\oplus(2N,1)\ . \tag{7}$$

The $(2N+2,0)$ irrep contains $L=0,2,\ldots$, and, therefore, total angular momenta $J=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$, while the nonsymmetric $(2N,1)$ irrep contains $L=1,2,3,\ldots$, and $J=\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2},\ldots$. All these states have opposite parity with respect to the ground-state band. Since the ground-state spin is $J=\frac{1}{2}$, only GDR components with J values $\frac{1}{2}$ and $\frac{3}{2}$ can be excited by the dipole operator (2). Therefore, the photoabsorption peak splits into five components, two of them belonging to $(2N+2,0)$ irrep, the remaining three to $(2N,1)$. The energy splitting can be evaluated by considering the Hamiltonian

$$\hat{H}=\hat{H}(s,d,a_j)+\varepsilon_p\hat{n}_p+\alpha\hat{Q}^{\,\mathrm{BF}}\cdot\hat{Q}^{\,p}\ , \tag{8}$$

to be compared with Eq. (1) for even-even nuclei. Here, $a_j^\dagger$ (a_j) creates (annihilates) one nucleon in the single-particle level $j=\frac{1}{2}$ or $\frac{3}{2}$, and $\hat{H}(s,d,a_j)$ is the usual IBFA Hamiltonian.[16,17] In actual cases it resembles the $\mathrm{SU}^{\mathrm{BF}}(3)$ exact symmetry. As in Eq. (1), the quadrupole-quadrupole interaction is responsible for GDR fragmentation, while other coupling terms are less important and therefore neglected in Eq. (8). The quadrupole operators in Eq. (8) are so defined, in order to satisfy the commutation rules of su(3) algebra:

$$\hat{Q}^{p}=-\frac{\sqrt{3}}{2}(p^{\dagger}\times\tilde{p})^{(2)}\,, \tag{9a}$$

$$\hat{Q}^{\rm BF}=(d^{\dagger}\times\tilde{s}+s^{\dagger}\times\tilde{d})^{(2)}-\frac{\sqrt{7}}{2}(d^{\dagger}\times\tilde{d})^{(2)} \mp\frac{1}{\sqrt{2}}[(a^{\dagger}_{1/2}\times\tilde{a}_{3/2})^{(2)}-(a^{\dagger}_{3/2}\times\tilde{a}_{1/2})^{(2)} -(a^{\dagger}_{3/2}\times\tilde{a}_{3/2})^{(2)}]\,, \tag{9b}$$

where the sign $-$ or $+$ is chosen depending on particle- or hole-coupling character, respectively [see Eqs. (6a) and (6b)].

Since the quadratic Casimir operator, $\mathcal{C}_2$, of SU(3) is given by

$$\mathcal{C}_2=2\hat{Q}\cdot\hat{Q}+\tfrac{3}{4}\hat{L}\cdot\hat{L}\,, \tag{10}$$

the Hamiltonian (8) can be expressed in terms of relevant Casimir operators for the considered groups in Eq. (5). In adiabatic limit which holds in the present case, the part proportional to angular momentum operator, $\hat{L}$, which is not diagonal in the coupled SU(3) basis (5), can be safely neglected. In fact, as previously outlined in Ref. 3, the rotational energies within a given band, which depend on the $\hat{L}\cdot\hat{L}$ term, are small ($\sim$30 keV) in comparison with the dipole splitting induced by the quadrupole-quadrupole interaction ($\sim$ 2000 keV).

The energies of two $J=\frac{1}{2}$ GDR components are then given by

$$E[(2N+2,0),\tfrac{1}{2}]=\varepsilon_p-\frac{\alpha}{\sqrt{3}}(2N+1)\,,\qquad E[(2N,1),\tfrac{1}{2}]=\varepsilon_p+\frac{\alpha}{\sqrt{3}}(N+2)\,. \tag{11}$$

Most correctly, the preceding result can be recovered by solving the corresponding two-mixed level problem and diagonalizing the Hamiltonian which couples $J=\frac{1}{2}$ GDR components to $J=\frac{1}{2}$ and $\frac{3}{2}$ states in the ground-state rotational band:

$$\hat{H}=\varepsilon_p\hat{I}+\begin{bmatrix}0 & -\frac{\alpha}{2\sqrt{6}}(4N+5)\\ -\frac{\alpha}{2\sqrt{6}}(4N+5) & E(\frac{3}{2}{}^{\rm g.s.b.}_{1})-\frac{\alpha}{4\sqrt{3}}(4N+5)\end{bmatrix}, \tag{12}$$

where the excitation energy of the first $\frac{3}{2}$ state in the ground-state (g.s.) band is negligible with respect to the GDR energy splitting.

It results

$$E[(2N+2,0),\tfrac{1}{2}]=\varepsilon_p-\frac{\alpha}{2\sqrt{3}}(4N+5)\,,\qquad E[(2N,1),\tfrac{1}{2}]=\varepsilon_p+\frac{\alpha}{4\sqrt{3}}(4N+5)\,. \tag{13}$$

The energ litting between these two GDR states with spin $\frac{1}{2}$ is then given by $\Delta E=\alpha\sqrt{3}(N+\frac{5}{4})$, to be compared with the corresponding quantity for $J^{\pi}=1^-$ GDR components in even-even deformed nuclei,[3] $\Delta E=\bar{\alpha}\sqrt{3}(N+\frac{1}{4})$. If $\alpha=\bar{\alpha}$, an acceptable assumption for neighboring isotopes since the quadrupole-quadrupole interaction strength is almost constant in a given mass region,[7-10] this fact amounts to a larger splitting in odd-A nuclei than in even-even adjacent ones, by a factor proportional to $1/N\cong5-10\%$.

The dipole transition strengths from ground state to each GDR component can be easily evaluated in adiabatic approximation and are proportional to the reduced Wigner coefficients[22] of the $\mathrm{SU}^{\mathrm{BF}p}(3)\supset\mathrm{SO}^{\mathrm{BF}p}(3)$ decomposition. In the actual case, connecting the ground state in $(2N+1,0)$ irrep with GDR states in $(2N+2,0)$ and $(2N,1)$ irreps, they are equal to 1 in absolute value and differences arise only because of geometric SO(3) factors:

$$|\langle(2N+2,0),\tfrac{1}{2}\|\hat{D}^{(1)}\|(2N+1,0),\tfrac{1}{2}\rangle|^2=2D_0/3\,,\qquad |\langle(2N,1),\tfrac{1}{2}\|\hat{D}^{(1)}\|(2N+1,0),\tfrac{1}{2}\rangle|^2=4D_0/9\,, \tag{14}$$

where D_0 is an adjustable parameter, with the physical meaning of effective transition charge.

The energy splittings of three $J=\frac{3}{2}$ GDR components are obtained analogously by considering the coupling to levels with spins $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ in the ground-state band and solving the relevant three-mixed level problem:

$$\hat{H}=\varepsilon_p\hat{I}-\alpha\begin{bmatrix}0 & \frac{2K+3}{4\sqrt{15}} & \frac{1}{2}[\frac{3}{5}(K-1)(K+4)]^{1/2}\\ \frac{2K+3}{4\sqrt{15}} & \frac{E(\frac{3}{2}{}^{\rm g.s.b.}_{1})}{\alpha}+\frac{2K+3}{5\sqrt{3}} & -\frac{1}{10}[3(K-1)(K+4)]^{1/2}\\ \frac{1}{2}[\frac{3}{5}(K-1)(K+4)]^{1/2} & -\frac{1}{10}[3(K-1)(K+4)]^{1/2} & \frac{E(\frac{5}{2}{}^{\rm g.s.b.}_{1})}{\alpha}-\frac{2}{35}(2K+3)\end{bmatrix}, \tag{15}$$

with $K=2N+1$. Approximate diagonalization of matrix (15), where terms of order greater than $1/N$ are neglected, yields the following eigenvalues:

$$E[(2N+2,0),\tfrac{3}{2}]=\varepsilon_p-0.4427\frac{\alpha}{\sqrt{3}}(4N+5)\ ,$$
$$E[(2N,1),\tfrac{3}{2}{}_1]=\varepsilon_p+0.2500\frac{\alpha}{\sqrt{3}}(4N+5)\ , \qquad (16)$$
$$E[(2N,1),\tfrac{3}{2}{}_2]=\varepsilon_p+0.2938\frac{\alpha}{\sqrt{3}}(4N+5)\ ,$$

where the energy splitting between the $\frac{3}{2}$ state in $(2N+2,0)$ irrep and the mean value of two components belonging to $(2N,1)$ irrep is

$$\Delta E=0.715\frac{\alpha}{\sqrt{3}}(4N+5)=\alpha\sqrt{3}(0.953N+1.192)\ ,$$

nearly equal to the GDR $\Delta E(J=\frac{1}{2})$ splitting. The relevant dipole transition strengths are given according to the Wigner coefficients:

$$|\langle(2N+2,0),\tfrac{3}{2}\|\hat{D}^{(1)}\|(2N+1,0),\tfrac{1}{2}\rangle|^2 =2(2N+5)D_0/[15(N+1)]\ ,$$
$$|\langle(2N,1),\tfrac{3}{2}{}_1\|\hat{D}^{(1)}\|(2N+1,0),\tfrac{1}{2}\rangle|^2=2D_0/9\ , \qquad (17)$$
$$|\langle(2N,1),\tfrac{3}{2}{}_2\|\hat{D}^{(1)}\|(2N+1,0),\tfrac{1}{2}\rangle|^2=2ND_0/[5(N+1)]\ .$$

Now, we return to Eq. (6b) in order to deal with the hole-coupling mode and compare the relevant results with those already obtained in the case of particle coupling. The following product representation and decomposition hold:

$$(2N,1)\otimes(1,0)=(2N+1,1)\oplus(2N-1,2)\oplus(2N,0)\ , \qquad (18)$$

where the symmetric irrep $(2N,0)$ contains the angular momentum values $L=0,2,\ldots,$ and $J=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$. The GDR states which belong to $(2N+1,1)$ irrep have $L=1,2,3,\ldots,$ and $J=\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2},\ldots$. Finally, in the $(2N-1,2)$ irrep two sets of pseudo-orbital L values are present: $L=1,3,\ldots,$ which give rise to $J=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots,$ and $L=2,3,\ldots,$ with corresponding spins $J=\frac{3}{2},\frac{5}{2},\ldots$. Therefore, eight GDR components can be excited from the ground state by dipole operator (2). Three of them have spin $J=\frac{1}{2}$, parity opposite to that of the ground-state rotational band, and belong to different irreps. Their excitation energies are given by expectation values of Casimir operators (10), in terms of which Hamiltonian (8) can be defined. The following eigenvalues are so obtained:

$$E[(2N+1,1),\tfrac{1}{2}]=\varepsilon_p-\frac{\alpha}{2\sqrt{3}}(4N+1)\ ,$$
$$E[(2N-1,2),\tfrac{1}{2}]=\varepsilon_p+\frac{\alpha}{\sqrt{3}}(N+1)\ , \qquad (19)$$
$$E[(2N,0),\tfrac{1}{2}]=\varepsilon_p+\frac{\alpha}{\sqrt{3}}(N+4)\ .$$

It is worth noting that the energy splitting between the lower state and the mean value of the other two is $\Delta E=\alpha\sqrt{3}(N+1)$, roughly equal to that obtained between the two $J=\frac{1}{2}$ GDR components in the case of particle coupling.

The transition strengths from nuclear ground state to GDR components (19) are

$$|\langle(2N+1,1),\tfrac{1}{2}\|\hat{D}^{(1)}\|(2N,1),\tfrac{1}{2}\rangle|^2 =4(N+1)D_0/[9(2N+1)]\ ,$$
$$|\langle(2N-1,2),\tfrac{1}{2}\|\hat{D}^{(1)}\|(2N,1),\tfrac{1}{2}\rangle|^2 =4ND_0/[9(2N+1)]\ , \qquad (20)$$
$$|\langle(2N,0),\tfrac{1}{2}\|\hat{D}^{(1)}\|(2N,1),\tfrac{1}{2}\rangle^2=2D_0/3\ .$$

Finally, the five $J=\frac{3}{2}$ GDR states couple to low-lying levels in the ground-state irrep with spins $J=\frac{1}{2}$, $\frac{3}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, and $\frac{5}{2}$; their excitation energies can be obtained from the solution of the associated five-mixed level problem.[23]

In conclusion, we have presented a simple algebraic model for the description of giant dipole resonances in odd-mass nuclei; when nuclei exhibit static axially symmetric deformations, analytic expressions for both GDR excitation energies and dipole transition strengths can be obtained. In general cases the present phenomenological model provides us with a suitable basis for feasible numerical calculations. Moreover, further degrees of freedom can be added to the present formulation, like g-boson excitations or isospin, in order to take into account more complicated situations.

Useful discussions with Professor F. Iachello are gratefully acknowledged.

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