PHYSICAL REVIEW C VOLUME 40, NUMBER 6 DECEMBER 1989

Triton model calculation test of the Bonn W-matrix rank-one approximation

B. F. Gibson and B. C. Pearce
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

G. L. Payne
Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa 52242
(Received 27 October 1988)

The rank-one separable part of the W-matrix representation of the t matrix for the Malfliet-Tjon I-III nucleon-nucleon potential has been shown to lead to results for the triton binding energy and low-energy ($E_{\rm lab} \leq 50$ MeV) nucleon-deuteron scattering that agree well with results for that potential obtained by solving the full local-potential equations. We explore why this prescription appears to work so well for the Malfliet-Tjon I-III model and test it further using the Reid-soft-core spin-singlet interaction, which possesses a very strong short-range repulsion (a stiff core). For this model, a calculation using the rank-one separable part of the W-matrix representation of the two-body t matrix overbinds the triton by 25%.

I. INTRODUCTION

The traditional approach to solving the two-body problem in momentum space has been in terms of the Lippmann-Schwinger equation—a homogeneous equation for the bound state and an inhomogeneous equation for the continuum. Bartnik, Haberzettl, and Sandhas[1,2] have proposed an alternative formulation. They have outlined a unified description of the bound-state and continuum regimes in terms of a single real, nonsingular inhomogeneous integral equation, the solution of which is the W matrix. One advantage of this prescription is that one can construct an exact representation of the two-body off-shell t matrix in which the bound-state pole and scattering-cut information are contained in a single separable term (a rank-one expansion). The remainder term, which is nonseparable, is a real, nonsingular function that vanishes half on shell.

The hope was expressed in Ref. 1 that the single separable term in the W matrix might, when used as input in exact three-body equations, provide a reasonable approximation to the experimental data. That is, the remainder term in the W-matrix representation of the two-body t matrix might make a negligible contribution. This hope was bolstered by results obtained in Ref. 3 for the triton bound state using the rank-one approximation to the Malfliet-Tjon[4] model MT I-III (a spin-singlet, spin-triplet interaction that acts only in $l=0$ partial waves) and for neutron-deuteron scattering using that potential and the Alt, Grassberger, and Sandhas[5] separable-potential formulation of the Faddeev[6] equations. In the former case, the binding energy obtained using the rank-one separable term in the W matrix had a minimum at $E_t(k)=-8.595$ MeV (as a function of the parameter k in the W-matrix solution at negative two-body energies), which compared very well with the full, local-potential solution[7] of -8.58 ± 0.1 MeV. Similarly, neutron-deuteron scattering length results for the W matrix ($^4a_{nd}=6.41$ fm and $^2a_{nd}=0.86$ fm) compared well with the local-potential numerical results of Kloet and Tjon[8] ($^4a_{nd}=6.35$ fm and $^2a_{nd}=0.9$ fm).

Our purpose is to apply the W-matrix prescription to a "more realistic potential" (one possessing strong short-range repulsion) and attempt to understand whether this promising rank-one separable-potential approximation to local potentials might be used to make four-body and five-body calculations more tractable. To that end we briefly outline the rank-one W-matrix approximation in Sec. II. We present triton binding energy results for several Malfliet-Tjon models in Sec. III obtained using this W matrix along with full local-potential solutions for the same models. We also explore the three-body bound state for a modified Reid-soft-core[9] (RSC) spin-singlet potential. Our conclusions are stated in Sec. IV.

II. SUMMARY OF THE BONN W-MATRIX FORMALISM

For completeness we summarize here the elements for the W-matrix representation of the two-body t matrix which we require to carry out our investigation. The off-shell t matrix for partial wave l at energy E is given by

$$t_l(p,p';E)=W_{kl}(p,k;E)\Delta_{kl}(E^+)W_{kl}(p',k;E)+R_{kl}(p,p';E)\ , \tag{1}$$

where $E^+=E+i\epsilon$, $R_{kl}(p,p';E)$ is the remainder function that vanishes half on shell, and k is a parameter specified below. The function $\Delta_{kl}(E^+)$ in Eq. (1) is given by

$$\Delta_{kl}(E^+)=\frac{k^l}{W_{kl}(k,k;E)\hat{F}_{kl}(E^+)}\ , \tag{2}$$

where $\hat{F}_{kl}(E^+)$ is defined in terms of a simple integral as

$$\hat{F}_{kl}(E^+)=1-\int_0^\infty \frac{q^l W_{kl}(q,k;E)}{E^+-q^2/m}q^2dq\ . \tag{3}$$

The W matrix satisfies the nonsingular integral equation

$$W_{kl}(p,p';E)=U(p,p')+\int_0^\infty \frac{[U_l(p,q)-U_l(p,k)]}{E-q^2/m} \times q^l W_{kl}(q,p';E)q^2dq \qquad (4)$$

in which the parameter k satisfies the following constraints:

$$\text{(I)}\ k^2=mE,\ E\geq 0 \qquad (5a)$$

$$\text{(II)}\ k \text{ arbitrary},\ \ E<0\ . \qquad (5b)$$

The $U_l(p,q)$ in Eq. (4) is a simple function of the partial wave projection of the potential,

$$U_l(p,q)=q^{-l}V_l(p,q)\ . \qquad (6)$$

Here the q^{-l} factor compensates for the q^l behavior of $V_l(p,q)$ as $q \rightarrow 0$ and ensures that $U_l(p,q)$ does not vanish identically at $q=0$. It was demonstrated in Ref. 1 that, at negative energies ($mE=-\alpha^2$), for solutions $W_{kl}(q,k;-\alpha^2/m)$ which satisfy

$$\int_0^\infty \frac{q^l W_{kl}(q,k;-\alpha^2/m)}{-(\alpha^2+q^2)/m} q^2dq=1\ , \qquad (7)$$

$-\alpha^2/m$ is equal to the binding energy E_n (i.e., $-\alpha_n^2/m$) of one of the bound states. The corresponding wave function is given by

$$\psi_{nl}=C_{nl}\frac{W_{kl}(p,k;-\alpha_n^2/m)}{-(\alpha_n^2+p^2)/m}\ , \qquad (8)$$

where C_{nl} is the normalization constant.

Clearly, the leading term on the right-hand side of Eq. (1) is of a form similar to that arising from a rank-one separable potential such as

$$V(p,p')=g(p)\Lambda g(p')\ , \qquad (9)$$

where Λ is the potential strength and $g(p)$ is the potential form factor. If the remainder $R_{kl}(p,p';E)$ in Eq. (1), which vanishes half on shell, can be shown to make a negligible contribution in the calculation of three-nucleon observables, then one might utilize such a prescription to simplify few-body calculations with realistic potentials.

To investigate this possibility for models other than that discussed in Refs. 1 and 3, we utilize potentials composed of sums of Yukawa forms

$$V(r)=\sum_i V_i\frac{\exp(-\mu_i r)}{\mu r} \qquad (10)$$

or

$$V(q)=\sum_i \frac{V_i}{2\pi^2}\frac{1}{q^2+\mu_i^2} \qquad (11a)$$

$$=\sum_i V_i(\mathbf{p},\mathbf{p}'),\ \ q^2=(\mathbf{p}-\mathbf{p}')^2\ . \qquad (11b)$$

The $l=0$ term in the partial wave expansion of Eq. (11b) is

$$V_{i0}(p,p')=\int V_i(\mathbf{p},\mathbf{p}')d\Omega = \frac{V_i}{2\pi pp'}\ln\frac{(p+p')^2+\mu_i^2}{(p-p')^2+\mu_i^2}\ , \qquad (12)$$

and for $l=0$ one has the trivial relation

$$U_0(p,p')=V_0(p,p')\ . \qquad (13)$$

The s-wave separable potential Faddeev equation that determines the three-boson bound state is

$$\psi(\mathbf{p},\mathbf{P})=2G_0\int d\mathbf{p}'\, t\left[\mathbf{p},\mathbf{p}'+\tfrac{1}{2}\mathbf{P};E-\frac{3P^2}{4m}\right] \times \psi(\tfrac{1}{2}\mathbf{p}'+\mathbf{P},\mathbf{p}')\ . \qquad (14)$$

Expressing the t matrix in terms of the W matrix and neglecting the remainder function, then Eq. (14) is satisfied by the ansatz

$$\psi(\mathbf{p},\mathbf{P})=G_0W_{k0}\left[p,k;E-\frac{3P^2}{4m}\right]u(P)\ . \qquad (15)$$

Here G_0 is the free-particle three-nucleon propagator and $u(P)$ is the spectator function describing the dynamics of the third nucleon moving relative to the center of mass of the interaction pair. (Our Jacobi coordinates are $\mathbf{p}$ for the pair and $\mathbf{P}$ for the momentum of the spectator relative to the pair.) After a modest amount of algebra, one can demonstrate that $u(P)$ satisfies the homogeneous integral equation

$$u(P)=2\Delta_{k0}\left[E-\frac{3P^2}{4m}\right]\int_0^\infty I_{00}(p',P;E)u(p')p'^2dp'\ , \qquad (16)$$

where the kernel is

$$I_{00}(p,P;E)=\int_{-1}^{1}dx\frac{W_{k0}\left[|\mathbf{p}+\tfrac{1}{2}\mathbf{P}|,k;E-\frac{3P^2}{4m}\right]W_{k0}\left[|\tfrac{1}{2}\mathbf{p}+\mathbf{P}|,k;E-\frac{3P^2}{4m}\right]}{(p^2+P^2-pPx)/m-E} \qquad (17)$$

Equation (16) is the usual separable-potential formulation in which the form factors $g(p)$ in the separable potential of Eq. (9) have been replaced by the W matrix. The generalization to include spin and isospin follows the conventional three-nucleon analysis. It is these three-body equations that we solve numerically to explore the utility of approximating the t matrix by the leading separable term in the W-matrix representation.

III. NUMERICAL RESULTS

In addition to verifying the bound-state results for the MT I-III potential quoted in Ref. 3, we have explored a number of the Malfliet-Tjon[4] models. The particular parameters used[10] are quoted in Table I. They differ slightly from those quoted originally in Ref. 4. Models I and II correspond to spin singlet, whereas models III and IV correspond to spin triplet. Model V is a spin-averaged interaction. (Note that these potentials possess an implied projection operator and act only in the $l=0$ partial wave.)

Because we are dealing with the bound-state problem, the parameter k in Eq. (4) is arbitrary, as is indicated by the condition expressed in Eq. (5b). As found in Ref. 3, the trinucleon bound-state energy has its minimum as a function of k very close to the value that one obtains solving the configuration-space local-potential Faddeev equation discussed in Ref. 10. We list those triton bound-state energies in Table II, where it is clear that for the MT I-III model the W-matrix prescription duplicates almost exactly the local-potential result. A similar close approximation of the local-potential result by the W-matrix approximation was found for the spin-averaged MT V model.

We were disappointed to discover that the W-matrix prescription did not work nearly as well for the simpler (one term Yukawa) MT II-IV model. In this case the difference between the ^{3}H binding energies was more than 1 MeV, or some 10%. Combining two term (MT I or MT III) potentials with one term (MT II or MT IV) potentials led to reasonable W-matrix results. However, it was clear from examining the MT II potential (multiplied by a factor of 1.1 to increase the strength of the potential sufficiently to support a three-body bound state with a reasonably large binding energy) and MT IV potentials separately, that neglecting the remainder term $R_{k0}(p,p';E)$ does not provide a satisfactory representation of the t matrix for a single Yukawa. The fact that the triton bound-state and neutron-deuteron scattering calculations using the W-matrix prescription for the MT I-III model work so well would appear to indicate that a cancellation occurs in that case which makes the contribution of the remainder term negligible.

To test that supposition, we investigated a simple model based upon the RSC spin-singlet interaction. That potential alone produces a three-boson bound state ($V^t \equiv V^s$) having about 1 MeV binding. We multiplied the strength of the midrange attraction by a factor of 1.08 to increase the three-body binding energy to about 7 MeV. The potential parameters were

$$V_1 = -10.463 \text{ MeV} ,$$

$$V_2 = -1815.66 \text{ MeV} ,$$

$$V_3 = 6484.2 \text{ MeV} ,$$

with ranges of $\mu(=0.7 \text{ fm}^{-1})$, 4μ, and 7μ. The configuration-space Faddeev equation yields a three-body binding energy of $E_3 \simeq -7.1$ MeV. The W-matrix prescription yielded a minimum at $E_3(k \approx 0.85) = -9.1$ MeV. Apparently the error made in neglecting the remainder due to the very strong (stiff) repulsive short-range Yukawa term overcompensates for that made in neglecting the remainder due to the two longer-range attractive terms. Regardless, the W-matrix prescription fails to reproduce the local-potential result in the region of the parameter k that minimizes the three-body binding energy, in contradiction to the supposition made in Ref. 3.

One comment of a technical nature is in order. Whereas the Malfliet-Tjon models have a sufficiently soft repulsive character at short range that one can solve the W-matrix three-body equations by means of the simple power method, such is not the case for the RSC spin-singlet model. In that case one must resort to a more powerful approach such as the Lanczos procedure described in detail in Ref. 11 or a full eigenvalue solution, in order to avoid the three-body eigenvalue corresponding to the strong short-range repulsion. Spline methods[12] are most efficient in obtaining the W matrix itself.

In attempting to understand why use of just the separable term in the W-matrix representation of the t matrix led to significant underbinding of the triton for the single-term (purely attractive) Yukawa potential model and overbinding for the RSC spin-singlet (strong short-range repulsion) potential model, we examined norms of the remainder term R_{k0} $(p,p';E)$ for all potential models investigated. (We used a Gaussian weighting about the on-shell momentum to obtain convergence.) However,

TABLE I. Potential parameters for the Malfliet-Tjon models, from Ref. 10.

Model	V_A (MeV fm)	μ_A (fm^{-1})	V_R (MeV fm)	μ_R (fm^{-1})
I	513.968	1.55	1438.720	3.11
II	52.49	0.809	0	
III	626.885	1.55	1438.720	3.11
IV	65.120	0.633	0	
V	570.3316	1.55	1438.4812	3.11

TABLE II. Triton binding energies obtained by solving the configuration-space Faddeev equations and by solving the W-matrix separable equations for selected Malfliet-Tjon potential models.

Model	E_3 (local) (MeV)	E_3 (W matrix) (MeV)
MT I-III	-8.54	-8.53
MT V	-7.54	-7.50
MT II-IV	-11.8	-10.5
MT II-III	-10.2	-9.9
MT I-IV	-8.5	-8.4
MT II ($\times 1.1$)	-5.6	-4.8
MT IV	-24.9	-21.4

there was no obvious characteristic which would allow us to predict *a priori* which would make a small contribution and whether the triton would be overbound or underbound. This is likely because the separable term in the W matrix does provide a good first order representation of the t matrix (the remainder term vanishes half on shell). [The observed discrepancy in the binding ($\sim 1-2$ MeV) is but a small fraction of the total potential energy of the trinucleon bound state ($\sim 40-50$ MeV).] Thus, in the three-body bound-state calculation the remainder term provides only a small (off-shell) correction to the leading separable term, and it is difficult to know *a priori* what the properties of such an off-shell term will be.

IV. CONCLUSIONS

The Bonn W-matrix representation of the two-body t matrix has two very attractive features: the leading separable term contains the bound-state pole and scattering-cut information, and the nonseparable remainder term vanishes half on shell. This property of the remainder term leads one to hope that corrections from this term in few-body calculations will be small, as was the case for the MT I-III potential. However, the remarkably accurate three-nucleon bound-state results obtained for the MT I-III model using solely the leading separable term of the W-matrix representation are not found for other potentials of the Yukawa form—the MT II-IV and RSC spin-singlet potential models. The former is purely attractive and the latter has a very stiff, short-range repulsive core. The implication is that the error made in neglecting the W-matrix remainder term for the MT I-III model (which is composed of a long-range attractive term and a shorter-range repulsive term) is small due to cancellations. Furthermore, we do not see in the RSC spin-singlet case the remarkable bounding from below of the local-potential triton energy by the rank-one W-matrix results (for all values of the arbitrary parameter k) that was seen with the MT models. We conclude that, although the leading separable term of the W-matrix representation of the t matrix provides a very good first order approximation, its use alone in few-body bound-state calculations (that is neglecting the nonseparable remainder term) is not justified for all potentials.

ACKNOWLEDGMENTS

The work of B.F.G. and B.C.P. was performed under the auspices of the U.S. Department of Energy, while that of G.L.P. is supported in part by the U.S. Department of Energy. We thank H. Haberzettl for introducing us to the Bonn W matrix and providing numerical checks of our MT results.

[1]E. A. Bartnik, H. Haberzettl, and W. Sandhas, Phys. Rev. C **34**, 1520 (1986).
[2]A related approach to the bound-state problem was considered by S. K. Adhikari and L. Tomio, Phys. Rev. C **24**, 1189 (1981).
[3]E. A. Bartnik, H. Haberzettl, T. Januschke, U. Kerwath, and W. Sandhas, Phys. Rev. C **36**, 1678 (1987).
[4]R. A. Malfliet and J. A. Tjon, Nucl. Phys. **A127**, 161 (1969); Ann. Phys. (N.Y.) **61**, 425 (1970).
[5]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. **B2**, 167 (1967).
[6]L. D. Faddeev, Zh. Eksp. Teor. Fiz. **39**, 1459 (1960) [Sov. Phys.—JETP **12**, 1014 (1961)].
[7]B. L. G. Bakker and P. Ruig, in *Few Particle Problems in the Nuclear Interaction,* edited by I. Slaus *et al.* (North-Holland, Amsterdam, 1972), p. 351.
[8]W. M. Kloet and J. A. Tjon, Ann. Phys. (N.Y.) **79**, 407 (1973).
[9]R. V. Reid, Ann. Phys. (N.Y.) **50**, 411 (1968).
[10]G. L. Payne, J. L. Friar, B. F. Gibson, and I. R. Afnan, Phys. Rev. C **22**, 823 (1980).
[11]C. R. Chen, G. L. Payne, J. L. Friar, and B. F. Gibson, Phys. Rev. C **33**, 1740 (1986).
[12]P. M. Prenter, *Splines and Variational Methods* (Wiley, New York, 1975).