Collisional relaxation in simulations of heavy-ion collisions using Boltzmann-type equations

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We compare three test-particle methods currently used in numerical simulations of Boltzmann-type equations for the analysis of intermediate-energy heavy-ion collisions with an exact solution of the Krook-Wu model. These methods are the full ensemble, parallel ensemble, and hybrid techniques. We find that collisional relaxation is sensitive to the method of simulation used. The full ensemble approach is found to agree with the exact results of the Krook-Wu model. The parallel ensemble procedure provides a reasonable approximation to the analytical relaxation rate for a wide range of systems, while the hybrid method overestimates the relaxation rate. We further compare transverse flow data from the first two of these methods in a cascade simulation of heavy-ion collisions, and find reasonable agreement provided the two-body cross section is not enhanced by a large factor over its free space value. This has implications for quantitative comparisons of calculations to experimental data.

I. INTRODUCTION

Heavy-ion experiments below $E_{\rm lab}=2$ GeV/nucleon are currently being analyzed in terms of Boltzmann-type kinetic equations. One such equation for the time evolution of the phase-space distribution function $f(\mathbf{r},\mathbf{p},t)$ of a nucleon that incorporates both the mean-field U and a collision term with Pauli blocking of final states is (see, for example, Ref. 1)

$$\frac{\partial f}{\partial t}+\nabla_p U\cdot\nabla_r f-\nabla_r U\cdot\nabla_p f = -\frac{\rho}{(2\pi)^6}\int d^3p_2 d^3p_{2'} d\Omega \frac{d\sigma_{NN}}{d\Omega} g\{[ff_2(1-\rho f_{1'})(1-\rho f_{2'})-f_{1'}f_{2'}(1-\rho f)(1-\rho f_2)](2\pi)^3\delta^3(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_{1'}-\mathbf{p}_{2'})\}\ . \tag{1.1}$$

Above, ρ is the density of nucleons, $d\sigma_{NN}/d\Omega$ is the differential nucleon-nucleon cross section, and g is the relative velocity. Equation (1.1) has been variously referred to as the Boltzmann-Uehling-Uhlenbeck (BUU), Vlasov-Uehling-Uhlenbeck (VUU),[2] Boltzmann-Nordheim,[3] or Landau-Vlasov[4,5] equation. In general, the mean-field U depends on both the density ρ and the momentum $\mathbf{p}$ of the nucleon. The cascade model[6] ignores mean-field effects, the particles moving without interaction between collisions. The opposite extreme involves dropping the hard collisions, but retaining soft interactions, such as in the Vlasov Equation (see, for example, Ref. 1). Equation (1.1) contains effects due to both hard collisions and soft interactions, albeit at a semiclassical level.

In the several efforts to date,[1,7,8] one of the main objectives has been to pin down the equation of state of dense nuclear matter. Simple parametrizations (see, for example, Refs. 9 and 10) that mimic results of more microscopic calculations[11] of the mean-field U are often used as inputs in simulating $f(\mathbf{r},\mathbf{p},t)$ using Eq. (1.1). Another input is the energy-dependent differential nucleon-nucleon cross section which is usually taken[1] from experiments. Recently, the importance of medium modifications of the free space cross sections has also been emphasized.[4,12,13] Where possible, comparisons of results from numerical simulations of Eq. (1.1) with the observed patterns of matter, momentum, and energy flow have been made and have led to some useful insights.[1,14,15] With increasing selectivity in the data and improved theoretical analyses the fulfillment of the stated objective appears promising.

Our aim in this paper is to provide checks on the accuracy of the numerical methods used in the simulation of Eq. (1.1). Many of the current methods (see, for example, Refs. 1 and 5) rely on Monte Carlo simulations using the test-particle method. Reliable, i.e., statistically significant estimates of observables usually require a large number of test particles. Equally important is the intrin-

sic accuracy of the numerical method. Consider, for example, the evaluation of the collision integral in Eq. (1.1) that involves the calculation of a five-dimensional integral in momentum space together with a localization of the collision partners in three-dimensional position space. In view of the many dimensions involved and the structure of the kernel, accuracy tests of the techniques currently employed are required.

In answering some of these questions concerning the test-particle method of solving a Boltzmann-type equation, it is useful to look at instances for which exact, closed-form solutions of the Boltzmann transport equation exist. An example is provided by the Krook-Wu model.[16,17] In this model, the relaxation to a Maxwell distribution is studied based on an exact solution of the nonlinear classical Boltzmann equation. Since the relaxation will be determined by the collision rate, and hence by the accuracy with which this rate is calculated, a comparison of a particular simulation to the exact solution will be instructive. We choose two simulation procedures currently being used in the literature for such a study. Specifically, we examine the outcome from the classical Boltzmann equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla_r f=-\frac{\rho}{(2\pi)^6}\int d^3p_2 d^3p_{2'} d\Omega \frac{d\sigma_{NN}}{d\Omega} g\{[ff_2-f_{1'}f_{2'}](2\pi)^3\delta^3(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_{1'}-\mathbf{p}_{2'})\} \tag{1.2}$$

using the numerical schemes adopted in these two cases. Our study focuses on a specific form of the collision integral with the intent to provide checks with exact solutions of a "homework" problem. Similar checks addressing effects due to the mean field and also to the Pauli blocking of final states are desirable supplements[1,18–20] to this study.

In Sec. II we introduce the analytic Krook-Wu solution to the Boltzmann equation for an isotropic, homogeneous classical gas, with an isotropic elastic cross section proportional to the inverse of the relative speed of the collision partners. Section III outlines three test-particle techniques for simulating the general Boltzmann equation: the full ensemble (FE), parallel ensemble (PE), and hybrid methods. All three have been used in transport model simulations of heavy-ion collisions. The results section (IV) is divided into two parts: Firstly, we consider a systematic comparison of the numerical methods with the exact Krook-Wu solution for systems in which the size, density, and interaction range are varied. It is found that the FE and PE approaches lead to the correct relaxation rate for a wide range of systems, but that the PE method breaks down when the interaction range becomes comparable to the linear dimensions of the system. The hybrid technique does not reproduce the analytical relaxation rate. In the second part of Sec. IV we compare the FE and PE methods in cascade simulations of intermediate-energy heavy-ion collisions. Once again, it is found that they lead to the same transverse flow provided that the interaction range does not become comparable to the size of the nuclei. In Sec. V we summarize these results.

II. THE KROOK-WU MODEL

For completeness, we summarize here the main results of the Krook-Wu model.[16,17] Consider a nonrelativistic classical one-component gas, described at time t by an isotropic homogeneous one-particle distribution function $\rho f(v,t)$, where ρ is the constant number density and $v=|\mathbf{v}|$ is the magnitude of the velocity. For an isotropic elastic differential cross section given by[21]

$$\sigma(g,\chi)=\frac{\kappa}{g}\ , \tag{2.1}$$

where χ is the center-of-mass system scattering angle, g the relative velocity, and κ a constant independent of g and χ, Krook and Wu show that the Boltzmann equation

$$\partial_\tau f(v,\tau)=-f(v,\tau)+\frac{1}{4\pi}\int d^3w\int_0^\pi d\chi\sin\chi\int_0^{2\pi} d\epsilon\, f(w',\tau)f(v',\tau)\ , \tag{2.2}$$

where

$$v'^2=\tfrac{1}{2}(v^2+w^2)+\tfrac{1}{2}(v^2-w^2)\cos\chi+|\mathbf{v}\times\mathbf{w}|\sin\chi\cos\epsilon\ , \tag{2.3}$$

$$w'^2=\tfrac{1}{2}(v^2+w^2)-\tfrac{1}{2}(v^2-\omega^2)\cos\chi-|\mathbf{v}\times\mathbf{w}|\sin\chi\cos\epsilon\ , \tag{2.4}$$

and $\tau=4\pi\rho\kappa t$ is a dimensionless time variable, has a solution

$$f(v,\tau)=\frac{e^{-v^2/(2K\beta^2)}}{2(2\pi\beta^2K)^{3/2}}\left\{\frac{5K-3}{K}+\frac{1-K}{K^2}\frac{v^2}{\beta^2}\right\},\ \beta^2\equiv\frac{T}{m}\ , \tag{2.5}$$

with $K=1-e^{-\tau/6}$ and $\tau\geq\tau_0\equiv 6\ln(\frac{5}{2})$. T and m are the temperature and mass, respectively. The boundary conditions are the equilibrium Maxwell-Boltzmann distribution

$$f(v,\tau)\underset{\tau\to\infty}{\to} f_0(v)=\frac{e^{-v^2/(2\beta^2)}}{(2\pi\beta^2)^{3/2}}\ , \tag{2.6}$$

and, for all τ,

$$\int f(v,\tau)d^3v=1\ , \tag{2.7}$$

$$\int f(v,\tau)v^2d^3v=3\beta^2\ . \tag{2.8}$$

Equations (2.7) and (2.8) are mass and energy conservation, respectively. The Boltzmann equation (2.2) is

equivalent to the sequence of moment equations (see Ref. 17)

$$\partial_\tau M_k + M_k = \frac{1}{1+k}\sum_{m=0}^{k} M_m M_{k-m}, \quad k=0,1,2,\ldots, \tag{2.9}$$

where the $M_k(\tau)$ are defined by

$$M_k(\tau)=\frac{\sqrt{\pi}}{2(2\beta^2)^k\Gamma(k+\frac{3}{2})}\int d^3v\, v^{2k} f(v,\tau)\,, \tag{2.10}$$

normalized so that the boundary conditions read

$$M_0(\tau)=M_1(\tau)=1\,, \tag{2.11}$$

$$M_k(\infty)=1,\ k=0,1,2,\ldots. \tag{2.12}$$

For purposes of convenience, definition (2.10) of M_k may be extended to encompass negative and "odd" moments of the velocity (absolute value):

$$\mathcal{M}_n(\tau)=\frac{\sqrt{\pi}}{2(2\beta^2)^{n/2}\Gamma[(n+3)/2]}\int d^3v\, v^n f(v,\tau),$$
$$n=-2,-1,0,1,\ldots. \tag{2.13}$$

For $n=2k$ even, (2.13) coincides with definition (2.10). The higher moments probe the high velocity tail of the distribution function, while the low and negative ones describe the function near $v=0$. Inserting the solution (2.5) for $f(v,\tau)$ into definition (2.13) one obtains an expression describing the time evolution of each moment:

$$\mathcal{M}_n(\tau)=\frac{n}{2}(1-e^{-\tau/6})^{n/2-1}-\left[\frac{n}{2}-1\right](1-e^{-\tau/6})^{n/2},$$
$$n=-2,-1,0,1,\ldots, \tag{2.14}$$

where we recall that $\tau=4\pi\rho\kappa t$ measures time in units of the mean collision time at equilibrium. Of particular relevance to this work are the relaxation rates given by

$$\partial_\tau \mathcal{M}_n=\frac{1}{6}\frac{n}{2}\left[\frac{n}{2}-1\right]e^{-\tau/3}(1-e^{-\tau/6})^{n/2-2},$$
$$n=-2,-1,0,1,\ldots \tag{2.15}$$

for intermediate times τ.

III. THE TEST-PARTICLE METHOD

A. Methods of simulation

In the test-particle method for solving (for example) the Boltzmann equation, the phase-space distribution function $f(\mathbf{v},\mathbf{r})$ at some time t is represented by a set of $N=\tilde{N}A$ pointlike test particles, where A is the number of physical particles:

$$\rho f(\mathbf{v},\mathbf{r})\cong\frac{A}{V}\frac{1}{N}\sum_{i=1}^{N}\delta^{(3)}(\mathbf{v}-\mathbf{v}_i)\delta^{(3)}(\mathbf{r}-\mathbf{r}_i)\,. \tag{3.1}$$

Above, ρ is the total number density of physical particles. The quantities $(\mathbf{v}_i,\mathbf{r}_i)$ are chosen at random from a specified distribution function. The test-particle fluid is then evolved according to the Boltzmann equation with a two-body scattering cross-section $\sigma_{NN}/\tilde{N}$, and propagation between stochastic collisions is determined by Hamilton's equations. The above procedure ensures that the system's phase space is sampled in a sufficiently local fashion, in accordance with the pointlike nature of binary collisions in the Boltzmann equation approach. Henceforth, we will refer to this approach as the full ensemble (FE) method.

Because computation times involved in a $\tilde{N}A$ test-particle ensemble collision may be prohibitively large, it has often been the practice to divide the test-particle fluid into $\tilde{N}$ separate ensembles of A physical point nucleons and permitting only intraensemble collisions with the two-body cross sections σ_{NN}. We will refer to this approach, which is used in, for example, Refs. 1,9, and 10, as the parallel ensemble (PE) method.

Clearly, there are many cases intermediate to the two methods above, viz. those in which each nucleon is represented by N' test nucleons, colliding with cross-section σ_{NN}/N' and one considers $\tilde{N}/N'$ noncommunicating parallel ensembles.

The algorithm employed in, for example, Refs. 4 and 5 follows the spirit of the FE method, but without the factor $\tilde{N}$ in the collision prescription. Instead of dividing the cross section by $\tilde{N}$, multiple collisions between test particles are prevented; i.e., two test particles are allowed to collide only once during a given time step. In referring to this approach below, we shall use the phrase "hybrid method."

We propose to test these various methods of dealing with the collision term using the exact (but classical and nonrelativistic) solution of the Boltzmann equation outlined in the previous section. In particular, we will focus our attention on the rate at which the moments of the distribution function calculated from these methods relax in comparison with those from the Krook-Wu model. Another aim will be to determine the extent to which interensemble communication is required in, for example, obtaining the correct collisional relaxation. Our numerical computations for the PE method were performed with the techniques described in Ref. 1. For calculations using the FE method we employ a straightforward generalization of the techniques used in the PE method. Simulations with the hybrid approach were performed following Refs. 4 and 5.

B. Calculational details

A set of $N=\tilde{N}A$ test nucleons is distributed uniformly in a cube of linear dimension L. The physical number density is then $\rho=A/L^3$, and is chosen within the range of densities achieved in heavy ion collisions, viz. $(1-4)\rho_0$, where $\rho_0=0.16\ \mathrm{fm}^{-3}$ is the density of nuclear matter at equilibrium. To ensure the constancy of ρ in the spirit of the Krook-Wu model, periodic boundary conditions are implemented. For a given temperature T, initial particle velocities are distributed isotropically using the rejection method (see, for example, Ref. 22) according to

$$P(v)dv = 4\pi v^2 f(v,\tau_i)dv \ , \tag{3.2}$$

where $f(v,\tau_i)$ is given by Eq. (2.5) and $\tau_i \geq 6\ln(\frac{5}{2})$ is the initial time. The initial velocity distribution was considered unacceptable if its moments differed by more than a specified tolerance from the exact moments, given by Eq. (2.14) with $\tau=\tau_i$. Typically, the relative error chosen in $\mathcal{M}_n(\tau_i), n=1,\ldots,4$, was of the order of 0.1% for $N=1000-10000$. For subsequent times a set of various moments calculated from a numerical test-particle evolution of the phase space density may be compared to the exact solution, providing a sensitive check on the numerical solution in its various implementations, as a function of the parameters of the system.

In the numerical simulation of the FE method, two test particles collide if in the entrance channel they pass each other at distances r_o less than $\sqrt{\sigma_{NN}/(\tilde{N}\pi)}$, where σ_{NN} is the total cross section. In the PE method, the particles collide with the physical cross section and the above distance is given by $\sqrt{\sigma_{NN}/\pi}$. Either way, but especially so for the latter method, the numerical collision process is nonlocal and, in particular, finite-size effects of the system may distort the numerical evolution of the moments of the distribution function.

In a Maxwell-Boltzmann gas, i.e., at times t large compared to 6 $(4\pi\rho\kappa)$, the average value of r_0 is given by

$$\langle r_0\rangle^2_{t\to\infty} = \frac{1}{\tilde{N}}\frac{[\Gamma(\frac{1}{4})]^2}{2\pi}\frac{\kappa}{\beta} \approx \frac{2.09}{\tilde{N}}\frac{\kappa}{\beta} \ . \tag{3.3}$$

The factor $1/\tilde{N}$ is not present in a parallel ensemble simulation ($\langle r_0\rangle_{\rm PE} = \sqrt{\tilde{N}}\langle r_0\rangle_{\rm FE}$), clearly showing the

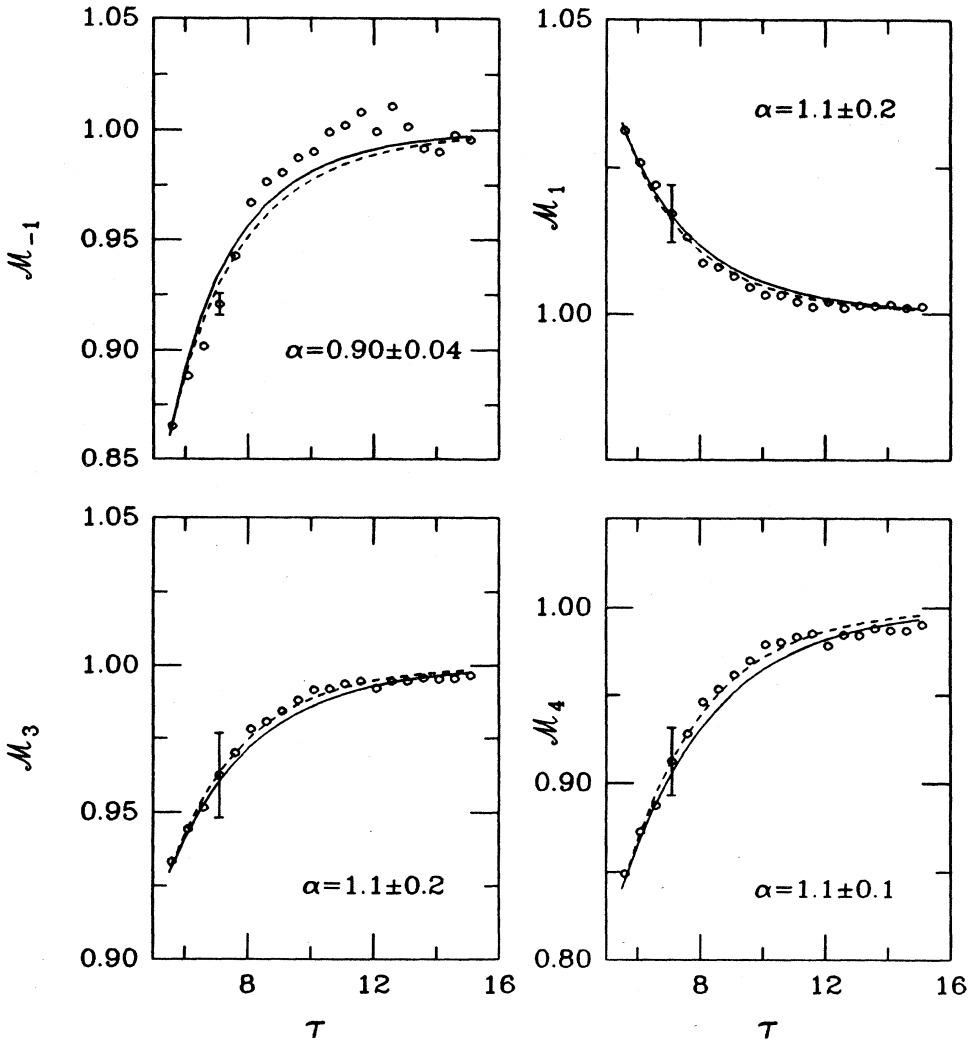

FIG. 1. Relaxation of the moments $\mathcal{M}_n$, $n=-1$, 1, 3, 4, as a function of dimensionless time from numerical simulations (open circles) using the full ensemble (FE) method, compared with the exact solutions of the Krook-Wu model (solid lines). These results were obtained with $L=6$ fm, $\kappa=0.024$ fm$^2\,c$, $\rho=4\rho_0$, $\tilde{N}=30$, and $T=40$ MeV.

localizing effect of the full ensemble technique. Note that the probability of two particles having zero relative velocity g is vanishingly small.

The numerical simulation can only be expected to approximate the exact solution if the linear dimension of the system, L, is much greater than $\langle r_0 \rangle$. If this condition is not met, one might reasonably expect the system to relax to the equilibrium value $f_0(v)$, Eq. (2.6), at a slower rate than that given by Eq. (2.14).

To further minimize the effects of the periodic boundary conditions, one might require the Knudsen number $\mathrm{Kn}_\infty \equiv \lambda_\infty / L$, to be much less than unity, where λ_∞ is mean free path of particles at large times τ. We use

$$\lambda_\infty = \frac{\sqrt{2}\beta}{4\pi\rho\kappa} , \tag{3.4}$$

where $\sqrt{2}\beta$ is the most probable particle velocity for a Maxwell-Boltzmann gas.

IV. RESULTS

A. Analytical model

The exact solution of the Krook-Wu model corresponds to a homogeneous infinite system. Results of a numerical simulation may therefore contain effects due to the finite size of the system, as outlined in the preceding

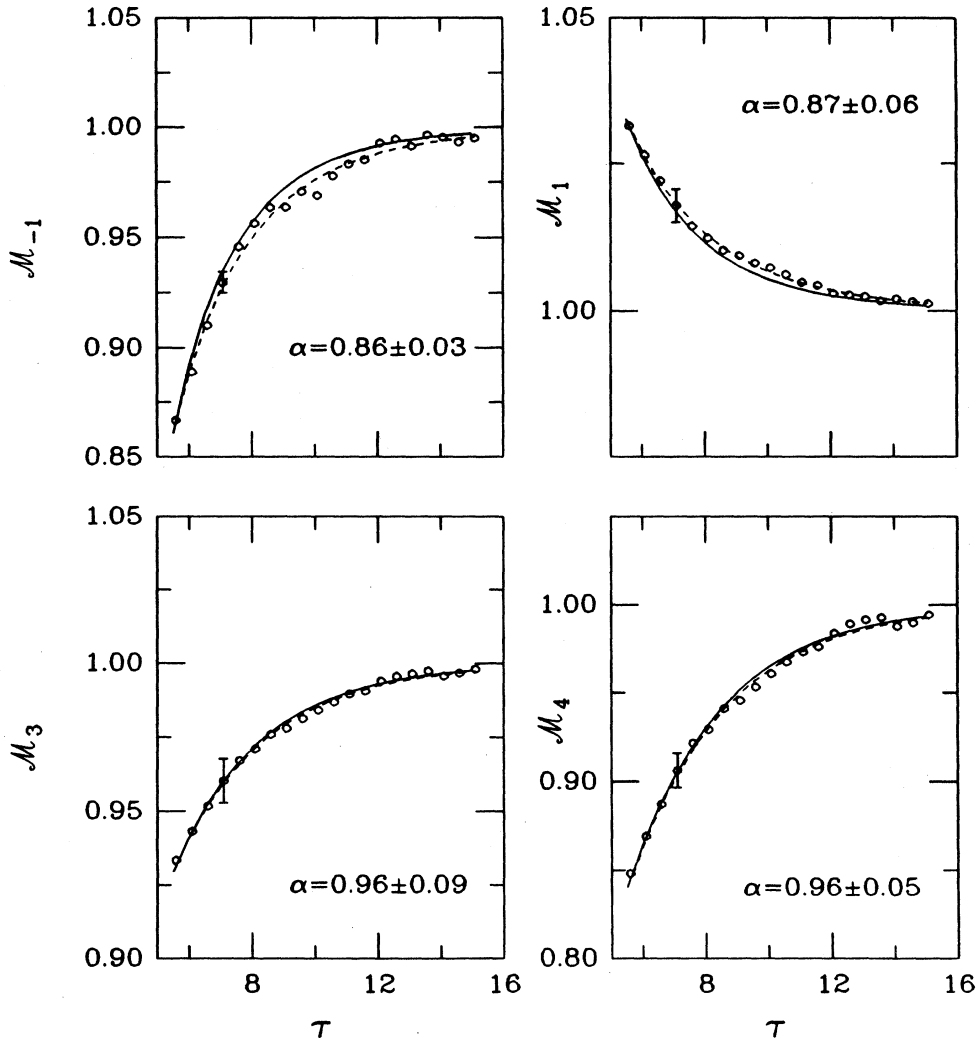

FIG. 2. Same as in Fig. 1 but for the parallel ensemble (PE) method with $\tilde{N} = 120$.

section. We now discuss these by comparing the relaxation rates to equilibrium of moments obtained from numerical calculation with the exact solution Eq. (2.14). One way to quantify any differences in relaxation is to change the input parameter κ in Eq. (2.14) by a scale factor $\alpha(\kappa \rightarrow \kappa' = \alpha\kappa)$, with the constraint that at the initial time τ_i the κ and κ' solutions coincide. A value of α greater than unity then corresponds to a numerical simulation that relaxes to equilibrium at a rate faster than the analytical solution, while if α is less than unity the simulation approaches equilibrium too slowly.

In Figs. 1–3 the time evolution of moments $\mathcal{M}_n$ for $n = -1$, 1, 3, and 4 from the PE, FE, and hybrid methods are shown as a function of τ. The moments $\mathcal{M}_0$ and $\mathcal{M}_2$ are unity for all times τ by construction and the numerical simulations preserve this feature. These results were obtained with $L = 6$ fm, $\kappa = 0.024$ fm$^2 c$, $\rho = 4\rho_0$, $A = 138$, and $T = 40$ MeV. For these values the relevant physical attributes with dimensions of length have the values $\langle r_0 \rangle_{\rm FE} \simeq 0.1$ fm, $\langle r_0 \rangle_{\rm PE} \simeq 0.5$ fm and $\lambda_\infty \simeq 1.5$ fm.

In the figures, the exact solution is drawn as a solid curve. Results of numerical simulations are shown as open circles, together with a typical error bar. For the PE method, this error was calculated using $e_{\rm PE} = \sigma_y / \sqrt{\tilde{N}}$, where

$$\sigma_y = \left[\sum_{i=1}^{\tilde{N}} (\mathcal{M}_n^{(i)} - \mathcal{M}_n)^2 / (\tilde{N} - 1) \right]^{1/2} ,$$

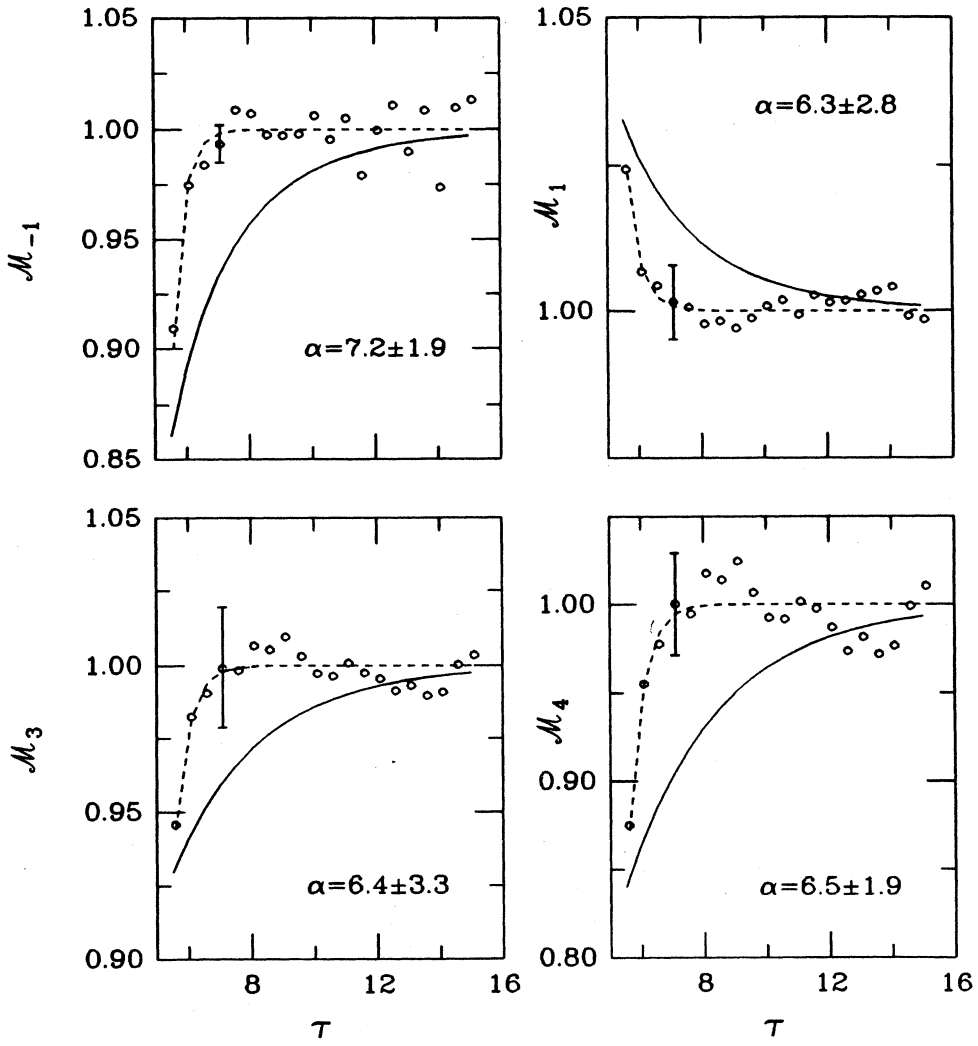

FIG. 3. Same as in Fig. 1 but for the hybrid method (see Refs. 4 and 5).

where $\mathcal{M}_n^{(i)}$ is the moment calculated from the ith ensemble, and $\mathcal{M}_n$ is the average for the entire ensemble set. For the FE and hybrid methods, the error bars were calculated similarly, with the full ensemble randomly divided into $\tilde{N}$ artificial "subensembles." Also given in each figure is the value of α calculated via a χ-squared fit to the numerical data. The dashed lines are the result of this fitting procedure. The error quoted for α is one standard deviation divided by the square root of the number of ensembles $\tilde{N}$ considered. For the FE and hybrid methods, this number is dictated by numerical computation times involved and was chosen to be 30. In the PE case, statistics were increased by choosing $\tilde{N}=120$.

From Figs. 1 and 2, we conclude that both the FE and PE methods yield relaxation rates consistent with the exact results for the system considered here. Any apparent differences between the moments from the two methods and analytical result cannot be considered statistically significant.

The moments calculated from the hybrid method (see Fig. 3) relax much too fast to the equilibrium distribution. This must be attributed to the algorithm where the collision rate is calculated by suppressing multiple collisions during each time step with the physical cross section. The general FE method, where cross sections are divided by $\tilde{N}$ but where multiple collisions are not withdrawn, turns out to give more satisfactory relaxation times. This result is consistent with comparisons[18] of

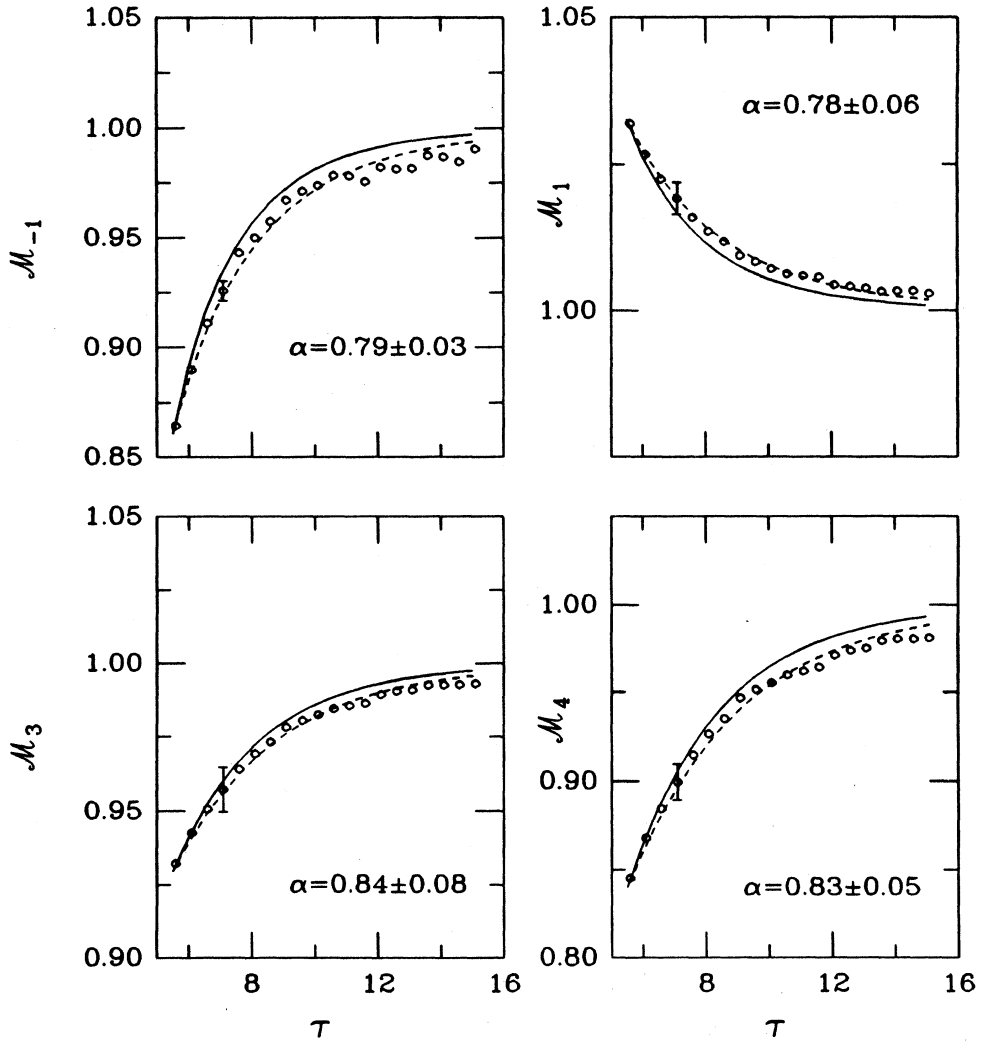

FIG. 4. As in Fig. 2, but with $L=3$ fm, $\rho=4\rho_0$, $\tilde{N}=976$, and $T=40$ MeV.

these two methods using the Toepffer-Wong model[23] in which the time evolution of some nonequilibrium momentum distributions are studied numerically by solving the Uehling-Uhlenbeck equation. This conclusion is also consistent with the calculation in Ref. 20.

We turn now to the case when the linear dimension $L/2$ is comparable to the lengths $\langle r_0 \rangle_{\rm FE}$, $\langle r_0 \rangle_{\rm PE}$, and λ_∞. Choosing $L=3$ fm, $\kappa=0.024$ fm$^2 c$, $\rho=4\rho_0$, $\tilde{N}=976$, and $T=40$ MeV, moments from the PE method were compared with the exact moments (see Fig. 4). The PE method underestimates the relaxation rates by about 30–40 %. This discrepancy can be translated into an effective reduction in either the cross section or the density by about 30–40 %. In contrast, results obtained from the FE method are typically as accurate as those shown in Fig. 1. In general, therefore, from the magnitude of α, we conclude that the FE method approximates the exact results better than the PE method.

We therefore turn to a more systematic investigation of when the PE method becomes inaccurate. Conceivably, this may occur when the differential cross section κ or the mean free path λ become too large in relation to the system, or when the number of physical particles used to calculate the collision integral becomes too small. Throughout the following discussion, $T=40$ MeV and $L=6$ fm. Also, from here on, the value of α used to characterize the deviation from the exact solution is an average obtained from a relaxation of the moments $\mathcal{M}_n$, $n=-1$, 1, and 3.

In Fig. 5, the cross section κ is varied in a PE simulation, keeping the density fixed at $4\rho_0$. Therefore $A=138$, and the mean free path λ_∞ (shown by the solid line in Fig. 5) is less than 3 fm in the range of values of κ considered. When κ becomes large, i.e., $\langle r_0 \rangle_{\rm PE}/L \gtrsim 0.1$, the relaxation rate to the equilibrium distribution is too slow. Clearly, the collision process in the numerical simulation is not localized enough in relation to the size of the system. The presence of a surface diminishes the collision frequency $4\pi\rho\kappa$ by a factor $\alpha<1$. Such reductions could arise in heavy-ion collisions involving light systems.

The above nonlocalization effect is also apparent in Fig. 6, where $\lambda_\infty \propto (\rho\kappa)^{-1}$ is fixed at 0.36 fm, i.e., κ is varied by changing the density of the system. As $\langle r_0 \rangle_{\rm PE}$ increases, α drops to values well below unity. This decrease is consistent with that in Fig. 5, where A is fixed at 138. Thus the decrease is not attributable to a small number of physical particles A at small ρ, i.e., the number of Monte Carlo points used to perform the integration of the collision term. We note that in order to ensure reasonable statistics, $\tilde{N}$ is changed to keep N fixed at $\sim 17\,000$.

To investigate the possible effect of small A, we fix $\kappa=0.1$ fm$^2 c$ in Fig. 7 and vary A by changing the density ρ. At this value of κ, a nonlocality effect corresponding to $\alpha \sim 0.7$ is expected and indeed is seen in Fig. 7. We choose this value to retain a Knudsen number ${\rm Kn}_\infty \equiv \lambda_\infty/L$ of less than unity for small particle numbers $A \sim 10$. From Fig. 5, it is apparent that the effect of ${\rm Kn}_\infty$ up to $\frac{1}{2}$ does not significantly affect the relaxation rate. From Fig. 7, we therefore conclude that down to

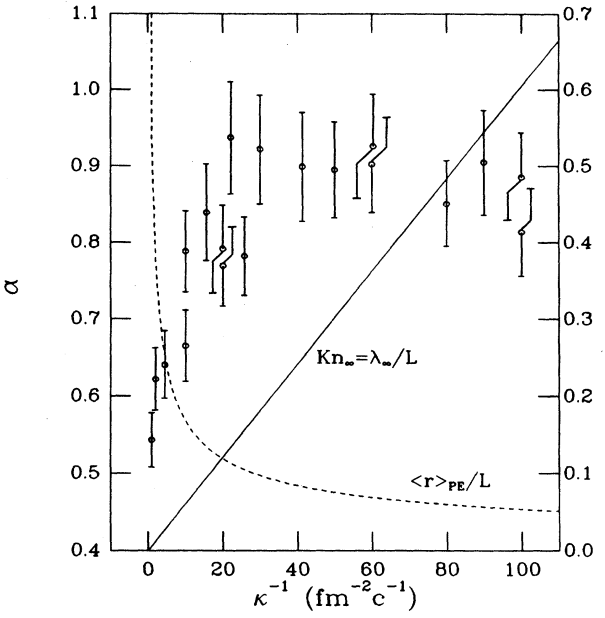

FIG. 5. Average value (see the text) of α vs the inverse cross section κ^{-1} (open circles), as obtained from a PE simulation. The density, temperature, and size are fixed at $\rho=4\rho_0$, $T=40$ MeV, and $L=6$ fm, respectively. The solid line is the corresponding Knudsen number ${\rm Kn}_\infty$, while the dashed line is the ratio of $\langle r_0 \rangle_{\rm PE}$ to L (right-hand scale). The number of ensembles used is $\tilde{N}=120$.

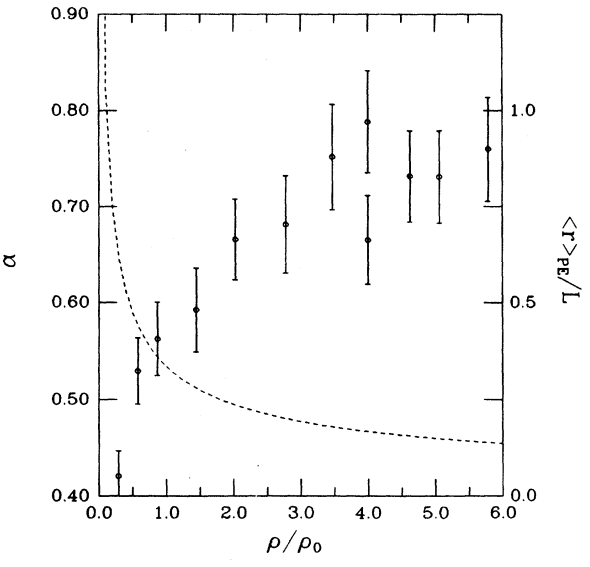

FIG. 6. Average value of α vs the density from a PE simulation (open circles). The temperature, size, and mean free path at large τ are fixed to be $T=40$ MeV, $L=6$ fm, and $\lambda_\infty=0.36$ fm, respectively. The dashed line is the ratio $\langle r_0 \rangle_{\rm PE}/L$ (see the text).

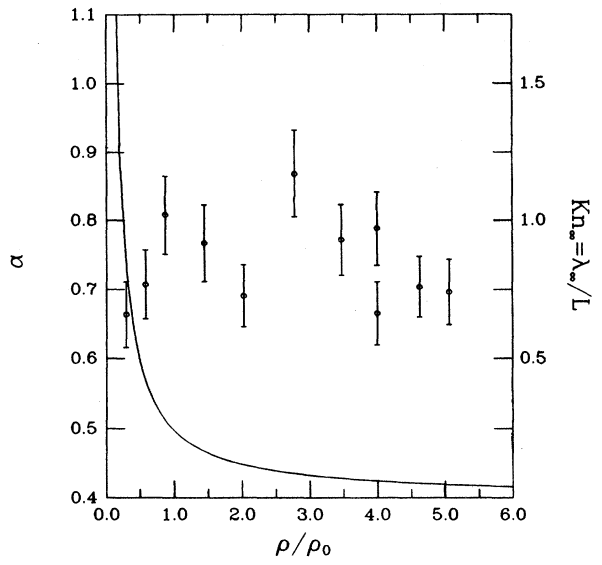

FIG. 7. Average value of α vs density from a PE simulation (open circles). The cross section, temperature, and size of the system are fixed at $\kappa=0.1$ fm$^2 c$, $T=40$ MeV, and $L=6$ fm, respectively. The solid line is the Knudsen number. For this system, $\langle r_0 \rangle_{PE}/L=0.17$.

$A=10$ ($\rho \sim 0.3\rho_0$) there is no discernable effect on α due to decreased particle number. Once again, as in Fig. 6, $\tilde{N}$ has been increased for small A, to keep $N \sim 17\,000$.

To conclude this section, we have seen that the FE method provides a reliable technique for simulating the Boltzmann equation. We further see that for Kn_∞ up to at least $\frac{1}{2}$ and for densities as low as $0.3\rho_0$, the PE method is also sufficiently accurate. However, for $\langle r_0 \rangle_{PE}/L \gtrsim 0.1$, for which more than 40% of the particles lie within a distance $\langle r_0 \rangle_{PE}$ of the surface of the system, a sizable surface effect due to the nonlocality of the collision process in the PE method can be expected. We turn next to the question of whether in a heavy-ion collision simulated by the Boltzmann equation (1.1), a PE method is sufficiently accurate.

B. Heavy-ion collisions

For a free space nucleon-nucleon cross section of ~ 40 mb in a geometrical picture, $r_0 \sim 1$ fm for a PE simulation, while the interaction region in near-central heavy-ion collisions before freezeout is of the order of 10 fm. For collisions involving smaller mass systems, and especially if one considers in-medium processes that significantly enhance the two-body cross section, r_0/L may well be large enough for a PE evolution of the system to be different from a FE simulation.

We thus consider differences between the PE and FE methods in flow parameters of heavy-ion collision simulations at impact parameters $b=0.46R$, where $R=1.14A^{1/3}$ fm is the radius of the colliding nuclei. Since we are concerned with the accuracy of the collision integral, the mean field in Eq. (1.1) is turned off; thus, no interensemble communication is present in the calculations, and the nucleons have no Fermi momentum in the initial state. A "pseudo" Pauli blocking principle[6] is, however, retained by prohibiting collisions between two nucleons with a c.m. kinetic energy of less than 25 MeV. We take the nucleon-nucleon cross section from Ref. 1.

Table I summarizes the results obtained. Shown are the average in-plane transverse momentum $\langle \omega_i p_x^i \rangle$, where $\omega_i=\mathrm{sgn}(y_{\mathrm{cm}}^i)$, and the slope F at midrapidity of the function $p_x(y)$. F was obtained by fitting a cubic polynomial to all p_x with $|y_{\mathrm{cm}}/y_{\mathrm{cm}}^{\mathrm{proj}}|<1$. The error given for $\langle \omega_i p_x^i \rangle$ is one standard deviation calculated from

TABLE I. Flow parameters from collision simulations using the Boltzmann equation (1.1), with $U \equiv 0$. The impact parameters are given by $b=0.46R$, where $R=1.14A^{1/3}$ fm is the radius of the colliding nuclei. The nucleon-nucleon cross-section σ_{NN} is taken from Ref. 1, and $r_0^2=\sigma_{NN}^{\max}/\pi$ for the PE method, while for the FE method, $r_0^2=\sigma_{NN}^{\max}/(\tilde{N}\pi)$. The flow parameter F is the midrapidity slope of $p_x(y)$, while $\langle \omega_i p_x^i \rangle$ is the average in-plane transverse momentum (see the text).

System	E/A (MeV)	$\sigma_{NN}/\sigma_{NN}^{\mathrm{free}}$	r_0/R	Method	$\langle \omega_i p_x^i \rangle$ (MeV/c)	F (MeV/c)
^{12}C+C	200	1	0.03	FE	12±1	34±6
			0.51	PE	12±1	39±1
		2	0.04	FE	27±1	62±4
			0.72	PE	23±1	69±3
^{20}Ne+Ne	800	1	0.03	FE	21±2	70±6
			0.43	PE	18±1	57±3
		2	0.04	FE	42±2	122±6
			0.60	PE	40±2	118±4
^{139}La+La	800	1	0.03	FE	40±2	108±13
			0.22	PE	41±2	116±3
		4	0.06	FE	68±2	179±11
			0.45	PE	61±2	179±14

$\tilde{N} \sim 20\,000-30\,000$ events, divided by $\sqrt{\tilde{N}}$. The error quoted for F is the standard deviation of four equal samples taken from the quoted number of events. Differences between the FE and PE simulations are seen to arise if the two-nucleon cross section is increased well beyond its free space value. For $\sigma_{NN}/\sigma_{NN}^{\text{free}} \gtrsim 2$, the average in-plane transverse momenta are slightly underestimated by the PE method in comparison with the FE results. On the other hand, the PE method yields a reasonable estimate of the midrapidity flow F, even for $\sigma_{NN}/\sigma_{NN}^{\text{free}} \sim 4$. In view of the values of r_0/R and the analysis of the Krook-Wu model, this result is somewhat surprising. However, several effects that prevent a more direct comparison with the analytical model may be playing a role, as for example, (i) the varying size of the interaction region; (ii) the energy dependence of the cross section; and (iii) the sensitivity of collective flow to the relaxation rate.

V. SUMMARY

We have considered the numerical simulation of the Boltzmann equation using the test-particle method in three forms. In the parallel (PE) ensemble method each nucleon is represented by one test particle and $\tilde{N}$ separate ensembles are considered. The nucleons collide with the physical cross section σ_{NN}. In the full (FE) ensemble method, $\tilde{N}$ test particles represent one physical particle. The test particles collide with two-body cross section $\sigma_{NN}/\tilde{N}$ and we consider only one ensemble. The hybrid method[4,5] allows for collisions between all pairs of test particles with cross section σ_{NN}, and suppresses multiple collisions within one time step.

We have shown that the FE technique yields an accurate estimate of the collision integral Eq. (1.2). Comparison of the PE method with analytical results in the Krook-Wu model show that for a wide range of physical situations, the PE technique provides a reasonable numerical method for simulating the Boltzmann equation. However, if a sizable fraction ($\sim 40\%$) of particles lie within a distance $\sim\sqrt{\sigma_{NN}/\pi}$ of the surface of the system, the nonlocal nature of the collisional simulation drastically affects the relaxation rate to equilibrium. On the other hand, the relaxation is negligibly affected by the size of the mean free path in relation to the system, up to Knudsen numbers of at least $\frac{1}{2}$. Similarly, the Krook-Wu collision integral is seen to be calculated reasonably well by as few as 10 physical particles. The third technique we have considered, the hybrid method, is found not to reproduce the correct relaxation rate.

The comparisons of FE and PE simulations in a cascade model of heavy-ion collision show that the PE method provides a reasonable technique for calculating the mid rapidity flow F. For large enhancements of σ_{NN} over the free space value, the PE method tends to underestimate the average in-plane transverse momenta in comparison with the FE method. The Krook-Wu analysis shows that this effect may be attributed to the artificial decrease in the collision rate in the PE method for small systems ($\langle r_0 \rangle \sim L$), since the collision rate affects the transverse flow produced.[12] Quantitative comparisons of the BUU model, in which in-medium enhancements of σ_{NN} play a role, to experimental data should therefore take this numerical effect into account.

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