# Path integral and boson-fermion expansion in many-fermion systems: Lipkin model

Kazunari Kaneko
*Department of Physics, College of Liberal Arts, Kyushu Sangyo University, Kashii, Fukuoka 813, Japan*
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In a previous paper, a quantum-mechanical formulation involving both mean fields and independent-particle fields in many-fermion systems was proposed using the path-integral technique. Then the semiclassical calculation of the energy spectra was performed, and the quantization rule was derived by applying a stationary phase approximation on the path integral. In this paper, a boson-fermion expansion is derived from our formulation using Dirac quantization. As an illustration, the Lipkin model is utilized.

The author has recently proposed a quantum-mechanical treatment using the path-integral technique for the description of the interplay between collective and independent-particle motions.[1,2] The path integral was then written as a functional integral over collective and independent-particle fields. In a previous paper,[2] the path integral over the independent-particle fields was evaluated using the quantum adiabatic approximation, and then the effective Lagrangian only of collective fields was obtained. Furthermore, the semiclassical quantization method led to the quantization condition analogous to the Bohr-Sommerfeld quantization rule derived by Shanker, except that it included the independent-particle degrees of freedom. Then the quantum numbers characterizing the excitations of the independent particles denoted the seniority numbers.

The boson expansion (BE) method[3] is a useful tool in analyzing the anharmonicity effects in transitional even-even nuclei. Furthermore, this method has been extended to the case of the odd nuclei [we call it the boson-fermion expansion (BFE) method[4]]. The entire fermion Hilbert space can be mapped into both boson and ideal-fermion spaces, and then fermion operators are described by the boson and ideal-fermion operators. The ideal-fermion operators do not obey the usual anticommutation relations. This is due to some constraints of ideal fermions. On the other hand, Yamamura and Kuriyama[5] have recently succeeded in obtaining BFE using Dirac quantization[6] of the classical theory with the ordinary and the Grassmann variables. The classical version of the constraints was obtained from the canonicity condition.

In this paper, the relationship between our formulation and BFE method will be examined. Then BFE is derived using Dirac quantization. As an illustration, the Lipkin model is utilized.

We first start from the path integral of Lipkin model in Ref. 2:

$$K(T)=N\int D[a]D[a^*]D[b]d[b^*]\exp\left[i\int_0^T L(t)dt\right] , \tag{1}$$

where the Lagrangian is written as

$$L(t)=\frac{i}{2}\sum_m[(a_m^*\dot a_m+a_m\dot a_m^*)+(b_m^*\dot b_m+b_m\dot b_m^*)] -2\varepsilon J_z+\tfrac{1}{2}V(J_+J_++J_-J_-) , \tag{2}$$

$$J_+=\sum_m a_m^*b_m^*=(J_-)^\dagger ,$$

$$J_z=\tfrac{1}{2}\left[\sum_m(a_m^*a_m+b_m^*b_m)-2\Omega\right] , \tag{3}$$

where the overdot denotes the time derivative. As the Lagrangian $L$ is not quadratic in the Fermi fields, a direct path-integral treatment is difficult. It is, therefore, useful to introduce an equivalent Lagrangian which is quadratic in the Fermi fields. As is well known, a Gaussian method leads to the Lagrangian that is quadratic in the Fermi fields. Then the path integral (1) becomes

$$K(T)=N'\int D[\rho_+]D[\rho_-]D[\rho_z] \times\exp\left[-\frac{i}{2}\int_0^T V(\rho_+^2+\rho_-^2)dt\right]K_i(T) , \tag{4}$$

$$K_i(T)=\int D[a]D[a^*]D[b]D[b^*]\exp\left[i\int_0^T L_i dt\right] , \tag{5}$$

where the Lagrangian $L_i(t)$ is given by

$$L_i(t)=\frac{i}{2}\sum_m[(a_m^*\dot a_m+a_m\dot a_m^*)+(b_m^*\dot b_m+b_m\dot b_m^*)] -2\varepsilon J_z+V(\rho_+J_++\rho_-J_-) . \tag{6}$$

Thus, new fields are introduced as independent integration variables in addition to the original Fermi fields. If we use the stationary phase approximation to the path integral (4), $(\rho_+,\rho_-,\rho_z)$ are related to the original Fermi fields by

$$\rho_+=J_+,\quad \rho_-=J_-,\quad \rho_z=J_z .$$

In the connection with the mean-field theory, it is convenient to parametrize auxiliary fields (collective fields) as follows:

$$\rho_+=2r\sigma^*(1-\sigma^*\sigma)^{1/2},$$
$$\rho_-=2r(1-\sigma^*\sigma)^{1/2}\sigma, \tag{7}$$
$$\rho_z=2r(\sigma^*\sigma-\tfrac{1}{2}),$$

where $\sigma$ is the complex variable and $r$ is the real variable. Thus, the auxiliary fields are expressed by two types of degrees of freedom, i.e., one is collective variables $(\sigma,\sigma^*)$ and the other is the independent variable $r$. By introducing this auxiliary field the double counting of degrees of freedom occurs in the canonical form representation. We will discuss this later.

To proceed further, we go to the "body-fixed frame" of fermions by means of unitary transformation,

$$\alpha_m=ua_m-vb_m^*,\quad \beta_m=ub_m+va_m^*, \tag{8}$$

where $\alpha_m$ and $\beta_m$ are the Grassmann numbers and the coefficients are

$$u=(1-\sigma^*\sigma)^{1/2},\quad v=\sigma. \tag{9}$$

The path integral (1) can then be written as

$$K(T)=N'\int D[r]D[\sigma]D[\sigma^*]\times\exp\left[i\int_0^T L_0(t)dt\right]K_f(T), \tag{10}$$

$$K_f(T)=\int D[\alpha]D[\alpha^*]D[\beta]D[\beta^*]\times\exp\left[i\int_0^T L_f(t)dt\right]. \tag{11}$$

Here the Lagrangians are defined by

$$L_0(t)=2\Omega A+2\varepsilon\Omega-2Vr^2(1-\sigma^*\sigma)(\sigma^{*2}+\sigma^2), \tag{12}$$

$$L_f(t)=\frac{i}{2}\sum_m[(\alpha_m^*\dot\alpha_m+\alpha_m\dot\alpha_m^*)+(\beta_m^*\dot\beta_m+\beta_m\dot\beta_m^*)]-A\sum_m(\alpha_m^*\alpha_m+\beta_m^*\beta_m)+B\sum_m\alpha_m^*\beta_m^*+B^*\sum_m\beta_m\alpha_m, \tag{13}$$

where the coefficients $A$ and $B$ are

$$A=\frac{i}{2}(\sigma^*\dot\sigma-\dot\sigma^*\sigma)+\varepsilon(1-2\sigma^*\sigma)+2Vr(1-\sigma^*\sigma)(\sigma^{*2}+\sigma^2), \tag{14}$$

$$B=i[\dot\sigma+\tfrac{1}{2}\sigma(\dot\sigma^*\sigma-\sigma^*\dot\sigma)]/(1-\sigma^*\sigma)^{1/2}-2\varepsilon(1-\sigma^*\sigma)^{1/2}\sigma+2Vr(1-\sigma^*\sigma)^{1/2}[\sigma^*(1-\sigma^*\sigma)-\sigma^3]. \tag{15}$$

In Ref. 2, the adiabatic approximation was used for evaluating the path integral (11) over the independent-particle fields, and then the path integral of effective action was obtained. Furthermore, the quantization rule was derived by applying the semiclassical quantization method. The energy levels were characterized by two integers $m$ and $n$. The quantum number $m$ represented the number of full waves fitted along the time-dependent Hartree-Fock (TDHF) orbit, while $n$ labeled the excitations of the independent particles and denoted the seniority numbers.

In this paper, let us perform the canonical quantization of the classical system expressed by the Lagrangian $L(t)$. To do such a quantization, we should construct the canonical formulation from the Lagrangian. Since the dynamical variables are $r$, $\sigma$, $a_m$, and $\beta_m$, the canonical conjugate momenta are defined as

$$\pi=\frac{\partial L}{\partial\dot r}=0, \tag{16}$$

$$\pi^\sigma=\frac{\partial L}{\partial\dot\sigma}=2iS\sigma^*+\frac{\partial B}{\partial\dot\sigma}\sum_m\alpha_m^*\beta_m^*+\frac{\partial B^*}{\partial\dot\sigma}\sum_m\beta_m\alpha_m, \tag{17}$$

$$\pi_m^\alpha=\frac{\partial L}{\partial\dot\alpha_m}=-i\alpha_m^*,\quad \pi_m^\beta=\frac{\partial L}{\partial\dot\beta_m}=-i\beta_m^*, \tag{18}$$

where $2S=2\Omega-\sum_m(\alpha_m^*\alpha_m+\beta_m^*\beta_m)$. Then the Hamiltonian is given by the following Lagrange transform:

$$H=\tfrac{1}{2}(\pi^\sigma\dot\sigma-\dot\pi^\sigma\sigma)+\tfrac{1}{2}\sum_m[(\pi_m^\alpha\dot\alpha_m+\alpha_m\dot\pi_m^\alpha)+(\pi_m^\beta\dot\beta_m+\beta_m\dot\pi_m^\beta)]-L. \tag{19}$$

The equation of motion for an arbitrary physical quantity $F$ is given by

$$i\dot F=[F,H]_P, \tag{20}$$

where $[\ ,\ ]_P$ is the Poisson bracket involving Grassmann variables given by Gasalbuoni.[7]

Let us now consider Eq. (16). Here $\pi=0$ is regarded as a *first-class constraint*.[6] The consistency condition is then given by

$$\dot\pi=[\pi,H]_P=-\frac{\partial H}{\partial r}=\frac{\partial L}{\partial r}=-4V(1-\sigma^*\sigma)(\sigma^{*2}+\sigma^2)\left[\Omega-r-\tfrac{1}{2}\sum_m(\alpha_m^*\alpha_m+\beta_m^*\beta_m)\right]-2V(1-\sigma^*\sigma)^{1/2}\left\{[\sigma^*(1-\sigma^*\sigma)-\sigma^3]\sum_m\alpha_m^*\beta_m^*+[\sigma(1-\sigma^*\sigma)-\sigma^{*3}]\sum_m\beta_m\alpha_m\right\}\approx 0. \tag{21}$$

Since the Poisson bracket must be worked out before we make use of the constraint equations, we use a different equality sign $\approx$ from the usual $=$.[6] Therefore, for the arbitrary $(\sigma,\sigma^*)$ the following conditions should hold:

$$\phi_1=\Omega-\tfrac{1}{2}\sum_m(\alpha_m^*\alpha_m+\beta_m^*\beta_m)-r=S-r\approx 0, \tag{22}$$

$$\phi_2=\sum_m\alpha_m^*\beta_m^*\approx 0,\quad \phi_3=\sum_m\beta_m\alpha_m\approx 0. \tag{23}$$

As mentioned before, there is the double counting of degrees of freedom due to the introduction of the auxiliary fields. From the consistency condition (21), the double counting is eliminated by the constraints (22) and (23). The constraints are identical to those obtained by Yamamura and Kuriyama[5] who were led to these constraints from the canonicity conditions The constraints (22) and (23) will play an important role in the canonical quantization later on. The Poisson brackets of these constraints satisfy the following relations:

$$[\phi_1,\phi_2]_P \approx 0, \quad [\phi_1,\phi_3]_P \approx 0 , \tag{24}$$

$$[\phi_2,\phi_3]_P \approx \tfrac{1}{2}\sum_m [(\alpha_m\alpha_m^* - \alpha_m^*\alpha_m) + (\beta_m\beta_m^* - \beta_m^*\beta_m)] . \tag{25}$$

Equation (25) shows that the constraints $\phi_2$ and $\phi_3$ are *second class.*[6] Following the standard procedure of Dirac, it is useful to introduce the Dirac bracket defined by

$$\begin{aligned}[F,G]_D = {}&[F,G]_P - [F,\phi_2]_P[\phi_2,\phi_3]_P^{-1}[\phi_3,G]_P \\ &-[F,\phi_3]_P[\phi_3,\phi_2]_P^{-1}[\phi_2,G]_P .\end{aligned} \tag{26}$$

Here the Dirac brackets satisfy the following relations:

$$[\alpha_m,\pi_{m'}^\alpha]_D = -i\delta_{mm'} - \pi_m^\beta(2J)^{-1}\beta_{m'} , \tag{27}$$

$$[\beta_m,\pi_{m'}^\beta]_D = -i\delta_{mm'} - \pi_m^\alpha(2J)^{-1}\alpha_{m'} , \tag{28}$$

$$[\alpha_m,\pi_{m'}^\beta]_D = \pi_m^\beta(2J)^{-1}\alpha_{m'} , \tag{29}$$

$$[\alpha_m,\alpha_{m'}]_D = [\beta_m,\beta_{m'}]_D = [\alpha_m,\beta_{m'}]_D = 0 , \tag{30}$$

$$[\pi_m^\alpha,\pi_{m'}^\alpha]_D = [\pi_m^\beta,\pi_{m'}^\beta]_D = [\pi_m^\alpha,\pi_{m'}^\beta]_D = 0 , \tag{31}$$

$$[\sigma,\pi^\sigma]_D = i, [\sigma,\sigma]_D = [\pi^\sigma,\pi^\sigma]_D = 0 , \tag{32}$$

$$[\alpha_m,\sigma]_D = [\alpha_m,\pi^\sigma]_D = [\beta_m,\sigma]_D = [\beta_m,\pi^\sigma]_D = 0 , \tag{33}$$

$$[\pi_m^\alpha,\sigma]_D = [\pi_m^\alpha,\pi^\sigma]_D = [\pi_m^\beta,\sigma]_D = [\pi_m^\beta,\pi^\sigma]_D = 0 , \tag{34}$$

where $J$ is defined by

$$2J = \frac{i}{2}\sum_m [(\alpha_m\pi_m^\alpha - \pi_m^\alpha\alpha_m) + (\beta_m\pi_m^\beta - \pi_m^\beta\beta_m)] . \tag{35}$$

Furthermore, the auxiliary variables $(\rho_+,\rho_-,\rho_z)$ of Eqs. (7) are written as

$$\begin{aligned}\rho_+ &= -i\pi^\sigma(1+i\pi^\sigma\sigma/2S)^{1/2} , \\ \rho_- &= 2S(1+i\pi^\sigma\sigma/2S)^{1/2}\sigma ,\end{aligned} \tag{36}$$

$$\rho_z = -i\pi^\sigma\sigma - S , \tag{37}$$

where we used the constraint (22) $r \approx S$. From the inverse relations of Eqs. (8) $a_m$ and $b_m$ are expressed as

$$a_m = \left[1+\frac{i\pi^\sigma\sigma}{2S}\right]^{1/2}(\alpha_m + \sigma\beta_m^*) , \tag{38}$$

$$b_m = \left[1+\frac{i\pi^\sigma\sigma}{2S}\right]^{1/2}(-\sigma\alpha_m^* + \beta_m) . \tag{39}$$

In the connection with the BFE, it is convenient to take another canonical form instead of $(\sigma,\pi^\sigma)$, $(\alpha_m,\pi_m^\alpha)$, and $(\beta_m,\pi_m^\beta)$:

$$\begin{aligned}X &= \sqrt{2S}\,\sigma , \\ X^* &= -i\pi^\sigma/\sqrt{2S} , \\ \alpha_m^* &= i\pi_m^\alpha , \\ \beta_m^* &= i\pi_m^\beta .\end{aligned} \tag{40}$$

Furthermore, let us perform a canonical quantization by the replacements

$$[\ ,\ ]_D \longrightarrow [\ ,\ ]_\pm , \tag{41}$$

$$X,X^* \longrightarrow \hat{X},\hat{X}^* , \tag{42}$$

$$\alpha_m,\alpha_m^* \longrightarrow \hat{\alpha}_m,\hat{\alpha}_m^* , \tag{43}$$

$$\beta_m,\beta_m^* \longrightarrow \hat{\beta}_m,\hat{\beta}_m^* , \tag{44}$$

where $[\ ,\ ]_\pm$ denotes the anticommutation (+) and commutation (−). Then Eqs. (27)–(35) are written by the respective equations

$$[\hat{\alpha}_m,\hat{\alpha}_{m'}^*]_+ = \delta_{mm'} - \hat{\beta}_m^*(2\hat{S})^{-1}\hat{\beta}_{m'} , \tag{45}$$

$$[\hat{\beta}_m,\hat{\beta}_{m'}^*]_+ = \delta_{mm'} - \hat{\alpha}_m^*(2\hat{S})^{-1}\hat{\alpha}_{m'} , \tag{46}$$

$$[\hat{\alpha}_m,\hat{\beta}_{m'}^*]_+ = \hat{\beta}_m^*(2\hat{S})^{-1}\hat{\alpha}_{m'} , \tag{47}$$

$$[\hat{\alpha}_m,\hat{\alpha}_{m'}]_+ = [\hat{\beta}_m,\hat{\beta}_{m'}]_+ = [\hat{\alpha}_m,\hat{\beta}_{m'}]_+ = 0 , \tag{48}$$

$$[\hat{X},\hat{X}^*]_- = 1, \quad [\hat{X},\hat{X}]_- = [\hat{X}^*,\hat{X}^*]_- = 0 , \tag{49}$$

$$[\hat{\alpha}_m,\hat{X}]_- = [\hat{\alpha}_m,\hat{X}^*]_- = [\hat{\beta}_m,\hat{X}]_- = [\hat{\beta}_m,\hat{X}^*]_- = 0 , \tag{50}$$

$$\begin{aligned}\hat{\rho}_+ &= \hat{X}^*(2\hat{S} - \hat{X}^*\hat{X})^{1/2} , \\ \hat{\rho}_- &= (2\hat{S} - \hat{X}^*\hat{X})^{1/2}\hat{X} , \\ \hat{\rho}_z &= \hat{X}^*\hat{X} - \hat{S} ,\end{aligned} \tag{51}$$

$$\hat{a}_m = \left[1-\frac{\hat{X}^*\hat{X}}{2\hat{S}}\right]^{1/2}\hat{\alpha}_m + \frac{\hat{X}}{\sqrt{2\hat{S}}}\hat{\beta}_m^* , \tag{52}$$

$$\hat{b}_m = \left[1-\frac{\hat{X}^*\hat{X}}{2\hat{S}}\right]^{1/2}\hat{\beta}_m - \frac{\hat{X}}{\sqrt{2\hat{S}}}\hat{\alpha}_m^* . \tag{53}$$

These equations are identical with those (BFE) given by several authors.[4,5] In this simple Lipkin model, the expressions (51)–(53) are identical to operators obtained by the Holstein-Primakoff mapping of seniority states. In their mapping, the constraints

$$\sum_m \hat{\alpha}_m^*\hat{\beta}_m^* = \sum_m \hat{\beta}_m\hat{\alpha}_m = 0$$

play an important role in order to select the physical subspace, and are nothing but the quantized version of constraints (23). In our formulation the constraints (22) and (23) were naturally derived from the consistency condition (21), and were necessary for avoiding the double counting of degrees of freedom. Therefore, it is clear that the independent-particle fields $(\alpha_m,\beta_m)$ are the classical version of ideal quasiparticle operators in their attempts.

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