Seniority mapping of single fermion operators

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We discuss the construction of ideal space seniority images of single fermion operators which proceeds via the generalized Dyson mapping and a subsequent similarity transformation. Particular attention is paid to the implications of a consistent treatment of single-particle and collective degrees of freedom. The results are compared with expressions for the mapped single fermion operators previously obtained, respectively, within the Otsuka-Arima-Iachello framework and by using the prescriptions of nuclear field theory. We point out certain inconsistencies in existing constructions of interacting-boson-fermion model operators in general and the interacting-boson-fermion model quadrupole operator in particular.

I. INTRODUCTION

The phenomenological interacting-boson-fermion model (IBFM) (Ref. 1) utilizes single nucleon degrees of freedom in addition to the boson degrees of freedom which appear in the interacting-boson model (IBM) (Ref. 2) where they are interpreted as representing collective angular momentum zero and two fermion pairs. The actual substantiation of this interpretation has recently been the focus of many microscopic investigations[1,3–7] which generally do lend some support to the phenomenological models, but also point to various difficulties (some to be addressed below) which arise on the road from a microscopic shell-model description to the traditional phenomenological IBM description.

A microscopic approach to the IBFM is, of course, confronted with all those considerations which one has to address when a microscopic basis for the IBM is pursued. In addition, a proper discussion of a framework in which a model such as the IBFM can arise has to address the problem of a consistent treatment of collective degrees of freedom and the single-particle degrees of freedom out of which they are constructed to the first place. It is this somewhat neglected aspect on which we focus here, especially in connection with the notion that the boson image of a product of fermion operators can, in general, be obtained as the product of the images of the individual operators.

The microscopic calculation of realistic IBM and IBFM parameters is at least partially successful in the sense that qualitative trends and some quantitative values are reproduced.[1,5] However, when some of the procedures involved in these calculations are scrutinized in simpler situations as discussed below, they may show inconsistencies and deficiencies which make any proclaimed "microscopic derivation" of these parameters suspect. This paper can therefore be viewed as a contribution towards a program where one knows at various stages in the transition from microscopy to phenomenology why certain procedures and/or approximations are justified and can be expected or trusted to work.

The paper is organized as follows. In Sec. II we consider the general problem of constructing Marumori images in a truncated subspace. What clearly emerges is the general incorrectness of simply multiplying ideal space images in contrast to constructing the image of the associated product in the original space. Section III contains, at first, a discussion and comparison of procedures by which the seniority images of single fermion operators can be constructed. We continue by addressing special problems connected to the structure of these images and the legitimate ways in which they can be used. Section IV finally contains an assessment of some constructions in microscopic investigations of the IBFM and what will be required of further investigations along similar lines.

II. GENERAL PROBLEM

Microscopic approaches to the IBFM are mostly formulated in the framework of the Otsaka-Arima-Iachello (OAI) method[8] which, in the language of boson mappings, can be termed a Marumori mapping,[9–10] more specifically, a truncated Marumori-type mapping. In this formalism the initial focus is on the correspondence between fermion *states* and ideal space *states*, the latter comprised of *independent* bosons and ideal fermions. Ideal space operators (which in the above formalism are, in principle, all infinite series) are subsequently determined by equating matrix elements in the original and ideal spaces.

The construction of general operators in the IBFM usually proceeds by appropriately coupling ideal space images of single fermion operators.[1] Before commenting on this seemingly innocent step, it is useful to look at the general question of constructing ideal space operators in the OAI or truncated Marumori method.

To be specific we consider the original single-j-shell OAI mapping[8] and enquire about the validity of constructing ideal space operators (such as the Hamiltonian) in this formalism as the product of images of the individual "factorized" operators, i.e., we enquire whether

$$(\Theta_1\Theta_2)_{\rm OAI}=(\Theta_1)_{\rm OAI}(\Theta_2)_{\rm OAI}\ ,$$

where the OAI operators contain, by definition, only s- and d-boson operators. (For the discussion and remarks that follow we mostly have in mind that Θ_1 and Θ_2 are bifermion operators. In the case of single fermion operators the general line of argument remains valid, but additional considerations have to be taken into account as discussed in Sec. III.)

With the OAI picture in mind, let $|\Psi_{SD}\rangle$ denote a fermion state in the SD subspace and $|\Psi_Q\rangle$ a state in the orthogonal complement. Correspondingly, let $|\Psi_{sd}\rangle$ and $|\Psi_q)$ denote the associated ideal space states. The OAI boson image $(\Theta_1\Theta_2)_{\rm OAI}$ of the product of fermion operators is then determined through the requirement

$$(\Psi'_{sd}|(\Theta_1\Theta_2)_{\rm OAI}|\Psi_{sd})=\langle\Psi'_{SD}|\Theta_1\Theta_2|\Psi_{SD}\rangle \qquad (1)$$

which, in principle, determines the coefficients in an infinite series ansatz for $(\Theta_1\Theta_2)_{\rm OAI}$. The OAI images of Θ_1 and Θ_2 are determined, analogously to Eq. (1), through

$$(\Psi'_{sd}|(\Theta_1)_{\rm OAI}|\Psi_{sd})=\langle\Psi'_{SD}|\Theta_1|\Psi_{SD}\rangle \qquad (2)$$

with a similar expression for Θ_2.

Without at all addressing questions concerning the degree to which the SD subspace is indeed decoupled for a chosen Hamiltonian or the range of validity of the so-called zero-order approximation (a specific truncation in the sd-truncated OAI series ansatz[8]), we consider the most favorable situation where the SD subspace is indeed decoupled. Taking the form of the Hamiltonian to be the product $\Theta_1\Theta_2$ (a sum of products is a simple extension), this means that, in general, $\langle\Psi'_{SD}|\Theta_1\Theta_2|\Psi_{SD}\rangle\neq 0$, whereas

$$\langle\Psi_Q|\Theta_1\Theta_2|\Psi_{SD}\rangle=0=\langle\Psi'_{SD}|\Theta_1\Theta_2|\Psi'_Q\rangle\ .$$

It is clear that the decoupling conditions above dictate that the *complete* Marumori image $(\Theta_1\Theta_2)_B$ will contain terms comprised of s and d bosons only, implying $(\Theta_1\Theta_2)_B=(\Theta_1\Theta_2)_{\rm OAI}$. However, the same equality need not necessarily hold for the complete Marumori and OAI images of the *individual* operators Θ_1 and Θ_2, since the decoupling conditions above do not necessarily imply that $\langle\Psi_Q|\Theta|\Psi_{SD}\rangle=0$, where Θ can be either Θ_1 or Θ_2. $(\Theta_1)_B$ and $(\Theta_2)_B$ may therefore contain terms which couple sd states to states from the orthogonal complement. In products of such terms the non-sd bosons could sometimes be contracted leaving sd contributions which are effectively contained in $(\Theta_1\Theta_2)_{\rm OAI}$ but obviously absent from the product $(\Theta_1)_{\rm OAI}(\Theta_2)_{\rm OAI}$ defined by the equality (2) and its counterpart for Θ_2.

To be even more explicit, consider

$$\begin{aligned}(\Psi'_{sd}|(\Theta_1\Theta_2)_{\rm OAI}|\Psi_{sd})&\equiv\langle\Psi'_{SD}|\Theta_1\Theta_2|\Psi_{SD}\rangle\\&=\sum_{\Psi''_{SD}}\langle\Psi'_{SD}|\Theta_1|\Psi''_{SD}\rangle\langle\Psi''_{SD}|\Theta_2|\Psi_{SD}\rangle+\sum_{\Psi_Q}\langle\Psi'_{SD}|\Theta_1|\Psi_Q\rangle\langle\Psi_Q|\Theta_2|\Psi_{SD}\rangle\end{aligned} \qquad (3)$$

and

$$(\Psi'_{sd}|(\Theta_1)_{\rm OAI}(\Theta_2)_{\rm OAI}|\Psi_{sd}\rangle=\sum_{\Psi''_{sd}}(\Psi'_{sd}|(\Theta_1)_{\rm OAI}|\Psi''_{sd})(\Psi''_{sd}|(\Theta_2)_{\rm OAI}|\Psi_{sd})+\sum_{\Psi_q}(\Psi'_{sd}|(\Theta_1)_{\rm OAI}|\Psi_q)(\Psi_q|(\Theta_2)_{\rm OAI}|\Psi_{sd})\ . \qquad (4)$$

Since the second sum in Eq. (4) is zero by definition (OAI images contain only s and d bosons), it is clear that the operator equality

$$(\Theta_1\Theta_2)_{\rm OAI}=(\Theta_1)_{\rm OAI}(\Theta_2)_{\rm OAI}$$

can only hold if either Θ_1 or Θ_2 does not couple SD states with states from the orthogonal complement. Alternatively one could have the equality

$$(\Theta_1\Theta_2)_{\rm OAI}=[(\Theta_1)_B(\Theta_2)_B]_{sd\ \rm trunc}$$

by first retaining those terms in $(\Theta_1)_B$ and $(\Theta_2)_B$ which connect states $|\Psi_{sd})$ with states $|\Psi_q)$ (which typically contain g bosons or other non-sd bosons), and truncate to s and d bosons *only after* all the appropriate contractions had been performed.

As an illustrative example, consider the case of a quadrupole-quadrupole interaction which has often been discussed in this context in the literature. In Ref. 11 Faessler and Morrison noted from numerical studies that spectra obtained from the boson Hamiltonian $H_B=-Q_{\rm OAI}\cdot Q_{\rm OAI}$ compared very unfavorably with those obtained from the exact shell-model calculations for $H=-Q\cdot Q$. While this might still have reflected the inappropriateness of truncating to the SD subspace, a study of the $j=\frac{13}{2}$ shell by Halse[12] showed that this subspace indeed decoupled to a large extent and showed that $H_B=-(Q\cdot Q)_{\rm OAI}$ yielded much better results in this particular case.

From our general discussion above, it should now be clear that the previously noted disparity[11,12] is simply a reflection of the fact that the complete Marumori boson image Q_B of the quadrupole operator Q contains (schematically) terms such as, e.g., $(d^\dagger\tilde{g}+g^\dagger\tilde{d})$ and $(d^\dagger d^\dagger\tilde{d}\tilde{g}+g^\dagger d^\dagger\tilde{d}\tilde{d})$ which, in the product $Q_B\cdot Q_B$, could be contracted to terms containing only d bosons. As already stated in general terms, in this specific case these contributions are effectively contained in $(Q\cdot Q)_{\rm OAI}$ but are absent from $Q_{\rm OAI}\cdot Q_{\rm OAI}$.

In order to construct the OAI (or other truncated) image of a product operator in the ideal space from the images of the individual operators that constitute the product, it is therefore not sufficient that the original product leave some subspace invariant, but the same should be true for the individual operators themselves. This condition seems to be violated especially in the case where the individual operators are single fermion operators.

Before turning to a discussion of the IBFM where the above (erroneous) practice has most often been used or implied, it should be pointed out that this procedure was, in fact, never used nor advocated in the original paper by Otsuka, Arima, and Iachello[8] which deals with even-mass *sd* systems.

The above discussion would also be incomplete if we did not point out that if Θ_1 and Θ_2 above act in distinctly separate parts of the original fermion space, then, to the extent that the OAI method is applicable at all, it would be perfectly in order to construct the image of a product of operators as the product of the images. If, e.g., Θ_1 is the proton quadrupole operator $Q_\pi^{2\mu}$ and Θ_2 the corresponding neutron operator $Q_\nu^{2\mu}$, then the ideal space image of $Q_\pi^2\cdot Q_\nu^2$ could simply be taken as $(Q_\pi^2)_{\rm OAI}\cdot(Q_\nu^2)_{\rm OAI}$, the only remaining proviso being that it should be not necessary to introduce proton-neutron bosons as are required in extensions of the IBM to lighter nuclei.[2]

III. SENIORITY IMAGES OF SINGLE FERMION OPERATORS

A. Various mapping procedures

The images of single fermion operators in the context of the IBFM were first introduced by enlarging the ideal space to include independent ideal fermion degrees freedom in addition to the boson degrees of freedom of the IBM and by subsequently equating matrix elements in OAI fashion. (A recent review with further references is given by Scholten.[1])

For a single-j shell the results of Ref. 1 reduce to

$$\begin{aligned} c^{jm} &\to A^{jm} \\ &= \left[\frac{\Omega-N}{\Omega}\right]^{1/2} a^{jm}+\frac{1}{\sqrt{\Omega}}s^\dagger\tilde{a}_{jm} \\ &\quad+\left[\frac{5(\Omega-N)}{\Omega(\Omega-1)}\right]^{1/2}[d^\dagger\tilde{a}_j]^j_m \\ &\quad-\left[\frac{5}{\Omega(\Omega-1)}\right]^{1/2}s^\dagger[a^j\tilde{d}]^j_m\ , \end{aligned}\tag{5}$$

$$c_{jm}\to A_{jm}\equiv(A^{jm})^\dagger\ . \tag{6}$$

Here c^{jm} (c_{jm}) is a creation (annihilation) operator in the original fermion space and similarly the a's are the ideal fermion operators which, in addition to satisfying the standard fermion algebra, are defined to commute with all the boson operators.[1]

Before we compare these results with those obtained from a Dyson mapping and subsequently similarity transformation, it seems in order to comment on the reported construction of these operators within the framework of nuclear field theory (NFT).[13,14] First it should be noted that the NFT "derivations" are, in fact, based on a procedure where the original NFT rules are complemented with a further set of empirical rules "still lacking a first-principle derivation."[13] The typical boson number dependent factors introduced in this empirical way are similar to the factors which naturally arise in the Dyson mapping framework when a seniority-type similarity transformation is applied to the original pair boson images.[15,16] The *ad hoc* factors introduced in the NFT construction therefore clearly reflect some of the implications of transforming to a seniority basis.

However it is not at all clear that the "global" application[13,14] of the NFT empirical rules conform to such a basis transformation and NFT derivations of ideal space operators therefore seem to be in want of a more careful consideration of the implications of a change from a pair boson to seniority boson basis. This could account for the differences between the seniority images of, e.g., single fermion operators obtained with the NFT prescription[14] and those obtained either with the OAI method[1] or the equivalent results obtained from the formalism discussed below. It is not clear though whether such a remedy alone will suffer to bring all NFT results in line with the standard ones.

The Dyson-type mapping of single fermion operators utilized below was first introduced by Okubo[17] and later generalized by Geyer and Hahne[18,19] and Marshalek.[20] (A recent extensive discussion is given by Klein and Marshalek.[21]) Anticipating our application to a single-j shell, this nonunitary mapping is written as (summation convention implied)

$$\begin{aligned} c^{jm} &\to A^{jm} \\ &=(a^{jm}-a^{jm_1}B^{jmjm_2}B_{jm_1jm_2})Q+a_{jm_1}B^{jmjm_1}\ , \end{aligned}\tag{7}$$

$$\begin{aligned} c_{jm} &\to A_{jm} \\ &=a_{jm}+a^{jm_1}B_{jmjm_1}Q\ . \end{aligned}\tag{8}$$

The fermion operators are the same as those appearing in Eqs. (5) and (6) and the antisymmetric boson operators $B^{jmjm'}=-B^{jm'jm}$ are subsequently coupled to good angular momentum

$$B^{jmjm'}=\sqrt{2}\sum_{JM}\langle jmjm'|JM\rangle B^{JM}\ , \tag{9}$$

with a similar expression for the annihilation operator. The boson operators B_{JM} and $B^{J'M'}$ satisfy the standard boson algebra and the notation $s^\dagger\equiv B^{00}$, $d^\mu\equiv B^{2\mu}$, etc., is used below.

The operator Q above is the projector onto states with no ideal fermions and, in terms of the ideal fermion number operator $n_a=a^{jm}a_{jm}$, it is defined by

$$Q=1-a^{jm}(n_a+1)^{-1}a_{jm} \tag{10}$$

with the stated property

$$Q|\Psi\rangle=\begin{cases}|\Psi\rangle & \text{if } n_a|\Psi\rangle=0\\ 0 & \text{if } n_a|\Psi\rangle\neq 0\ .\end{cases} \tag{11}$$

The projector Q is needed in the mapping (7) and (8) as a result of the definition of the physical subspace which contains states with an arbitrary number of bosons but with one ideal fermion at most. (See Refs. 18–21 for a further discussion.)

In order to construct the seniority images of the operators c^{jm} and c_{jm} we now use the formalism developed in Refs. 15 and 16 where it was shown how a similarity transformation could be constructed which would induce the desired ideal space association between (fermion) seniority and the number of non-s ideal space bosons plus ideal fermions, namely[15] $v=2\sum' n_{2J}+n_a$. Full details about this transformation can be found in the paper by de Kock and Geyer[16] and here we simply give the results when the similarity transformation [see Eqs. (17)–(23) of Ref. 16] is applied to the Dyson images (7) and (8) of single fermion operators. Retaining only terms of the same order as those considered in expressions (5) and (6), we find

$$c^{jm} \to \Gamma^{jm} = \left[\frac{\Omega-N+n_d}{\Omega}\right] a^{jm}Q + \frac{1}{\sqrt{\Omega}} s^\dagger \tilde{a}_{jm} - \frac{\sqrt{5}}{\Omega}[d^\dagger a^j]^j_m sQ + \sqrt{5/\Omega}\left[\frac{\Omega+1-N-n_d-n_a}{\Omega+1-2n_d-n_a}\right][d^\dagger \tilde{a}_j]^j_m - \frac{\sqrt{5}}{\Omega} s^\dagger [a^j \tilde{d}]^j_m Q \ , \tag{12}$$

$$\tilde{c}_{jm} \to \tilde{\gamma}_{jm} = \tilde{a}_{jm} - \frac{1}{\sqrt{\Omega}} a^{jm} sQ - \left[\frac{\sqrt{5}}{\Omega+1-2n_d-n_a}\right][d^\dagger \tilde{a}_j]^j_m s - \sqrt{5/\Omega}[a^j \tilde{d}]^j_m Q \ . \tag{13}$$

B. Products of images—specific problems

We now turn to a discussion of how and with what limitations the various single fermion images considered in Sec. III A may be utilized. The first observation is that in Γ^{jm} and $\tilde{\gamma}_{jm}$ [Eqs. (12) and (13) above], a^{jm} is always accompanied by the projector Q. The projector can only be ignored [as in Eqs. (5) and (6)] if the terms involved are consistently allowed to act only on states with no ideal fermion. (See, however, the discussion below about possible generalizations without this restriction.)

Whereas Γ^{jm} contains a term on the type $[d^\dagger a^j]^j_m s$, $\tilde{\gamma}_{jm}$ does not contain the Hermitian conjugate $s^\dagger[\tilde{a}_j \tilde{d}]^j_m$. The appearance of $[d^\dagger a^j]^j_m s$ in Γ^{jm} seems to be in conflict with the seniority changing property $|\Delta v|=1$ of c^{jm}. (In the OAI image of c^{jm} the term $[d^\dagger a^j]^j_m s$ is excluded by choice from the very beginning[22].) However, one should remember that in the actual calculation of matrix elements of Dyson images (before and after the similarity transformation), a product of matrix elements of an operator image and the image of the conjugate operator is always involved (see Refs. 15 and 23 for details). In the present case the term $[d^\dagger a^j]^j_m s$ will therefore never contribute to a given matrix element because of the absence of $s^\dagger[\tilde{a}_j \tilde{d}]^j_m$ in $\tilde{\gamma}_{jm}$. In fact, in Ref. 15 it was argued that whenever the Dyson (seniority) image of an operator contained only terms that could couple different ideal space states, it is always possible to write down a Hermitian equivalent operator for which a matrix element could be calculated in the standard way. Since this is indeed just the case with Γ^{jm} and $\tilde{\gamma}_{jm}$, we can write down the Hermitian equivalent operators

$$c^{jm} \to \alpha^{jm} = \left[\frac{\Omega-N+n_d}{\Omega}\right]^{1/2} a^{jm} + \left[\frac{5}{\Omega}\left[\frac{\Omega+1-N-n_d-n_a}{\Omega+1-2n_d-n_a}\right]\right]^{1/2}[d^\dagger \tilde{a}_j]^j_m + \frac{1}{\sqrt{\Omega}} s^\dagger \tilde{a}_{jm} - \left[\frac{5}{\Omega(\Omega-2n_d-n_a)}\right]^{1/2} s^\dagger [a^j \tilde{d}]^j_m \ , \tag{14}$$

$$\tilde{c}_{jm} \to \tilde{\alpha}_{jm} = (-1)^{j-m}(\alpha^{j-m})^\dagger \ . \tag{15}$$

When compared to Eqs. (5) and (6) we see that for the lowest seniority matrix elements used to determine the coefficients in these equations, the above coefficients reduce to the ones given in Eqs. (5) and (6).

If α^{jm} and $\tilde{\alpha}_{jm}$ are written without the projector Q, as above, they retain their validity only when they are used consistently as already discussed. This also implies that one is *not* allowed to use expressions (14) and (15) [or, equivalently, (5) and (6)] to construct ideal space product operators containing two ideal fermions operators, such as would, e.g., be the case if one only simply mapped $c^\dagger c^\dagger cc$ onto $\alpha^\dagger \alpha^\dagger \alpha\alpha$ and retained, in the latter, product terms of the type $s^\dagger aa$, $d^\dagger aa$, or their Hermitian conjugates. [Apart from the inconsistency related to the neglect of Q, the inconsistency related to the neglect of contracted terms (as discussed in Sec. II) is also introduced in a product such as $\alpha^\dagger\alpha^\dagger\alpha\alpha$.]

The additional care that should be exercised when contemplating products of α's has its origin in the definition of ideal space states. In principle, one could, of course, visualize an ideal space with states containing collective bosons [s and d bosons only (say)] together with an arbitrary number of ideal fermions. For a simple SU(2) model this has, in fact, been done,[19,24] with the result that for the single fermion images one then finds [see Eqs. (4.1)–(4.5) of Ref. 24 with an appropriate change of notation]

$$c^{jm} \to \Lambda^{jm} = \beta^{jm}\left[\frac{\Omega-n_s-n_a}{\Omega-n_a}\right] + \frac{1}{\sqrt{\Omega}} s^\dagger \tilde{a}_{jm} \ , \tag{16}$$

$$\tilde{c}_{jm} \to \tilde{\lambda}_{jm} = \tilde{a}_{jm} + \sqrt{\Omega} s \beta^{jm} (\Omega - n_a)^{-1} , \tag{17}$$

with

$$\beta^{jm} = a^{jm} - (a^j \cdot a^j)(\Omega - n_a)^{-1} \tilde{a}_{jm} . \tag{18}$$

On comparison with expressions (12) and (13) one notes that while the typical seniority-type factors are common to the two sets of operators, no explicit projection operator Q is present in expressions (16) and (17) anymore. Instead in its place a suitably weighted component of the equivalent of a collective ideal fermion pair is removed whenever ideal space fermions are created, as is evident from the definition (18) of β^{jm}.

To appreciate the crucial observation to be made from this comparison we recall an important point from the Dyson boson mapping formalism. For the mapping of bifermion operators it is namely possible to find (without loss of generality) the ideal space images in terms of the collective bosons dictated by the fermion structure by first performing the generalized Dyson mapping followed by a truncation to the collective bosons. (See, e.g., the recent paper by Kim and Vincent[25] for a thorough discussion of this point.) For the mapping of single fermion operators the situation is quite different and even for situations where collectivity is completely prescribed by the algebraic structure (as, e.g., in the Ginocchio model[26]) the same truncation procedure fails to yield the proper single fermion images. Above this can also be seen from inspection of the first terms in Eqs. (12) and (16). The truncated version [when restricted to SU(2) where only s bosons are involved] has, apart from the seniority factor, the term $a^{jm}Q$ whereas the complete image correspondingly has β^{jm} and the two operators will give equivalent results only when operating on states with no ideal fermion.

What has been elaborated and placed in context above is that it is only the images (16) and (17) [or their appropriate, but still unknown, generalizations to $SO(2(2j+1))$] which could, in principle, be used to construct arbitrary ideal space products (although even then one has to distinguish carefully between operator identities and operator equalities which only hold in the physical subspace, as well as keep in mind possible contributions from contractions as in Sec. II). The images (12) and (13) do not constitute truncated versions of the above described generalizations and, in contrast, are only valid for the limited roles for which they had been constructed, namely the calculation of single-particle matrix elements in an ideal space where states are limited to one ideal fermion at most. Uninhibited multiplication and coupling of these images to obtain multiparticle operators in the ideal space, in particular, falls outside the range of validity for which they had been constructed.

In Ref. 27 two-quasiparticle degrees of freedom were, in fact, introduced by simply coupling the images of single fermion operators [the generalizations of expressions (5) and (6) to a multi-j-shell situation] appearing in a general one-body plus two-body Hamiltonian. To this extent the calculation[27] of strength coefficients of terms where a d boson and two quasiparticles are coupled must remain suspect in view of our discussion above. However, we would like to comment further on the handling[27] of terms where an s boson couples to two quasiparticles. In Ref. 27 it was argued that if the coefficients u and v used there were chosen to be Bardeen-Cooper-Schrieffer (BCS) solutions, then the coefficient of this particular coupling would vanish in the image (obtained from the coupling of single fermion images) of the pure pairing part of the Hamiltonian. This state of affairs is indeed also reflected when the images (16) and (17) [which are exact but limited to cases where only s bosons are (and need to be) introduced] are used to construct the image of the pairing Hamiltonian, namely $\Lambda^j \cdot \Lambda^j \tilde{\lambda}_j \cdot \tilde{\lambda}_j$, will only contain $s^\dagger s$ and $s^\dagger s^\dagger ss$ terms (no $s^\dagger aa$ or $a^\dagger a^\dagger s$ terms) when it is operative in the physical subspace. (See Ref. 19 for a discussion of this point.) The above connection can also be understood from the point of view that expressions (16) and (17) represent a quantized form of the Bogoliubov transformation (see also Ref. 28) in the sense that the "transformation coefficients" satisfy

$$1 \cdot \left[\frac{\Omega - n_s - n_a}{\Omega - n_a} \right] + \frac{1}{\sqrt{\Omega}} s^\dagger \sqrt{\Omega} s (\Omega - n_a)^{-1} = 1 . \tag{19}$$

One has to keep in mind, however, that β^{jm} is a modified creation operator and that it is the conjugate of a_{jm} only in the subspace with no ideal fermions.

From these observations it follows that to the extent that BCS coefficients, combined with a static ground-state approximation, approximate the operator "coefficients" in expressions (16) and (17), the "uninhibited multiplication" of the multi-j extensions of the single fermion images (5) and (6) to determine the coupling of an s boson to two quasiparticles can be justified, although the approximation will get worse for excited states.

Having vindicated the handling of $s^\dagger aa$ and $a^\dagger a^\dagger s$ terms in Ref. 27, it should be repeated that a similar determination[1,27] of the other boson-two-quasiparticle coupling strengths falls outside the range of validity of the formalism. To calculate these coefficients with the same level of microscopic support that was demonstrated for the $s^\dagger aa$ and $a^\dagger a^\dagger s$ coupling, one will first have to await an extension of the formalism either on the OAI level or on the level of the Dyson mapping plus seniority transformation[15,16] which properly includes all implications of an enlarged boson-plus-two-quasiparticle space. An OAI extension will, e.g., require the enlargement of the space used for equating matrix elements to include orthogonal states with at least two noncollective fermions in addition to S and D pairs.

The objections raised above also apply to the construction of the ideal space quadrupole operator by simply coupling[1,22] $[A^j, \tilde{A}_j]^2_\mu$ in Eqs. (5) and (6) or $[\Gamma^j \tilde{\gamma}_j]^2_\mu$ in Eqs. (12) and (13). That this (invalid) procedure leads to an operator which differs from the one obtained by equating matrix elements of the complete quadrupole operator was also realized in Ref. 29 where the difference was attributed to an "(improper) handling of intermediate states," as was also discussed in general in Sec. II above.

We illustrate the difference by showing the two non-Hermitian operators obtained first by applying the seniority transformation to the complete Dyson image of the quadrupole operator and second by coupling $[\Gamma^j \tilde{\gamma}_j]^2_\mu$ from Eqs. (12) and (13). We choose to illustrate a difference on this level because writing down Hermitian equivalent operators would, in both cases, require unavoidable further approximations, as mentioned in Ref. 29 and discussed in more detail in Ref. 15.

The expression obtained by transforming the complete quadrupole operator [see Eq. (35) of Ref. 16] is

$$
\begin{aligned}
(Q^{2\mu})_{\text{sen}} = & -\sqrt{2/\Omega}\left[s^\dagger \tilde{d}_\mu + d^\mu s \left[\frac{\Omega - N - n_d - n_a}{\Omega - 1 - 2n_d - n_a} \right] \right] + 10 \begin{Bmatrix} 2 & 2 & 2 \\ j & j & j \end{Bmatrix} \left[\frac{\Omega - 2N - n_a}{\Omega - 2n_d - n_a} \right] [d^\dagger \tilde{d}]^2_\mu \\
& + \left[\frac{\Omega - 2N - n_a}{\Omega - 2n_d - n_a} \right] [a^j \tilde{a}_j]^2_\mu + \frac{2(\Omega - 2N - n_a)}{(\Omega + 2 - 2n_d - n_a)(\Omega - 2n_d - n_a)} \sum_L \frac{1}{\sqrt{5}} \hat{L} [[d^\dagger \tilde{d}]^L [a^j \tilde{a}_j]^2]^2_\mu \\
& + \frac{2\sqrt{10\Omega}}{\Omega + 2 - 2n_d - n_a} \sum_J{}' (-1)^J \hat{J} \begin{Bmatrix} 2 & 2 & J \\ j & j & j \end{Bmatrix} [[d^\dagger s]^2 [a^j \tilde{a}_j]^J]^2_\mu ,
\end{aligned} \tag{20}
$$

while the one which results from $[\Gamma^j \tilde{\gamma}_j]^2_\mu$, where $Qa^{jm} = 0$ has consistently been taken into account, is given by

$$
\begin{aligned}
(Q^{2\mu})_{\text{coup}} = & -\sqrt{2/\Omega}\left[s^\dagger \tilde{d}_\mu + d^\mu s \left[\frac{\Omega - N - n_d - n_a}{\Omega - 1 - 2n_d - n_a} \right] \right] + 10 \begin{Bmatrix} 2 & 2 & 2 \\ j & j & j \end{Bmatrix} \left[\frac{\Omega + 1 - N - n_d - n_a}{\Omega + 1 - 2n_d - n_a} \right] [d^\dagger \tilde{d}]^2_\mu \\
& + \sqrt{10/\Omega} \left[\frac{\Omega - N + n_d}{\Omega + 2 - 2n_d - n_a} \right] \sum_L (-1)^L \hat{L} \begin{Bmatrix} 2 & 2 & L \\ j & j & j \end{Bmatrix} [[d^\dagger s]^2 [a^j \tilde{a}_j]^L]^2_\mu \\
& + \sqrt{10/\Omega} \sum_L \hat{L} \begin{Bmatrix} 2 & 2 & L \\ j & j & j \end{Bmatrix} [([d^\dagger s]^2 + [s^\dagger \tilde{d}]^2)[a^j \tilde{a}_j]^L]^2_\mu \\
& + \frac{10 n_s}{\Omega - 2n_a - n_a} \sum_{KL} (-1)^K \hat{K} \hat{L} \begin{Bmatrix} 2 & j & j \\ 2 & j & j \\ K & L & 2 \end{Bmatrix} [[d^\dagger \tilde{d}]^K [a^j \tilde{a}_j]^L]^2_\mu \\
& + \left[\frac{\Omega - N + n_d}{\Omega} - \frac{10 n_s}{\Omega - 2n_d - n_a} \begin{Bmatrix} j & j & 2 \\ j & j & 2 \end{Bmatrix} \right] [a^j \tilde{a}_j]^2_\mu .
\end{aligned} \tag{21}
$$

It is clear that even with the proper inclusion of the projector Q in obtaining expression (21), detailed differences in the respective coefficients of the two images remain, as is, of course, to be expected from the general discussion in Sec. II. The different coefficients of $[d^\dagger \tilde{d}]^2_\mu$ ($-2N$ vs $-N$ in the numerator) is the same as that which one gets from a comparison of the original OAI quadrupole operator in Ref. 8 and the simple coupling of the images (14) and (15) to obtain an image of the quadrupole operator. [Note that Eq. (3.17) in Ref. 1 indeed yields the correct numerator, $-2N$, in the single-j limit. This, however, requires an *explicit symmetrization* on the fermion level—schematically, $c^\dagger c = \frac{1}{2}(c^\dagger c - cc^\dagger + 1)$—*before* inserting these images. This *ad hoc* procedure, however, does not seem to be very satisfactory, since it is *a priori* unclear how it should be generalized for arbitrary products of c's.] Further comparison of differences for typical values of Ω and N shows that individual coefficients can differ from 10–20% to factors of 2–3. It is difficult to make general statements about the influence this will have when spectra and transition rates are calculated from these respective "microscopically determined" operators. The results given, e.g., by Halse[12] illustrate that the combined effect of differences for a number of coefficients can be quite large, accounting, e.g., to a large extent for the factor of 10 difference in level spacing originally observed in Ref. 11.

IV. ASSESSMENT AND CONCLUSIONS

Our presentation has a twofold purpose. In the first place it shows how seniority images of single fermion operators can be constructed from the Dyson mapping of such operators followed by an appropriate similarity transformation. The equivalent unitary mapping [Eqs. (14) and (15)] reduces to expressions already obtained for the lowest seniority states with the OAI method. The Dyson mapping procedure, however, explicitly illuminates the proviso that these operators may not act unrestrictedly on states already containing ideal fermions. (As discussed in Sec. III it is not quite clear whether the different expressions obtained with the nuclear field theory prescriptions can only be attributed to an inadequacy of the "empirical rules" adopted in this approach.)

In the second place we focus on the general validity of obtaining the ideal space image of a product of fermion operators simply as the product of the individual images. It is shown that if these individual Marumori images are determined only in a prechosen subspace, as is the practice in the OAI method, the two constructions (image of the product and product of the images) are generally quite different and that it is incorrect to construct ideal space operators as products of already constructed individual images.

Furthermore, by drawing on insight gained from the Dyson boson mapping formalism, specific limitations connected to the construction of single fermion images are pointed out. In particular, it is inappropriate to use existing images of single fermion operators to construct either the IBFM quadruple operator of operators for a two-quasiparticle extension of IBFM. Although single fermion images which, in principle, could be used for the construction of operators valid for states with an arbitrary number of ideal fermions are known for SU(2) models, their generalization is still wanting.

Finally, even the less ambitious undertaking of properly generalizing the IBFM to contain states with, at most, two quasiparticle degrees of freedom, together with a set of consistent operators, will first require an extension of the OAI basis states to include the set of orthogonal states in which a boson is replaced by two quasiparticles.

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