Two-level bosonic model that simulates the transition from a superconductive condensate to an alpha cluster condensate

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A simple bosonic model that has two different phases, one corresponding to a condensate of bosons, a situation that resembles spherical nuclei, and a second one of a condensate of pairs of bosons that corresponds to nuclei where the ground state is a condensate of alphalike clusters, is solved in several different approximate treatments and exactly. Some differences due to the finite number of bosons from the known results for large boson systems are obtained.

I. INTRODUCTION

The spectra of medium and heavy nuclei have frequently been described in terms of elementary modes of excitation involving clusters of nucleons.[1] Pairs of nucleons coupled to $J=0$, $T=1$ have been used, for instance, in nuclear superfluidity[2] as well as in pairing vibrations.[3] Excitation modes with a larger number of particles were introduced within the quartet model[4] some years ago. Within this picture the building blocks are α clusters. The reason for this choice was clearly suggested by the large stability of the α particles. Models of this type have been applied since the early years of nuclear physics.[5] For light nuclei where $N \simeq Z$ the α cluster has shown a reasonable success.[6] There is also, in connection with nuclear masses,[7-10] good evidence throughout the periodic table, suggesting a quartet picture.

It has been shown[11] that there may be a competition between condensates of pairs of like particles (i.e., a superconductive description of nuclei) and a state formed by a condensate of alphalike clusters (that may be associated with deformed states). It has also been shown that not only $T=1$, but also $T=0$ interacting correlated pairs would be needed to describe these alphalike clusters. For this one must include[12] R different two-particle "collective" excitations, where R is a number of the same order of magnitude than the number of active particles. In this way the four-body excitations are constructed as a coherent mixture of those pairs of two-body collective excitations. The quantum numbers of the four-body excitation that we are considering are similar to those of the alpha particle (i.e., $J=T=0$). In this case each collective pair will have only one partner when building the coherent state.

It is worthwhile to remark that in zero order of $1/\Omega$, where Ω is the number of effective particle-particle states, the collective pair can be treated as bosons because the Pauli corrections are of order $1/\Omega$.[13] A simple model that displays this competition can be provided by a bosonic model, where two levels have energies equal to $-D/2$ while the remaining ones, $2R$, have single boson energies $D/2$. The two levels with energy $-D/2$ simulate the proton and neutron pairing bosons that are used in the usual description of pairing vibrations. The condensate of this type of bosons can be described as a superconductive system. The $2R$ levels with energy $D/2$ simulate the two-particle excitations formed by a proton (p) and a neutron (n). If these p-n excitations interact via a pairinglike residual Hamiltonian, one obtains collective pairs of p-n pairs (that may have the same quantum numbers as alpha particles). In Ref. 11 it was shown that this type of residual Hamiltonian has some resemblance with the effective interaction for the ^{212}Po nucleus. A similar study, obtaining the same type of results was made in Ref. 14 for ^{204}Hg. This simple version has the advantage of displaying in a simple way the existence of a collective coherent state. The main effects that we want to point out are just related to the mere existence of this coherent state and therefore its structure seems to be not very important.

The main purpose of the present paper is to explore along this line the origin of the four-body correlations in heavy nuclei. We study the very simple model just discussed where the pairs of particles are replaced by bosons. We introduce in Sec. II a simple model for bosons that mock up many of the features that seem to be present in real nuclei, a method to solve it exactly and we also introduce the Holstein-Primakoff transformation between bosons that allows us to connect this simple model with one that has appeared before in connection with four-body correlations. In Sec. III we compare the results obtained for the ground state using different approximate treatments of this simple model with those obtained by the exact calculation. In Sec. IV we make a similar treatment for the excited states and the conclusions are drawn in Sec. V.

A secondary purpose of the present paper is to study with this simple model the competition between particle and pair condensation in Bose systems when the number of particles is not too large.

II. FORMULATION OF THE MODEL AND EXACT TREATMENTS

We will consider a system formed by M bosons interacting via the Hamiltonian

$$H=\frac{D}{2}\left[\sum_{q=1}^{R}(B_q^\dagger B_q+B_{\bar{q}}^\dagger B_{\bar{q}})-B_o^\dagger B_o-B_{\bar{o}}^\dagger B_{\bar{o}}\right] -G\left[B_o^\dagger B_{\bar{o}}^\dagger\left[\sum_{q=1}^{R}B_q B_{\bar{q}}\right]+\text{H.c.}\right], \tag{2.1}$$

where $B_q^\dagger$ creates a boson with quantum numbers q. There are $2R$ types of different bosons that have the same single-particle energy $D/2$, while bosons created by $B_o^\dagger$ and $B_{\bar{o}}^\dagger$ have single-particle energy $-D/2$. A boson of the type $B_q^\dagger$ is always created or destroyed by the Hamiltonian together with its conjugate boson $B_{\bar{q}}^\dagger$. This model is an attempt to mock up the competition between the pairinglike bosons (represented by $B_o^\dagger$, that creates a proton pairing phonon, and by $B_{\bar{o}}^\dagger$ that creates a neutron pairing phonon) and the alphalike excitations (represented by the coherent sum of pairs of conjugate bosons, where each of these bosons can be thought of as a p-n pair).

In order to solve exactly the Hamiltonian we can diagonalize it numerically. To simplify the evaluation of the matrix elements, it is convenient to introduce the concept of shell seniority, as in the fermionic case. If one considers for example the shell formed by the bosons of the type o and $\bar{o}$, one has a state of shell seniority s if (a) the state has s bosons of the type o or $\bar{o}$ and (b) the operator $B_o B_{\bar{o}}$ acting on the state cancels it. One can define in a similar way the seniority of the other shells. It will then be convenient to use the normalized states

$$|n,M,s,l\rangle=\left[\frac{(l+R-1)!}{n!(s+M-n)!(M-n)!l!(l+n+R-s)!}\right]^{1/2}(B_o^\dagger B_{\bar{o}}^\dagger)^{M-n}\left[\sum_{q=1}^{R}B_q^\dagger B_{\bar{q}}^\dagger\right]^n|s,l\rangle\ , \tag{2.2}$$

where s is the seniority associated with the shell o while l is the seniority associated with the shells q.

The matrix elements of the interacting part of the Hamiltonian can be written as

$$\langle n-1,M,s,l|H_{\text{int}}|n,M,s,l\rangle=-G\sqrt{n(R+l+n-1)(s+M-n+1)(M-n+1)}\ . \tag{2.3}$$

In zeroth order, when the interaction is negligible, the ground state of the system will be approximately described by a condensate of bosons formed by a mixture of $B_o^\dagger$ and $B_{\bar{o}}^\dagger$. If they represent, respectively, the proton and neutron pairing phonons the ground state will be given by a state of the form

$$B_o^{\dagger z/2}B_{\bar{o}}^{\dagger n/2}|0\rangle\ , \tag{2.4}$$

where z and n are the active number of protons and neutrons, respectively. For simplicity we will consider that $n=z$ in what follows.

When the splitting of the single-particle energies D is negligible compared with the interacting part of the Hamiltonian, the ground state will be described reasonably by a condensate of a coherent mixture of pairs of conjugate bosons. One will, therefore, have a phase transition as the interaction strength increases. Our main interest is to describe in a simple way this "phase transition," and the first thing to do is to find a parameter similar to an order parameter. As in the fermionic case, there are some operators that are important, such as

$$P_1^\dagger=B_o^\dagger B_{\bar{o}}^\dagger\ , \tag{2.5a}$$

$$P_2^\dagger=\frac{1}{\sqrt{R}}\sum_{q=1}^{R}B_q^\dagger B_{\bar{q}}^\dagger\ , \tag{2.5b}$$

and their matrix elements between ground states are good candidates to be the order parameters.

An important point when studying these simple models is the selection of the scaling parameters. Usually they are selected with the criterion that the matrix elements of the Hamiltonian are numbers of the order of 1 (such as $n/M, n/R$ or functions of these parameters as, for example, $[1+(n-1)/R]^{1/2}$). In our case we have two possible parameters, one related to the single-particle part of the Hamiltonian (MD) and the other one to the interaction part of the Hamiltonian ($GM\sqrt{RM}$). For reasons that will be clear when we study the approximate treatments, we choose as scaling parameter for the total energy of the system the one related to the interaction part of the Hamiltonian. From these two scaling parameters we can define a dimensionless parameter, $\xi=D/(G\sqrt{MR})$, that will allow us the scaling of the results as a function of the interaction. The scaling of the matrix elements of the operator (2.5) or the number of bosons operators is done with the same criterion, i.e., we extract the factors of M needed to make the matrix elements numbers of order 1.

If one performs a Holstein-Primakoff transformation, the matrix elements of the equivalent Hamiltonian are exactly the same ones obtained from the original Hamiltonian, and therefore one does not obtain any advantage by diagonalizing it numerically, but it will be very convenient in order to develop alternative approximate treatments. We therefore review in this section the essentials of the Holstein-Primakoff transformation (HPT).

The first step is to define a set of bosonic operators that satisfy

$$C_{ij}^\dagger=C_{ji}^\dagger\ , \tag{2.6a}$$

$$[C_{ij},C_{kl}^{\dagger}]=\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\ , \tag{2.6b}$$

$$[C_{ij}^{\dagger},C_{kl}^{\dagger}]=0\ . \tag{2.6c}$$

The vacuum state $|0\rangle_B$ of the new boson space is defined by

$$C_{ij}|0\rangle_B=0\ . \tag{2.6d}$$

The HP images of the bilineal forms in the original bosonic operators are obtained by requiring them to fulfill the commutation algebra of bibosonic operators:

$$(B_i^{\dagger}B_j)_{\rm HP}=\sum_k C_{ik}^{\dagger}C_{jk}=(C^{\dagger}C)_{ij}\ , \tag{2.7}$$

$$(B_i^{\dagger}B_j^{\dagger})_{\rm HP}=[C^{\dagger}(\mathbb{1}+C^{\dagger}C)^{1/2}]_{ij}\ , \tag{2.8a}$$

$$(B_iB_j)_{\rm HP}=[(\mathbb{1}+C^{\dagger}C')^{1/2}C]_{ij}\ . \tag{2.8b}$$

In Eqs. (2.8) we see the trademark of the HPT, the square-root operator.

In the present case it is convenient to make a HPT only for the bilineal operators $P_2^{\dagger}$ and

$$N_2=\sum_{q=1}^{R}B_q^{\dagger}B_q+B_{\bar{q}}^{\dagger}B_{\bar{q}}\ .$$

It is then convenient to define an operator $C^{\dagger}$ by the condition

$$C^{\dagger}=\tfrac{1}{2}\sum_{i,j}X_{ij}C_{ij}^{\dagger}\ . \tag{2.9}$$

The commutation relation associated with $C^{\dagger}$ yield the normalization condition

$$[C,C^{\dagger}]=\tfrac{1}{2}\sum_{ij}X_{ij}X_{ij}^{*}=1\ . \tag{2.10}$$

Defining $N_C=C^{\dagger}C$, one obtains for the HP image of $P_2^{\dagger}$ and N_2

$$(P_2^{\dagger})_{\rm HP}=C^{\dagger}\left[1+\frac{N_C}{R}\right]^{1/2}, \tag{2.11a}$$

$$(N_2)_{\rm HP}=2N_C\ . \tag{2.11b}$$

This expression allows us to write the HP image of the Hamiltonian as

$$(H_{\rm sp})_{\rm HP}=DN_C-\frac{D}{2}(B_o^{\dagger}B_o+B_{\bar{o}}^{\dagger}B_{\bar{o}})\ , \tag{2.12a}$$

$$(H_{\rm int})_{\rm HP}=-G\sqrt{R}\,(C^{\dagger}B_oB_{\bar{o}}\left[1+\frac{N_C}{R}\right]^{1/2}+\text{H.c.}\ . \tag{2.12b}$$

To diagonalize this Hamiltonian one can define as basis states forming the normalized basis

$$|n,K\rangle=\frac{1}{(K-n)!\sqrt{n!}}(B_o^{\dagger}B_{\bar{o}}^{\dagger})^{K-n}C^{\dagger n}|0\rangle \tag{2.13}$$

and the matrix elements of the nondiagonal part of the Hamiltonian can be written as

$$\langle n+1,K|H_{\rm int}|n,K\rangle=-G(K-n+1)\left[Rn\left[1+\frac{n-1}{R}\right]\right]^{1/2}. \tag{2.14}$$

It must be noted that the matrix elements of the Hamiltonian (2.12) given by (2.14) are the same as the one obtained from (2.3) when $s=l=0$.

III. APPROXIMATE TREATMENTS OF THE GROUND STATE AND COMPARISON WITH THE EXACT RESULTS

In this section we will study the ground state of bosonic systems in different types of approximations and compare the results obtained with the exact results. The first approximate method that will be used is the variational treatment of the HP image of the Hamiltonian, the second one will correspond to the equivalent of Bardeen-Cooper-Schrieffer (BCS) treatment for bosons, while the third one will be Hartree-Bose (HB) treatment of (2.1).

A. Variational treatment of the HP mapped Hamiltonian

To evaluate the mean value of the HP mapped Hamiltonian, it is convenient to define the condensate states

$$|\alpha M\rangle=(B_o^{\dagger}B_{\bar{o}}^{\dagger}+\alpha C^{\dagger})^M|0\rangle\ . \tag{3.1}$$

Their norm can be written as

$$N_{\alpha}(M)=(M!)^2\sum_{r=0}^{M}\frac{\alpha^{2r}}{r!} \tag{3.2}$$

and the expectation values of N_C and n_0

$$\langle\alpha M|n_o|\alpha M\rangle=(M!)^2\sum\frac{(M-r)}{r!}\alpha^{2r}=\left[M-\frac{\alpha}{2}\frac{\partial}{\partial\alpha}\right]N_{\alpha}(M)\ , \tag{3.3a}$$

$$\langle\alpha M|N_C|\alpha M\rangle=(M!)^2\sum\frac{r\alpha^{2r}}{r!}=\tfrac{1}{2}\alpha\frac{\partial}{\partial\alpha}N_{\alpha}(M)\ . \tag{3.3b}$$

In a similar way one can write for the expectation value of the interaction part of the mapped Hamiltonian

$$\langle\alpha M|B_o^{\dagger}B_{\bar{o}}^{\dagger}\left[1+\frac{N_C}{R}\right]^{1/2}C|\alpha M\rangle=(M!)^2\sum_{r=1}^{M}\frac{(M-r+1)}{(r-1)!}\left[1+\frac{r-1}{R}\right]^{1/2}\alpha^{2r-1}\ . \tag{3.4}$$

The energy can then be written as

$$E=\frac{\langle \alpha M|H|\alpha M\rangle}{\langle \alpha M|\alpha M\rangle}=\frac{2G\sqrt{RM}\sum\alpha^{2r-1}/r![(2r-M)\alpha\xi-(M-r+1)/\sqrt{M}\left[1+\frac{(n-1)}{R}\right]^{1/2}]}{\sum\frac{\alpha^{2r}}{r!}}, \tag{3.5}$$

while the minimization condition yields

$$\sum_{r=1}^{M}\frac{\alpha^{2r-1}}{(r-1)!}\left[(2r-M)\xi-\frac{E}{G\sqrt{MR}}-\frac{(M-r+1)(2r-1)}{\sqrt{M}\,\alpha}\left[1+\frac{r-1}{R}\right]^{1/2}\right]=0 . \tag{3.6}$$

This system of equations yields the value of α as well as the energy. This equation must be solved, in general, numerically. In order to obtain some insight on the meaning of its solutions it is convenient to look at Eqs. (3.4) and (3.5) in the limit $R\to\infty$ and M also going to infinity but considering that M/R is much smaller than one. This particular model is what was called the $\phi^\dagger AB$ model in Ref. 15.

In this case one can replace the sums by exponentials and powers, as for example,

$$\sum_{r=1}^{M}\frac{\alpha^{2r-1}}{(r-1)!}=\alpha\sum_{r=0}^{M-1}\frac{\alpha^{2r}}{r!}\simeq\alpha e^{\alpha^2} .$$

With this type of replacement the scaled energy of the system can be written as

$$E^N=\frac{E_{\text{g.s.}}}{GM\sqrt{MR}}=-\xi(1-2\eta^2)-2\eta(1-\eta^2) , \tag{3.7}$$

where $\eta=\alpha/\sqrt{M}$, while one obtains for the minimization condition

$$3\eta^2+2\xi\eta-1=0 . \tag{3.8}$$

The existence of this model was important to give a clue about how to choose the dimensionless interaction parameter ξ and the way in which energies have to be scaled.

The parameter η is a very natural order parameter. From its definition and (3.8) it follows that for $\xi\le -1$ one obtains $\eta=1$ (i.e., one only has pairs of bosons), while for $\xi>-1$ one obtains

$$\eta=-\frac{\xi}{3}+\frac{1}{3}\left[\xi^2+3\right]^{1/2} .$$

It is important to note that even for large values of ξ, η is non-negligible, and when ξ is very large (and positive) it can be approximately written as $\eta\simeq 1/2\xi$.

B. BCS-type treatment

In the BCS treatment one defines an operator that creates a pair of bosons as

$$\Gamma_o^\dagger\sum_{p=0}^{R}X_pB_p^\dagger B_{\bar p}^\dagger , \tag{3.9}$$

and the BCS wave function can be written as usual as a coherent state

$$|\phi\rangle=Ne^{z\Gamma_o^\dagger}|0\rangle . \tag{3.10}$$

In order to obtain the amplitudes of (3.9) one has two alternative ways. The first one is to obtain an operator

$$A_p^\dagger=U_pB_p^\dagger-V_pB_{\bar p} \tag{3.11}$$

such that, acting on the coherent state (3.10) it yields

$$A_p|\phi\rangle=0 .$$

The other alternative, that of course yields the same amplitudes, is to minimize the expectation value of the Hamiltonian in the coherent state (3.10).

To perform the BCS approximation it is convenient to introduce a chemical potential λ such that the average number of particles has the appropriate value. To simplify the description it will be convenient to introduce the parameters

$$\Delta_q=GU_oV_o,\quad \Delta_o=GRU_qV_q .$$

In our particular case, the BCS equations can be written as

$$\frac{|\frac{1}{2}+\lambda|}{[(\frac{1}{2}+\lambda)^2-\Delta_o^2]^{1/2}}+\frac{R|\frac{1}{2}-\lambda|}{[(\frac{1}{2}-\lambda)^2-\Delta_q^2]^{1/2}}=2M+R+1 , \tag{3.12a}$$

$$\Delta_0=\frac{GR\Delta_q}{2D[(\frac{1}{2}-\lambda)^2-\Delta_q^2]^{1/2}}=\frac{GR\Delta_q}{2E_q} , \tag{3.12b}$$

$$\Delta_q=\frac{G\Delta_o}{2D[(\frac{1}{2}+\lambda)^2-\Delta_o^2]^{1/2}}=\frac{G\Delta_o}{2E_o} . \tag{3.12c}$$

Using these equations one obtains for the amplitudes U and V

$$V_o^2=\frac{1}{2}\left[-1+\frac{D|\frac{1}{2}+\lambda|}{E_o}\right],\quad U_o^2=1+V_o^2 ,$$
$$V_q^2=\frac{1}{2}\left[-1+\frac{D|\frac{1}{2}-\lambda|}{E_q}\right],\quad U_q^2=1+V_q^2 , \tag{3.13}$$

and the scaled ground-state energy can be written as

$$E^N=E/(GM^{3/2}R^{1/2})=\frac{\xi}{M}(RV_q^2-V_o^2)-\frac{2}{M}\left[\frac{R}{M}\right]^{1/2}U_oU_qV_oV_q . \tag{3.14}$$

As $V_o^2=M(1-\eta^2)$ and $V_q^2=M\eta^2/R$ we can express all the relevant variables in terms of η. Some of them have rather simple expressions as, for example,

$$\Delta_o = GM\eta\left[\eta^2+\frac{R}{M}\right]^{1/2},$$

$$\Delta_q = GM(1-\eta^2)\left[1+\frac{1}{M(1-\eta^2)}\right]^{1/2},$$

or the ground-state energy

$$E^N = -\xi(1-2\eta^2) - 2\eta(1-\eta^2)\left[\left[1+\frac{M}{R}\eta^2\right]\left[1+\frac{1}{M(1-\eta^2)}\right]\right]^{1/2}. \tag{3.14'}$$

Some others, such as the chemical potential, have expressions that are more complex. These expressions help to choose the proper scaling for the different physical magnitudes and will also help to understand the comparison between the different approximation methods used.

C. Hartree-Bose treatment

One may start the Hartree-Bose treatment by closely following the formula developed in Ref. 16, but due to the very simple structure of the Hamiltonian it is more convenient to define

$$\Lambda_o^\dagger = \frac{1}{[2(c^2+b^2R)]^{1/2}}\left[c(B_o^\dagger+B_{\bar{o}}^\dagger)+b\sum_{q=1}^{R}(B_q^\dagger+B_{\bar{q}}^\dagger)\right], \tag{3.15}$$

and therefore the condensate state formed by N bosons can be written as

$$|N\rangle = \frac{1}{\sqrt{N!}}(\Lambda_o^\dagger)^{2M}|0\rangle . \tag{3.16}$$

The mean value of the single-particle and interacting parts of the Hamiltonian can be written as

$$\langle N|H_{\rm sp}|N\rangle = 2DM(X-Y)/(X+Y), \tag{3.17a}$$

$$\langle N|H_{\rm int}|N\rangle = -GM(M-\tfrac{1}{2})XY/(X+Y), \tag{3.17b}$$

where $X=b^2R$ and $Y=c^2$. The Hartree-Bose wave function is obtained by minimizing the ground-state energy with respect to these variational parameters. One obtains

$$r = \frac{1}{2(Y-X)} \quad \text{where } r = \frac{G(2M-1)}{2D}. \tag{3.18}$$

It can be shown that $X=0$ or $Y=0$ also correspond always to an extreme for the energy. The energy at the minimum can be written as

$$E = -|\xi|GM^{3/2}R^{1/2} \quad \text{if } |r|<1, \tag{3.19a}$$

$$E = -\frac{DM}{2r}(r^2+1) \quad \text{if } |r|>1. \tag{3.19b}$$

D. Comparison between exact and approximate treatments

All the comparisons were done for a rather large value of M ($M=50$) because we know that this is an essential condition for the validity of the different approximate methods used. Our main interest was to study the influence of R, and therefore we have used three different values of R/M which roughly correspond to the interesting limiting cases: 0.1, 1, and 10. We diagonalized the Hamiltonian for $M=50$ and 51, and we evaluate the energies as well as the matrix elements of $P^\dagger$ for $-2\le\xi\le 2$.

In Fig. 1 it is shown the ground-state energy obtained in the exact calculation for $M=50$ and the same energy obtained using the three approximate treatments discussed before. Figure 1(a) corresponds to $R/M=0.1$, Fig. 1(b) to $R/M=1$, and finally Fig. 1(c) to $R/M=10$. The most striking feature of the exact result is the big asymmetry in the ground-state energy obtained for positive and negative values of ξ for moderate and large values of R. For comparison the result obtained in the $\phi^\dagger AB$ model in the drawing corresponding to $R/M=1$ is

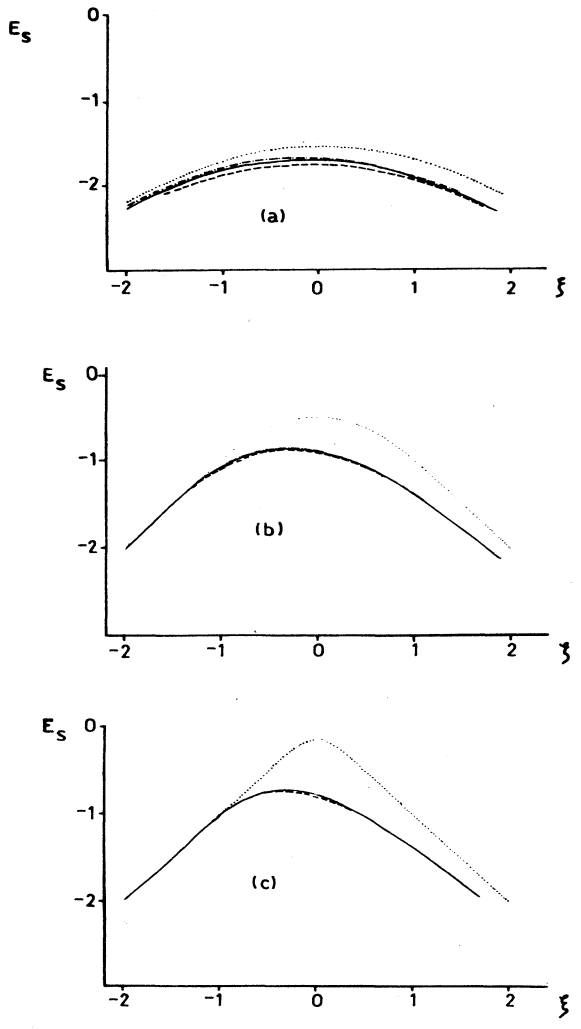

FIG. 1. Scaled ground-state energy ($E_s=E_{\rm g.s}/GM^{3/2}R^{1/2}$) for different approximations: Hartree-Bose is dotted line, Holstein-Primakoff is dashed and dotted line, BCS is dashed line, and the exact results is full line. In (a) $R/M=0.1$, (b) $R/M=1$, and (c) $R/M=10$. In all cases $M=50$.

also shown. The results of this model are almost indistinguishable from those corresponding to $R/M=10$ in the exact, HP and BCS calculations.

The Hartree-Bose description is very poor, and it is worse as the value of R/M increases. This method is completely unable to reproduce the most important features of the model, such as the already mentioned asymmetry or any type of pairlike coherence in the ground-state wave function. One may have guessed this behavior for strong interactions as the wave function proposed in the HB approximation does not exploit at all the main features of the residual interaction used. The main reason for the HB being so bad for positive (and rather large) values of ξ is that the HB wave function is not able to take into account the existence of correlated pairs of bosons [of the type shown in Eq. (2.5b) for example] even in small amounts. As these correlated pairs are in the main responsible for the lowering of the ground-state energy in the positive ξ region, HB fails as shown in Fig. 1.

As it can be guessed from expressions (3.7) and (3.14) for the BCS ground-state energy and the similar result obtained from the $\phi^\dagger AB$ model, HP and BCS yield similar descriptions of the ground state, and these similarities are more striking as R/M increases.

In Fig. 2 is shown the value of η^2 obtained in HP, BCS, and the $\phi^\dagger AB$ model as well as in the exact calculation, using the same convention as in Fig. 1. From the results obtained in the approximate treatments, we see that at $\xi=-1.0$ we have a well-defined phase transition. For $\xi<-1.0$, the system can be described essentially taking into account only a coherent pair of bosons in the R degenerate levels, while for ξ larger than -1.0 one needs to take into account also a pair of bosons in the levels o and $\bar{o}$. In a sense, one can say that there is a phase transition from a "normal" (for $\xi<-1.0$) to a "superconductive" phase. The equivalent phase transition for positive values of ξ does not happen for $\xi=1.0$ but occurs for a rather large value of the strength. The shape of the phase transition at -1.0 depends critically on the value of R, being much sharper for large M and R. As the $\phi^\dagger AB$ treatment is valid when M and R are large, one understands why the phase transition at $\xi=-1.0$ in this approximate treatment is much sharper than the one displayed by the exact calculation. We can conclude this section by saying that the HP and BCS treatments yield almost equally good results for the ground-state energies and that in practice they seem to be equivalent.

IV. APPROXIMATE TREATMENTS OF THE EXCITED STATES

This section is devoted to the study of the excited states using the same approximate treatments as in Sec. III, except Hartree-Bose on account of its poor description for the ground state. We will study the state corresponding to one- and two-body excitations. These states can be classified by the shell seniorities (s,l).

A. Holstein-Primakoff treatment

We will describe the $s=0$, $l=0$ excited state corresponding to two-body excitations in the HP picture. We assume that the excited state will have the form

$$|\alpha\beta M\rangle=(B_o^\dagger B_{\bar{o}}^\dagger+\beta C^\dagger)(B_o^\dagger B_{\bar{o}}^\dagger+\alpha C^\dagger)^{M-1}|0\rangle \ . \quad (4.1)$$

The parameter β is fixed by the orthogonality condition

$$\langle \alpha M|\alpha\beta M\rangle=0 \quad (4.2)$$

that yields

$$\beta=-\frac{\sum_{k=0}^{M-1}(M-k)(\alpha^{2k}/k!)}{\sum_{k=0}^{M-1}(\alpha^{2k+1}/k!)} \ . \quad (4.3)$$

To evaluate the energy we need the expression of the norm and the expectation values of $H_{\rm sp}$ and $H_{\rm int}$. The calculation is straightforward and one obtains

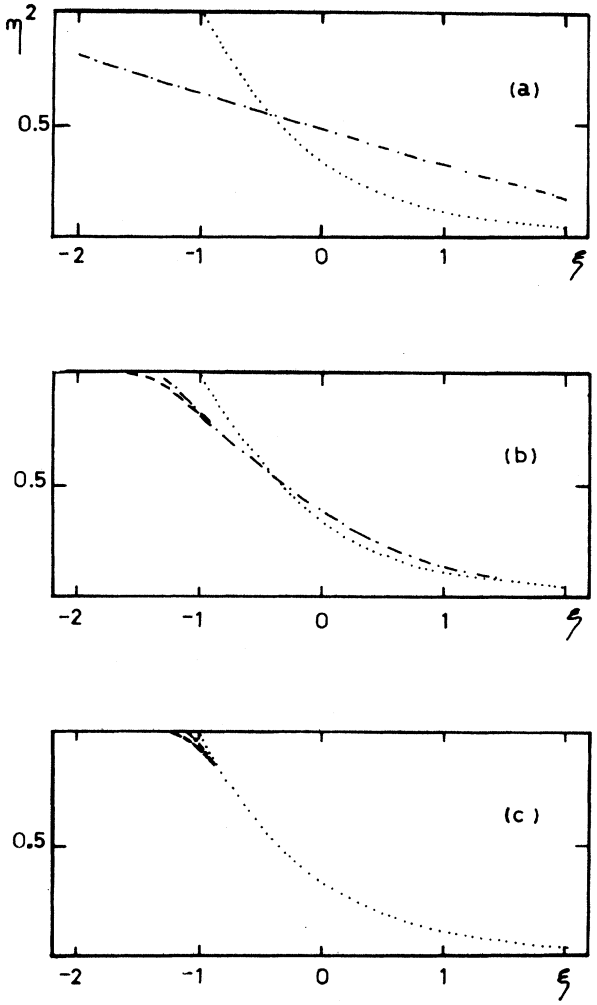

FIG. 2. The order parameter η^2 as a function of the scaled strength ξ. The $\phi^\dagger AB$ results are shown in dotted line. For HP and BCS the same convention as in Fig. 1 is used. The exact results are not shown as they are indistinguishable from the HP ones.

$$\langle \alpha\beta M|\alpha\beta M\rangle=(M-1)!^2\sum_{k=0}^{M-1}\frac{\alpha^{2k}}{k!}\left[(M-k)^2+\beta^2(k+1)+\frac{2\beta k(M-k)}{\alpha}\right], \tag{4.4}$$

$$\langle \alpha\beta M|H_{\rm sp}|\alpha\beta M\rangle=D(M-1)!^2\sum_{k=0}^{M-1}\frac{\alpha^{2k}}{k!}\left[\left[(M-k)^2+\frac{2\beta k(M-k)}{\alpha}\right](2k-M)+\beta^2(k+1)(2k+2-M)\right], \tag{4.5}$$

$$\langle \alpha\beta M|H_{\rm int}|\alpha\beta M\rangle=-2G\sqrt{R}\,(M-1)!^2\sum_{k=0}^{M-1}\frac{\alpha^{2k}}{k!}\left[\left[1+\frac{k}{R}\right]^{1/2}\beta(M-k)(k+1)\left[\frac{\beta k}{\alpha}-M-k\right]+\left[1+\frac{(k-1)}{R}\right]^{1/2}\frac{k}{\alpha}(M-k+1)(M-k)(M-k+1+\frac{(k-1)}{\alpha}\right]\right]. \tag{4.6}$$

As in Sec. III A, we can look at (4.3), (4.4), (4.5), and (4.6) in the limit $R\to\infty$, $M\to\infty$ with $M/R\ll 1$ ($\phi^\dagger AB$ model), replacing sums by exponentials. In such a way we obtain the norm β and the scaled excitation energy (when $\xi>-1$)

$$\langle \alpha\beta M|\alpha\beta M\rangle=(M-1)!^2M\frac{e^{\alpha^2}}{\eta^2}, \tag{4.7}$$

$$\beta=\frac{\eta^2-1}{\eta}\sqrt{M}, \tag{4.8}$$

$$E^N_{\rm exc}=\frac{E_{\rm exc}}{G\sqrt{MR}}=2\xi+4\eta=\tfrac{2}{3}\xi+\tfrac{4}{3}(\xi^2+3)^{1/2}. \tag{4.9}$$

B. BCS treatment

We describe initially the states corresponding to one-body excitations (i.e., $s=0$, $l=1$ or $s=1$, $l=0$). Within the BCS formalism they are the one-quasiparticle states, thus,

$$|1{\rm qp}\rangle=A_p^\dagger|\phi\rangle, \tag{4.10}$$

where $|\phi\rangle$ and $A_p^\dagger$ are defined by (3.10) and (3.11), respectively. Their energies are given by E_p (the quasiparticle energies), i.e.,

$$E_o=D[(\tfrac{1}{2}+\lambda)^2-\Delta_o^2]^{1/2}, \tag{4.11a}$$

$$E_q=D[(\tfrac{1}{2}-\lambda)^2-\Delta_q^2]^{1/2}. \tag{4.11b}$$

The states corresponding to two-body excitations are two-quasiparticles states. They can have four different structures, according to the (s,l) shell seniorities: ($s=0$, $l=2$), ($s=1$, $l=1$), ($s=2$, $l=0$), and ($s=0$, $l=0$). The first three have a simple structure in terms of quasiparticles and can be written as

$$|2{\rm qp}\rangle=\frac{1}{\sqrt{1+\delta_{pp'}}}A_p^\dagger A_{p'}^\dagger|0\rangle \tag{4.12}$$

and their energies are

$$E(2{\rm qp})=E_p+E_{p'}. \tag{4.13}$$

The description of the ($s=0$, $l=0$) states is more complicated as they can mix with the spurious state (see for instance Ref. 16). We have tried two different approximations: (i) the Tamm-Damcoff approximation (TDA) and (ii) the random-phase approximation (RPA). The first one usually has the advantage of having greater simplicity, while the second one isolates completely the spurious state.

One starts expressing the Hamiltonian in terms of quasiparticle operators. It can therefore be written as follows:

$$H=H_{00}+H_{11}+H_{22}+H_{31}+H_{13}+H_{40}+H_{04}, \tag{4.14}$$

where

$$H_{00}=E_{\rm g.s.}, \tag{4.15a}$$

$$H_{11}=E_qn_q+E_0n_o, \tag{4.15b}$$

$$H_{22}=-G(U_q^2U_o^2+V_q^2V_o^2)\sqrt{R}\,(\gamma_q^\dagger\gamma_o+\gamma_o^\dagger\gamma_q)-2G(U_qV_qU_oV_on_qn_o), \tag{4.15c}$$

$$H_{31}=(H_{13})^\dagger=G[(V_0^2+U_0^2)U_qV_q\gamma_o^\dagger n_q+U_oV_o(V_q^2+U_q^2)\sqrt{R}\,\gamma_q^\dagger n_o)], \tag{4.15d}$$

$$H_{40}=(H_{04})^\dagger=-G(U_o^2V_q^2+V_o^2U_q^2)\sqrt{R}\,\gamma_q^\dagger\gamma_o^\dagger. \tag{4.15e}$$

with

$$n_q=\sum_{r>0}(A_r^\dagger A_r+A_{\bar r}^\dagger A_{\bar r})\quad n_o=(A_o^\dagger A_o+A_{\bar o}^\dagger A_{\bar o}),$$

$$\gamma_q^\dagger=1/\sqrt{R}\sum_{r>0}A_r^\dagger A_{\bar r}^\dagger,\quad \gamma_o^\dagger=A_o^\dagger A_{\bar o}^\dagger.$$

To do the TDA we assume the

$$|2{\rm qp}\rangle=\Gamma_{\rm TDA}^\dagger|0\rangle, \tag{4.16}$$

where

$$\Gamma_{\rm TDA}^\dagger=X_q\gamma_q^\dagger+X_o\gamma_o^\dagger. \tag{4.17}$$

Applying the equation of motion method

$$[H_{00}+H_{11}+H_{22},\Gamma_{\rm TDA}^\dagger]=\omega_{\rm TDA}\Gamma_{\rm TDA}^\dagger \tag{4.18}$$

in the quasiboson approximation for $\gamma_p^\dagger$ and $\gamma_o^\dagger$ we obtain

$$\omega_{\rm TDA}=(E_q+E_0)\pm[(E_q-E_0)^2+G^2R(U_o^2U_q^2+V_q^2V_o^2)^2]^{1/2}. \tag{4.19}$$

In order to perform the RPA we assume that

$$|2{\rm qp}\rangle=\Gamma_{\rm RPA}^\dagger|\tilde 0\rangle,$$

where

$$\Gamma^{\dagger}_{\mathrm{RPA}}=X_q\gamma^{\dagger}_q+X_o\gamma^{\dagger}_o+Y_q\gamma_q+Y_o\gamma_o \tag{4.20}$$

and $|\tilde{0}\rangle$ is the RPA vacuum, i.e., it contains quasiparticle correlations.

The equation of motion in the quasiboson approximation for $\gamma^{\dagger}_q$ and $\gamma^{\dagger}_o$,

$$[H_{00}+H_{11}+H_{22}+H_{40}+H_{04},\Gamma^{\dagger}_{\mathrm{RPA}}]=\omega_{\mathrm{RPA}}\Gamma^{\dagger}_{\mathrm{RPA}}\ , \tag{4.21}$$

gives the frequencies

$\omega_{\mathrm{RPA}}=0$ (double root)

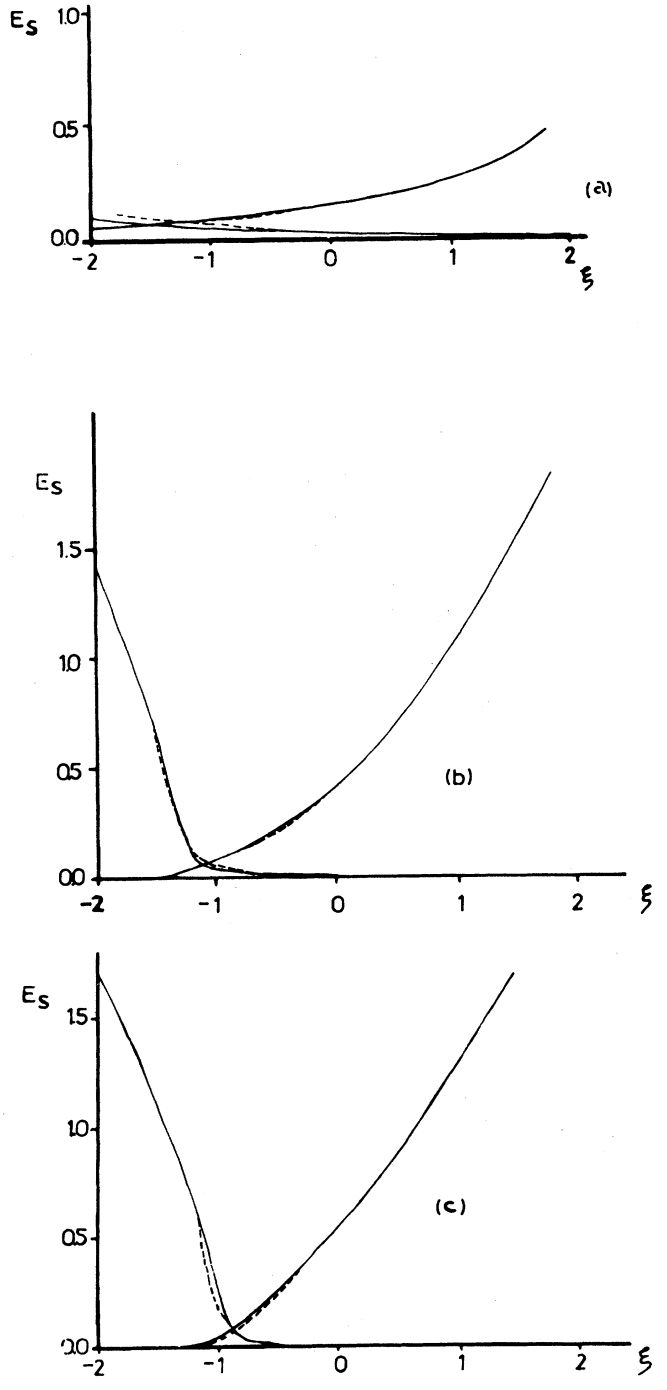

FIG. 3. Scaled excitation energy ($E_s=E_{\mathrm{g.s.}}/GM^{1/2}R^{1/2}$) for the lowest states with ($l=1$, $S=0$) and ($l=0$, $S=1$). The BCS results are shown in dashed line while the exact one is in full line.

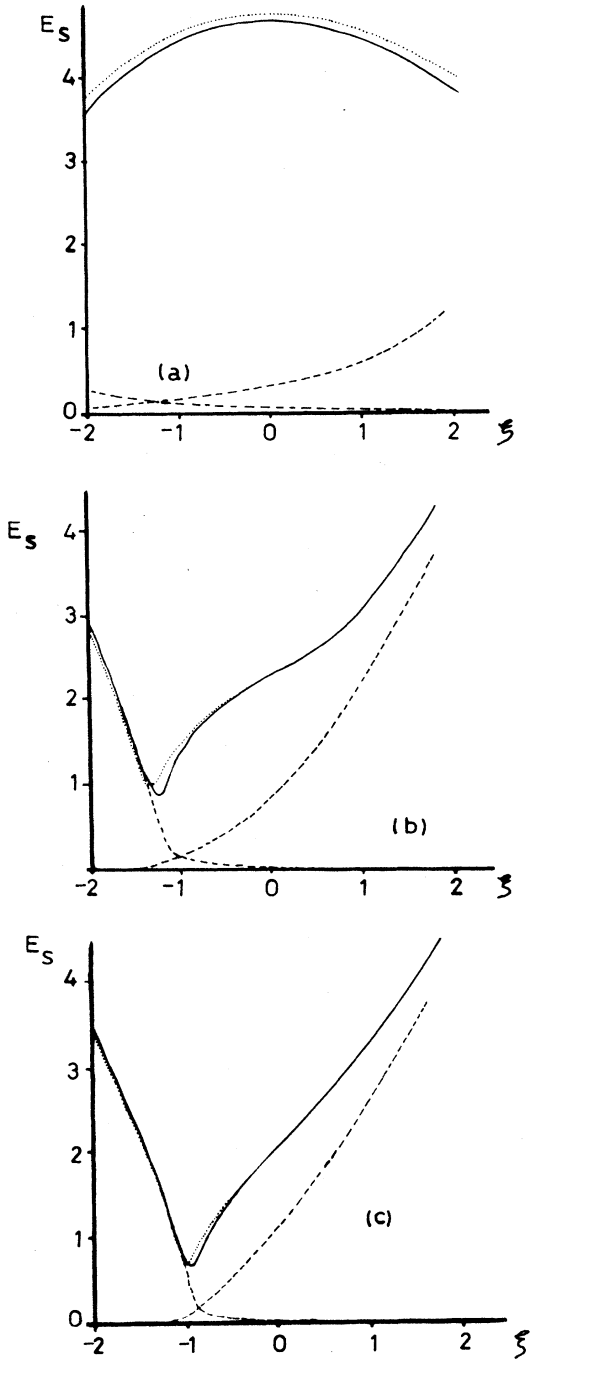

FIG. 4. Scaled excitation energy ($E_s=E_{\mathrm{g.s.}}/GM^{1/2}R^{1/2}$) for the lowest ($l=s=0$) states, in dashed line are shown the BCS results, in dotted line the RPA results, and in full line the exact ones.

and

$$\omega_{\mathrm{RPA}}=\pm[4(E_q^2+E_0^2) +2G^2R(U_o^2V_q^2+U_q^2V_o^2+U_o^2U_q^2+V_q^2V_o^2)]^{1/2}. \tag{4.22}$$

C. Comparison between exact and approximate treatments

In this section we compare the approximate treatments and exact calculation corresponds to the lowest excited states for systems with $M=50$ and $R/M=0.1$, 1, and 10, respectively, in the interval $-2\leq\xi\leq 2$.

In Fig. 3 we show the excitation energies of the lowest states with $(s=0,\ l=1)$ and $(s=1,\ l=0)$ for $R/M=0.1$, 1, and 10. We compute the exact excitation energies as

$$E_{\mathrm{exc}}^{N(M)}(\xi)=M[E_1^{(M)}(\xi)-E_{\mathrm{g.s.}}^{(M)}(\xi)]-\lambda(\xi)\ , \tag{4.23}$$

where $\lambda(\xi)$ is the "exact" chemical potential

$$\lambda(\xi)=\frac{1}{2}\left[\frac{(M+1)^{3/2}}{M^{1/2}}E_{\mathrm{g.s.}}^{(M+1)}(\xi^1) -ME_{\mathrm{g.s.}}^{(M)}(\xi)\right],\quad \xi^1=\xi\left[\frac{M}{M+1}\right]^{1/2}$$

and $E_1^{(M)}(\xi)$ is the lowest eigenvalue obtained by diagonalization of H in the space $(s=0,\ l=1)$ or $(s=1,\ l=0)$ for the system with $(2M+1)$ bosons. The full line represents the exact values (4.23) while the dashed lines correspond to the BCS bosonic quasiparticle energies (4.11a) and (4.11b). The results show that the BCS type treatment is appropriate to describe these one-body excitations and the goodness of such approximation increases when R/M increases.

In Fig. 4 we show the results obtained using the BCS (dashed), RPA (dotted), and exact (full line) for the $(l=s=0)$ states. It is particularly interesting to note how the RPA manages to isolate the spurious state at zero energy. This state has almost a vanishing energy in BCS, showing the advantage of taking properly into account the quasiparticle correlations with the RPA. The TDA results are not shown as they are not as good as the BCS ones. In general, we can conclude by saying that the BCS approximation is quite good for all the states that have different quantum number than the vacuum, while it is necessary to consider the RPA to properly treat those states (and in particular the spurious one).

V. CONCLUSIONS

We have studied a system that resembles on one side a condensate of proton and neutron pairing bosons, which is a reasonable model for spherical nuclei, and on the other side a condensate of alphalike clusters, which may modelate deformed nuclei. We have shown that for "spherical systems" the number of alphalike clusters is very high due to the large number (R) of bosons that contribute to this alpha cluster. Even for values of ξ as large as 2 one obtains a few percent of alphalike clustering in the wave function of the "pairing condensate." The model Hamiltonian that we have used in the bosonic space has similarities, with the effective one obtained in a multi-step-shell model calculation[11,14] for the ^{208}Pb region and therefore a mechanism like the one discussed here may be responsible for the large value of the preformation factor for alpha particles in this region.

Our last point will be related to the study within this simple model of the competition between particle and pair condensation in Bose systems when the number of particles is not too large. This competition has been studied in the thermodynamical limit.[17] In this case, if the interaction is attractive enough as to bind a pair of bosons in a cluster, one may have pair condensation. At low densities the ground state is a condensate of these clusters, but for high densities the one boson condensate is more stable. The main reason for this phase transition is the hard core that accounts for the saturation properties. The order of the transition between both condensates depends upon the behavior of the bosonic chemical potential as a function of the density. We have found that the situation is dramatically different in our simple model. The reason is the fact that the model studied does not have saturation properties, and therefore the hard core is absent in our model, at least for the number of particles (or equivalently densities) that are of interest in nuclei.

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[1] B. R. Mottelson, *Elementary Modes of Excitation in Nuclei,* Proceedings of the International School of Physics "Enrico Fermi," Course LXIX, Varenna, 1976, edited by A. Bohr and R. A. Broglia (North-Holland, Amsterdam, 1977).

[2] S. Belyaev, Mat. Fys. Medd. Dan. Vid. Selsk. **31**, 11 (1959); A. Bohr, B. R. Mottleson, and D. Pines, Phys. Rev. **110**, 936 (1958).

[3] D. R. Bes and R. A. Broglia, Nucl. Phys. **80**, 289 (1966).

[4] A. Arima and V. Gillet, Ann. Phys. (N.Y.) **66**, 117 (1971); A. Arima, V. Gillet, and J. Ginocchio, Phys. Rev. Lett. **25**, 1043 (1970).

[5]J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (Wiley, New York, 1952), Chap. 7.4 and references contained therein.
[6]A. Arima, H. Horiuchi, K. Kubodera, and N. Takigawa, Adv. Nucl. Phys. **5**, 345 (1972).
[7]M. Danos and V. Gillet, Z. Phys. **24B**, 294 (1972).
[8]P. Franzini and L. A. Radicati, Phys. Lett. **6**, 322 (1963).
[9]M. Cauvin, V. Gillet, F. Soulmagnon, and M. Danos, Nucl. Phys. **A361**, 192 (1981).
[10]G. G. Dussel, R. J. Liotta, and R. P. J. Perazzo, Nucl. Phys. **A388**, 606 (1982).
[11]G. G. Dussel, A. J. Fendrik, and C. Pomar, Phys. Rev. C **34**, 1969 (1986).
[12]G. G. Dussel and A. J. Fendrik, Phys. Rev. C **34**, 1097 (1986).
[13]D. R. Bes *et al.*, Phys. Lett. **52B**, 253 (1974); Nucl. Phys. **A260**, 1 (1976); **A260**, 27 (1976); **A260**, 77 (1976).
[14]M. Blatt, G. Dussel, and C. Pomar, private communication.
[15]G. G. Dussel and A. Zuker, *Le Modele $\phi^{\dagger}AB$, Rapport d'Activité* (Centre de Recherches Nucleaires de Strasborg, Strasbourg, 1985), p. 18.
[16]J. Dukelsky, G. G. Dussel, R. P. J. Perazzo, S. L. Reich, and H. M. Sofia, Nucl. Phys. **A425**, 93 (1984).
[17]P. Nozieres and D. Saint James, J. Phys. (Paris) **43**, 1113 (1982).