^{4}He binding energy calculation including full tensor-force effects

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The four-body equations of Alt, Grassberger, and Sandhas are solved in the version where the (2)+(2) subamplitudes are treated exactly by convolution, using one-term separable Yamaguchy nucleon-nucleon potentials in the $^1S_0$ and $^3S_1$-$^3D_1$ channels. The resulting $j^p=\frac{1}{2}^+$ and $\frac{3}{2}^+$ three-body subamplitudes are represented in a separable form using the energy-dependent pole expansion. Converged bound-state results are calculated for the first time using the full interaction, and are compared with those obtained from a simplified treatment of the tensor force. The Tjon line that correlates three-nucleon and four-nucleon binding energies is shown using different nucleon-nucleon potentials. In all calculations the Coulomb force has been neglected.

I. INTRODUCTION

Over the years considerable progress has been made on the solution of the quantum-mechanical four-body problem through integral equations. After the pioneering work of Yakubovsky[1] and Alt, Grassberger, and Sandhas[2] (AGS), a number of calculations were performed on the four-nucleon system. Extensive review articles have been written on the subject, covering not only the most recent work at the time,[3-7] but also explaining in great detail the equations, the different versions used in the calculations, the numerical techniques and the approximation methods.[6,7]

Although there are already many calculations of the ^{4}He binding energy using integral equations, together with local or nonlocal separable potentials between pairs, no attempts have been made to include the Coulomb force between protons, three-nucleon three-body forces, realistic one-boson exchange potential (OBEP) interactions or even full tensor-force effects. The most that has been achieved involves the use of a local potential between pairs[8] in the $^1S_0$ and $^3S_1$-$^3D_1$ channels, but neglecting the t-matrix d-wave components already at the two-body level. Since the tensor effects are included only through the s-wave two-nucleon t-matrix t_{00}, and the (3)+1 and (2)+(2) subsystem amplitudes are taken exclusively in the s wave, the resulting four-body equations conserve channel spin and channel angular momentum. The corresponding four-nucleon wave function has a very simple structure with only s-wave components between pairs, as well as zero relative orbital angular momentum between a nucleon and a pair, two pairs, and a nucleon and the remaining three nucleons.

Another major development[9] involved the use of one-term separable interactions in the $^1S_0$ and $^3S_1$-$^3D_1$ channels which as mentioned above are also treated in the "t_{00}" approximation, while taking into account p-wave (3)+1 subsystem amplitudes, in addition to the dominant s-wave ones, to conclude that negative parity (3)+1 states contribute less than 0.05% to the four-nucleon binding energy. The corresponding four-nucleon wave function now has a richer structure since p-wave states between a nucleon and a pair may subsist together with a p-wave relative orbital angular momentum between the extra fourth particle and the center of mass of the remaining three nucleons.

In the present work we attempt to include for the first time the full effect of the nucleon-nucleon tensor force through the exact treatment of all d-wave components at the two-, three-, and four-body levels. As a consequence, our calculations are at least 1 order of magnitude more time consuming than the simplest ones involving s-waves components at all levels. Although the potentials we use are relatively simple, the results constitute new bench marks that may also shed light on the effects of the tensor force on the binding energy of ^{4}He, and allow for a better understanding of the use of approximation methods in the solution of four-body equations. The four-nucleon wave function we now obtain has a considerably richer structure that results from all different two-, three-, and four-body channel components that couple to zero total angular momentum, zero total isospin and positive parity. This work also serves the purpose of setting up a very general framework, where improvements on the potential or number of (3)+1 or (2)+(2) subamplitudes that are included may be added at a later stage, making the calculation more realistic.

As mentioned in Ref. 6 most of the four-nucleon work that has been done in the past, uses three different, but equivalent formulations, all based on Yakubovsky or AGS original equations. They are: (a) a set of two-variable $[2V]$ integral equations that couple 3+1 and 2+2 wave-function components;[10] (b) a set of $N\times N$ one-variable $[1V]$ integral equations that also couple 3+1 and 2+2 wave-function components,[11] and where N is the total number of terms used in the separable representation of all (3)+1 and (2)+(2) subamplitudes; and (c) a set of $N'\times N'$ one-variable integral equations that only couple 3+1 wave-function components, and use the convolution method $[1V+C]$[12] to treat the (2)+(2) subamplitudes exactly, together with a number N' of terms to represent all (3)+1 subamplitudes in a separable form.

In the present work we use the later formulation which is the one we find more useful to extend our calculations into the scattering region. In Sec. II we write the equations and in Sec. III we present the results. Finally in Sec. IV we draw some conclusions.

II. THE EQUATIONS

In the present work we use the Alt, Grassberger, and Sandhas (AGS) formalism[2] in the form where the (2)+(2) subamplitudes are treated exactly by convolution.[31] A one-term separable two-nucleon interaction in the $^1S_0$ and $^3S_1$-$^3D_1$ channels is used together with the energy-dependent pole expansion[14] (EDPE), to set up a multiterm separable representation of the resulting three-nucleon subamplitudes. The four-nucleon equations we solve are, therefore, single-variable integral equations[6,7] for the 3+1 spectator function $q_u^{JIP}(k;SLjip)$, where J, I, and P are respectively, the total four-nucleon spin, isospin, and parity, $k(L)$ is the linear momentum (orbital angular momentum) of a single nucleon relative to the center of mass of the remaining three-nucleons which carry spin j, isospin i, and parity p. The channel spin is S and u denotes the separable terms of the corresponding (3)+1 subamplitude j^pi. Since in ^{4}He $I=0$, $J^P=0^+$, and as shown in a previous work,[9] negative-parity (3)+1 subamplitudes have little effect on four-nucleon binding energies, the (3)+1 subamplitudes of interest have isospin $i=\frac{1}{2}$, and $j^P=\frac{1}{2}^+$ and $\frac{3}{2}^+$. The integral equation we solve reads ($2m=\hbar=1$; m=nucleon mass)

$$q_u(k;SLj)=\sum_{vv'}^{Nj'}\sum_{S'L'j'}\int\frac{k'^2dk'}{2\pi^2}\mathcal{B}_{uv}(E;kk';SLj;S'L'j')\times D_{vv'}^{j'}(E-\tfrac{4}{3}k'^2)\times q_{v'}(k';S'L'j')\ , \tag{1}$$

where the driving term $\mathcal{B}$ is the sum of three terms

$$\mathcal{B}=B+X+Y\ , \tag{2}$$

$N^{j'}$ is the number of separable terms in the EDPE expansion of the $j'^+(3)+1$ subamplitude and $D_{vv'}^{j'}$ is the corresponding EDPE three-nucleon propagator given by Eq. (4) in Ref. 9. To simplify the notation we have dropped the indices J, I, P, and i which, as mentioned above, are fixed. As shown in Refs. 6 and 7, Eqs. (1) and (2) have a graphical representation which is depicted in Fig. 1. The first term in Fig. 1(b) is the Born term B, which involves the exchange of an interacting pair of nucleons between two 3+1 states, while the remaining three terms in Fig. 1(b) involve the exchange of two uncorrelated nucleons and the propagation of two noninteracting pairs; X corresponds to the first box diagram, and Y to the last two. Given our choice of two-nucleon channels, only identical (2)+(2) pairs can couple to 0^+. Therefore the resulting 3+1 and 2+2 four-body channels are shown in Table I, where $\Delta(\Gamma)$ is the pair spin (isospin).

Using the notation of Ref. 9, whenever possible, we now write explicit expressions for all terms in $\mathcal{B}$. The Born term is given by

$$B_{uu'}(E;kk';SLj;S'L'j')=\sum_{\nu\mathcal{L}}\sum_{sls'l'}\chi_{\mathcal{L}}^{\nu}(sljSL;s'l'j'S'L')\bar{B}_{\mathcal{L}}^{\nu}(E;kk';uslj;u's'l'j')\ , \tag{3}$$

where, ν is a triplet or singlet pair with spin Δ and isospin Γ,

$$\bar{B}_{\mathcal{L}}^{\nu}(E;kk';uslj;u's'l'j')=-\tfrac{1}{2}\int_{-1}^{1}dx\,P_{\mathcal{L}}(x)\left[\frac{k}{|\mathbf{p}'|}\right]^{l'}g_{u'}^{\nu}(E_{k'};|\mathbf{p}'|;s'l'j')\tau_{\nu}(T)g_u^{\nu}(E_k;|\mathbf{p}|;slj)\left[\frac{k}{3|\mathbf{p}|}\right]^{l}\ , \tag{4}$$

$$E_k=E-\tfrac{4}{3}k^2\ , \tag{5}$$

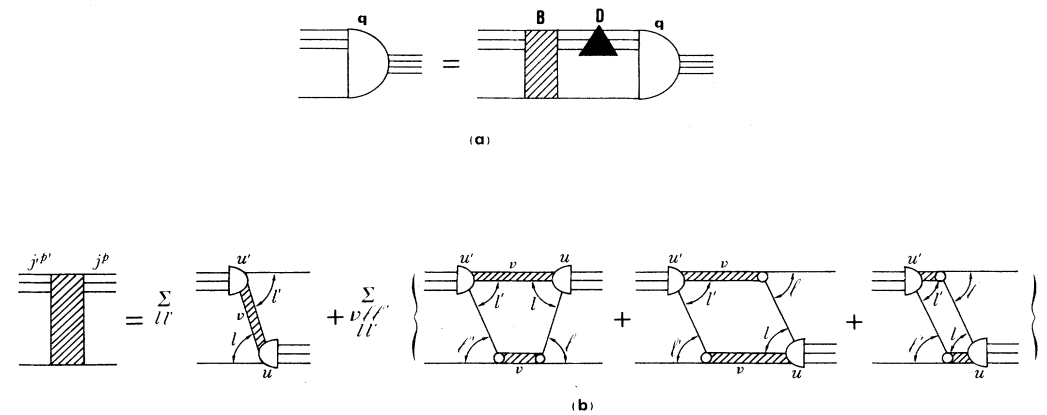

FIG. 1. (a) Graphical representation of the integral equation for the 3+1 spectator function. (b) Graphical representation of the driving term $\mathcal{B}$.

TABLE I. Four-nucleon, four-body channels that couple to $J^P=0^+$, $I=0$.

	(3)+1			(2)+(2)		
j^p	S	L	Δ	Γ	$\bar{S}$	$\bar{L}$
$\frac{1}{2}^+$	0	0	0	1	0	0
$\frac{3}{2}^+$	2	2	1	0	0	0
			1	0	2	2

$$T=E-k^2-\frac{(\mathbf{k}+\mathbf{k}')^2}{2}-k'^2\ , \tag{6}$$

$$\mathbf{p}=\frac{\mathbf{k}}{3}+\mathbf{k}'\ , \tag{7}$$

$$\mathbf{p}'=\frac{\mathbf{k}'}{3}+\mathbf{k}\ , \tag{8}$$

$$x=(\mathbf{k}\cdot\mathbf{k}')/|k|\,|k'|\ . \tag{9}$$

The $P_{\mathcal{L}}$ is a Legendre polinomial of order $\mathcal{L}$ and τ_ν is the triplet or singlet two-nucleon propagator shown in Eq. (2) of Ref. 9. The spin-isospin coefficient χ may be written as

$$\begin{aligned}\chi^\nu_{\mathcal{L}}(sljSL;s'l'j'S'L')=&\sum_{\lambda\lambda'}\sum_{\mathcal{L}'}\hat{\mathcal{L}}^2\hat{\mathcal{L}}'^2\hat{l}\hat{l}'\hat{L}\hat{L}'\widehat{(l-\lambda)}\widehat{(l'-\lambda')}\\
&\times\begin{bmatrix}2l+1\\2\lambda\end{bmatrix}^{1/2}\begin{bmatrix}2l'+1\\2\lambda'\end{bmatrix}^{1/2}\left[\frac{k'}{k}\right]^{\lambda+\lambda'}3^{\lambda-\lambda'}\begin{bmatrix}L'&\lambda'&\mathcal{L}'\\0&0&0\end{bmatrix}\begin{bmatrix}\mathcal{L}'&\lambda&\mathcal{L}\\0&0&0\end{bmatrix}\\
&\times SS'ss'jj'ii'W(\tfrac{1}{2}\Gamma I\tfrac{1}{2};i'i)\\
&\times\sum_{ABCD}\hat{A}^2\hat{B}^2\hat{C}^2\hat{D}^2W(CL'l'-\lambda'\lambda';l'\mathcal{L}')\begin{bmatrix}\mathcal{L}&D&L\\0&0&0\end{bmatrix}\begin{bmatrix}D&l-\lambda&l'-\lambda'\\0&0&0\end{bmatrix}\\
&\times\alpha\begin{Bmatrix}s'&A&l\\\frac{1}{2}&S&j\\\Delta&\frac{1}{2}&s\end{Bmatrix}\begin{Bmatrix}l&B&S'\\s'&l'&j'\\A&S&\frac{1}{2}\end{Bmatrix}\begin{Bmatrix}l&C&L\\S'&L'&J\\B&l'&S\end{Bmatrix}\begin{Bmatrix}\mathcal{L}&D&L\\\lambda&l-\lambda&l\\\mathcal{L}'&l'-\lambda'&C\end{Bmatrix},\end{aligned} \tag{10}$$

where $\hat{\mathcal{L}}=\sqrt{2\mathcal{L}+1}$, W is a Racah coefficient,

$$\alpha=(-1)^{S'-B+\mathcal{L}'+2s+\Delta-s'+1/2+i-i'+I+\Gamma+1}\ , \tag{11}$$

$i=i'=\frac{1}{2}$ and $I=J=0$. The quantum numbers s and l denote the channel spin and particle-pair relative orbital angular momentum of the three-body subsystem at each vertex. The g^ν_u are EDPE three-nucleon effective form factors as defined in Eqs. (3)–(7) of Ref. 9, but generalized to allow for a larger number of three-nucleon channels (three for $j^p=\frac{1}{2}^+$ and four with $j^p=\frac{3}{2}^+$).

Likewise the box-amplitude X is given by

$$X_{uu'}(E;kk';SLj;S'L'j')=\sum_{\bar{S}\bar{L}}\sum_{\nu\nu'}\delta_{\nu\nu'}\int\frac{q^2dq}{2\pi^2}A_{u'}(E;k'q;S'L'j';\bar{S}\bar{L}\nu\nu')G^D_{\nu\nu'}(E;\mathbf{k}'\mathbf{q}\mathbf{k})A_u(E;kq;SLj;\bar{S}\bar{L}\nu\nu')\ , \tag{12}$$

where

$$A_{u'}(E;k'q;S'L'j';\bar{S}\bar{L}\nu\nu')=\sum_{\mathcal{L}}\sum_{s'l'\sigma'\ell'}\bar{\chi}^\nu_{\mathcal{L}}(s'l'j'S'L';\sigma'\ell'\Delta'\bar{S}\bar{L})\bar{A}^\nu_{\mathcal{L}}(E;k'q;u's'l'j';\sigma'\ell'\nu')\ , \tag{13}$$

$$\bar{A}^\nu_{\mathcal{L}}(E;k'q;u's'l'j';\sigma'\ell'\nu')=(-1)^{l'}\tfrac{1}{2}\int_{-1}^{1}dx\,P_{\mathcal{L}}(x)\left[\frac{q}{|\mathbf{p}'|}\right]^{l'}\frac{g^\nu_{u'}(E_{k'};|\mathbf{p}'|;s'l'j')f^{\nu'}_{\sigma'\ell'}(|\mathbf{p}|)}{E+\varepsilon_\nu-k'^2-(\mathbf{k}'+\mathbf{q})^2-q^2/2}\left[\frac{1}{2}\frac{q}{|\mathbf{p}|}\right]^{\ell'}\ , \tag{14}$$

$$\mathbf{p}'=\tfrac{2}{3}\mathbf{k}'+\mathbf{q}\ , \tag{15}$$

$$\mathbf{p}=\mathbf{k}'+\tfrac{1}{2}\mathbf{q}\ , \tag{16}$$

$$x=(\mathbf{k}'\cdot\mathbf{q})/|\mathbf{k}'|\,|\mathbf{q}|\ , \tag{17}$$

and $\sigma(\ell)$ is the pair channel spin (angular momentum). The spin-isospin coefficient $\bar{\chi}^{\nu}_{\mathcal{L}}$ reads

$$\bar{\chi}^{\nu}_{\mathcal{L}}(s'l'j'S'L';\sigma'\ell'\Delta'\bar{S}\bar{L})=\sum_{\lambda\lambda'}\sum_{\mathcal{L}'}\hat{\mathcal{L}}^2\hat{\mathcal{L}}'^2\hat{l}'\hat{\ell}'\hat{L}'\hat{\bar{L}}\widehat{(\ell'-\lambda)}\widehat{(l'-\lambda')}$$
$$\times\begin{bmatrix}2\ell'+1\\2\lambda\end{bmatrix}^{1/2}\begin{bmatrix}2l'+1\\2\lambda'\end{bmatrix}^{1/2}\left[\frac{k'}{q}\right]^{\lambda+\lambda'}2^{\lambda+\lambda'}3^{-\lambda'}$$
$$\times\begin{bmatrix}L'&\lambda'&\mathcal{L}'\\0&0&0\end{bmatrix}\begin{bmatrix}\mathcal{L}'&\lambda&\mathcal{L}\\0&0&0\end{bmatrix}\hat{S}'\hat{\bar{S}}\hat{\sigma}'\hat{s}'\hat{\Delta}'\hat{j}'\hat{\Gamma}'\hat{i}'W(\Gamma\tfrac{1}{2}I\tfrac{1}{2};i'\Gamma')$$
$$\times\sum_{ABCD}\hat{A}^2\hat{B}^2\hat{C}^2\hat{D}^2W(CL'l'-\lambda'\lambda';l'\mathcal{L}')\begin{bmatrix}\mathcal{L}&D&\bar{L}\\0&0&0\end{bmatrix}\begin{bmatrix}D&\ell'-\lambda&l'-\lambda'\\0&0&0\end{bmatrix}$$
$$\times\alpha\begin{bmatrix}s'&A&\ell'\\\Delta&\bar{S}&\Delta'\\\frac{1}{2}&\frac{1}{2}&\sigma'\end{bmatrix}\begin{Bmatrix}\ell'&B&S'\\s'&l'&j'\\A&\bar{S}&\frac{1}{2}\end{Bmatrix}\begin{Bmatrix}\ell'&C&\bar{L}\\S'&L'&J\\B&l'&\bar{S}\end{Bmatrix}\begin{Bmatrix}\mathcal{L}&D&\bar{L}\\\lambda&\ell'-\lambda&\ell'\\\mathcal{L}'&l'-\lambda'&C\end{Bmatrix},\tag{18}$$

$$\alpha=(-1)^{\sigma'+S'-B+\mathcal{L}'+I+\Gamma}\,.\tag{19}$$

The Y term in (2) may be obtained from (12) by changing $G^{D}_{\nu\nu'}\rightarrow G^{E}_{\nu\nu'}$,

$$A_u(E;\mathbf{kq};SLj;\bar{S}\bar{L}\nu\nu')\rightarrow A_u(E;\mathbf{kq};SLj;\bar{S}\bar{L}\nu'\nu)$$

and multiplying by a phase

$$\beta=(-1)^{\bar{S}+\bar{L}+I+\Delta+\Delta'+\Gamma+\Gamma'}\,,\tag{20}$$

which, for identical pairs, reduces down to the usual factor of $(-1)^{\bar{S}+\bar{L}+I}$.

Although the intermediate (2)+(2) propagators are identical to those in Ref. 9 they are written here for completeness, and coherence of notation:

$$G^{D}_{\nu\nu'}(E;\mathbf{k}'\mathbf{qk})=\tau_{\nu'}(W-\varepsilon_{\nu'})\delta_{\nu t}-\frac{(W-Q)(W-Q')}{\pi}\int_0^{\infty}dx\frac{\mathrm{Im}[\tau_{\nu}(x)]\tau_{\nu'}(W-\varepsilon_{\nu}-\varepsilon_{\nu'}-x)}{(W-Q-\varepsilon_{\nu}-x)(W-Q'-\varepsilon_{\nu}-x)}\,,\tag{21}$$

$$G^{E}_{\nu\nu'}(E;\mathbf{k}'\mathbf{qk})=\frac{W-Q''}{-Q''}\tau_{\nu'}(W-\varepsilon_{\nu'})\delta_{\nu t}+\frac{(W-Q)(W-Q'')}{(W-Q-Q'')}\tau_{\nu}(Q''-\varepsilon_{\nu})\tau_{\nu'}(W-Q''-\varepsilon_{\nu'})$$
$$-\frac{(W-Q)(W-Q'')}{\pi}\int_0^{\infty}dx\frac{\mathrm{Im}[\tau_{\nu}(x)]\tau_{\nu'}(W-\varepsilon_{\nu}-\varepsilon_{\nu'}-x)}{(x+\varepsilon_{\nu}-Q'')(W-Q-\varepsilon_{\nu}-x)}\,,\tag{22}$$

where

$$W=E+\varepsilon_{\nu}+\varepsilon_{\nu'}-q^2\,,\tag{23}$$

$$W-Q=E+\varepsilon_{\nu}-q^2/2-(\mathbf{k}+\mathbf{q})^2-k^2\,,\tag{24}$$

$$W-Q'=E+\varepsilon_{\nu}-q^2/2-(\mathbf{k}'+\mathbf{q})^2-k'^2\,,\tag{25}$$

$$W-Q''=E+\varepsilon_{\nu'}-q^2/2-(\mathbf{k}'+\mathbf{q})^2-k'^2\,,\tag{26}$$

and $\delta_{\nu t}$ is zero unless ν is a triplet pair. Likewise ε_{ν} is the deuteron binding energy if $\nu=t$ and zero otherwise.

Since (1) is a set of coupled homogeneous integral equations that satisfy the Fredholm alternative, a solution only exists for discrete values of $E<-\varepsilon_t$, where ε_t is the underlying three-nucleon binding energy. A note of care should be added concerning the calculation of the box amplitudes X and Y. Since $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{q}$ appear in Eqs. (21) and (22), the integrals in the angles between $\hat{\mathbf{k}}$ and $\hat{\mathbf{q}}$ and between $\hat{\mathbf{k}}'$ and $\hat{\mathbf{q}}$ are performed simultaneously with the integration in q^2dq shown in (12); in other words, the integration in Eq. (14) cannot be done independently of the rest.

III. RESULTS

Given the parameters of the nucleon-nucleon interaction shown in Table II one proceeds to find the values of E for which the Fredholm determinant in Eq. (1) vanishes. Since these calculations are, as far as CPU time is concerned wise, very time consuming, it is worth studying first the convergence of the calculation with respect to the number of terms in the separable representation of (3)+1 subamplitudes, the number of d-state vertices included in the calculation of B, X, and Y amplitudes, and the importance of $j^p=\frac{3}{2}^+$ three-nucleon subamplitudes. For that we start by using a small number of mesh points M in the momentum variables k, q, and k', such as $M=14$, which, from previous experience,[6,9] is sufficient to obtain binding energies that are at least 99% con-

TABLE II. Parameters of the NN interaction using Yamaguchy form factors taken from Ref. 6.

Interaction	$\beta_0^t(f_m^{-1})$	$\beta_2^t(f_m^{-1})$	t	Percent D	$\beta_0^s(f_m^{-1})$	$\lambda_0^s(f_m^{-3})$
$Y1$	1.45		0	0	1.165	73.92
$Y2$	1.406		0	0	1.13	65.63
$Y4$	1.3134	1.5283	−1.689	4.0	1.13	65.63
$Y5$	1.2766	1.7610	−2.950	5.5	1.13	65.63
$Y6$	1.2412	1.9476	−4.4949	7.0	1.13	65.63

verged.

Since the calculation of B in Eq. (3) involves the summation over all channels of the underlying three-nucleon system (s,l,s',l') and the calculation of X and Y involves the summation over all channels of the three- and two-nucleon systems $(s,l,s',l',\sigma,l,\sigma',l')$, we show in Tables III and IV all two- and three-nucleon channels for the chosen two-nucleon interaction. Given that l and ℓ only take the values zero or two, one may define $L_T(L_T=l+l'$ in B and $L_T=\ell+\ell'+l+l'$ in X or $Y)$ as the sum of all angular momentum quantum numbers at each vertex in the amplitudes shown diagrammatically in Fig. 1(b). In an exact calculation L_T takes all possible values between zero and 8 in X or Y, but only up to 4 in B. Nevertheless, because d-wave vertices are weaker than s-wave vertices, one expects that terms with $L_T=8$ should contribute less to the binding energy than terms with $L_T=4$, and these less than $L_T=0$ ones. Therefore we study first how the four-nucleon binding energy changes as we allow the upper limit on L_T to increase from zero to 8. The results are shown, in Table V for the interaction $Y4$ and using a single term in the separable representation of the $j^p=\frac{1}{2}^+$ (3)+1 subamplitude. We find that by including all terms up to $L_T=4$ one gets 99.9% of the total binding energy. As expected $\max[L_T]=0$ and $\max[L_T]=2$ results are identical because in B one needs $l=l'$ to couple the initial $\frac{1}{2}^+$ 3+1 state in channel $S=L=0$ to the same final state with $S'=L'=0$. In addition in X and Y one needs at least $\max[L_T]=4$ to couple both $S=L=0$ and $S'=L'=0$ $\frac{1}{2}^+$ 3+1 states to an intermediate $\bar{S}=\bar{L}=2$ 2+2 state. The net effect of adding the d-wave vertices in the kernel, while increasing the upper limit on L_T from zero to 4 is an increased repulsion. Although the additional $l=l'=2$ term in B is attractive, much like the already existing $l=l'=0$ one, the new contributions coming from X and Y that result from $\ell=l=0$ and $\ell'=l'=2$, coupling $S=L=0$ and $S'=L'=0$ 3+1 states through an $\bar{S}=\bar{L}=0$ 2+2 intermediate state, are repulsive relative the dominant $\ell=l=\ell'=l'=0$ box amplitude. In addition to these terms there are those that involve, in intermediate states, two triplet pairs in channel $\bar{S}=\bar{L}=2$. These give rise to an attractive contribution as may be shown by switching off the $\bar{S}=\bar{L}=2$ channel. Keeping $\max[L_T]=4$, the resulting four-nucleon binding energy changes from 31.943 to 31.820 MeV which is 0.55 MeV smaller than the original $\max[L_T]=0$ result shown in Table V. Finally, by increasing $\max[L_T]$ from 4 to 8 one adds attraction leading to 0.1% extra binding.

Next in Table VI we show how the binding energy changes as we increase the number of terms in the separable representation of the $j^p=\frac{1}{2}^+$ subamplitude. For simplicity we take $\max[L_T]=0$, and increase $N^{1/2}$ from 1 to 4; while the first and third terms are attractive, the second and fourth terms are repulsive. Previous calculations where the tensor force was either neglected or included in some approximate form[6,11] indicate that four terms in the expansion of the subamplitudes is sufficient to calculate binding energies which are at least 99.9% converged. Since the convergence pattern with $N^{1/2}$ is no different here from what we found elsewhere, we take $N^{1/2}=4$ as the optimum value.

Although one expects the $\frac{1}{2}^+$ three-nucleon subamplitude that carries the quantum numbers of the triton to be dominant, there is no *a priori* reason to neglect the $\frac{3}{2}^+$ subamplitude which couples to $J^P=0^+$ through $S=L=2$ (see Table I). The $j^p=\frac{3}{2}^+$ subamplitude is predominantly made up of the $s=\frac{3}{2}$, $l=0$ three-nucleon subchannel which is Pauli repulsive (all nucleons have their spins aligned and are in relative s states). Therefore its contribution to ε_α is mainly repulsive. This is shown in Table

TABLE III. Two-nucleon channels for $^1S_0$ and $^3S_1$-$^3D_1$ nucleon-nucleon interactions.

ν	Δ	Γ	σ	ℓ
1	0	1	0	0
2	1	0	1	0
2	1	0	1	2

TABLE IV. Three-nucleon channels for $^1S_0$ and $^3S_1$-$^3D_1$ nucleon-nucleon interactions.

ν	s	l	j^p
1	$\frac{1}{2}$	0	
2	$\frac{1}{2}$	0	$\frac{1}{2}^+$
2	$\frac{3}{2}$	2	
1	$\frac{1}{2}$	2	
2	$\frac{1}{2}$	2	
2	$\frac{3}{2}$	0	$\frac{3}{2}^+$
2	$\frac{3}{2}$	2	

TABLE V. Four-nucleon binding energy ε_α (MeV) vs max$[L_T]$ for the interaction $Y4$. Only the $j^p=\frac{1}{2}^+$ (3)+1 subamplitude is included with $N^{1/2}=1$.

max$[L_T]$	ε_α (MeV)
0	32.370
2	32.370
4	31.943
6	31.962
8	31.978

TABLE VII. Four-nucleon binding energy ε_α (MeV) vs max$[L_T]$ for $N^{1/2}=1$ and $N^{3/2}=1$ and 4. The latter energies were calculated only for max$[L_T]=2.4$ to save CPU time. The interaction is $Y4$.

	ε_α (MeV)	
max$[L_T]$	$N^{3/2}=1$	$N^{3/2}=4$
2	32.369	32.368
4	31.943	31.942
6	31.961	
8	31.977	

VII where we display ε_α vs max$[L_T]$ for $N^{1/2}=1$ and $N^{3/2}$ equals 1 and 4. Comparing Tables V and VII one finds that the $\frac{3}{2}^+$ subamplitude only changes the α-particle binding energy by less than 0.01% which is 2 or 3 times less than what we have estimated in Ref. 9 to be the contribution of the negative-parity subamplitudes we have neglected already here. Therefore, from now on, we consider only the $j^p=\frac{1}{2}^+$ (3)+1 subamplitude. A note of care should nevertheless be added, concerning the effect of $\frac{1}{2}^-$, $\frac{3}{2}^-$, and $\frac{3}{2}^+$ three-nucleon states in the α-particle wave function. Although our findings seem to indicate that the contribution of these states to the binding energy is negligible, one cannot conclude that they may be discarded from the resulting α-particle wave function prior to a careful study of the weight of each component state. In particular we find that the contribution of the $\frac{3}{2}^+$ subamplitude is bigger in the absence of all 2+2 channels, than in their presence. This indicates that some cancellations may take place between terms, resulting from different components in the wave function which, though small, are not necessarily negligible. These findings are shown in Table VIII, where all calculations involve max$[L_T]=4$ and $N^{1/2}=4$ terms in the separable representation of the $\frac{1}{2}^+$ (3)+1 subamplitude. By switching off all (2)+(2) states in the calculation, the driving term $\mathcal{B}$ in Eq. (2) becomes the Born term B. In the presence of all (2)+(2) states, introducing the $\frac{3}{2}^+$ (3)+1 subamplitude hardly changes the four-nucleon binding energy, while in their absence its contribution rises to 0.4%. As for the (2)+(2) states they contribute roughly 45% to the ground-state binding energy independently of the presence or absence of the $\frac{3}{2}^+$ subamplitude.

Finally, keeping max$[L_T]=4$ and $N^{1/2}=4$, we increase M to 21 mesh points and calculate the ground-state energies of the α particle for the interactions shown in Table II. The energies corresponding to $Y1$ and $Y2$ interactions, which have no tensor force, have been reported previously,[6,12] and agree with the values obtained by Gibson and Lehman[10] or Sofianos *et al.*[11,15] The new "exact" ε_α results for Y_4, Y_5, and Y_6 potentials are given in Table IX, together with the old ones for $Y1$ and $Y2$ interactions. The corresponding triton binding energies ε_t are given for comparison. In addition we also give the energy of the excited 0^+ state in the α particle, which exists for all potentials we have used, but whose value approaches the corresponding ε_t threshold as the d-state component of the NN interaction increases. In the last two columns we also display ε_α and ε_α^* energies for max$[L_T]=0$. The difference between the third and fifth (or fourth and sixth) columns shows the contribution resulting from including d-state components in the kernel of Eq. (1). As mentioned before the net effect is an additional repulsion which increases slightly with the strength of the tensor component of the NN force.

Since Table V indicates that max$[L_T]=4$ results are 99.9% converged, we show in Table X the extrapolated ground-state energy for max$[L_T]=8$, and compare with the values obtained by previous calculations where the tensor force was included in some approximate way. As mentioned in the Introduction, four-body equations have been solved in the past using three different, but equivalent formulations. The work previously mentioned was done either through the solution of two-variable $[2V]$ integral equations[10] that couple 3+1 and 2+2 components in the wave function, or by making use of coupled one-variable integral equations for the 3+1 components, together with convolution $[1V+C]$ (Ref. 6) to include the (2)+(2) subamplitudes in the form of an integral over independent pair t matrices. These two formulations yield the same results when used in the context of an exact calculation, but differ[16] even when the same type of approximation is used to simplify their respective

TABLE VI. Four-nucleon binding energy ε_α (MeV) vs $N^{1/2}$ for the interaction $Y4$, and max$[L_T]=0$.

$N^{1/2}$	ε_α (MeV)
1	32.370
2	32.368
3	32.675
4	32.666

TABLE VIII. Four-nucleon binding energy ε_α (MeV) for max$[L_T]=4$, $N^{1/2}=4$, and the $Y4$ interaction. The $\frac{3}{2}^+$ subamplitude is included with $N^{3/2}=2$ in the presence and absence of all (2)+(2) states.

	All (2)+(2) states		No (2)+(2) states	
	$N^{3/2}=0$	$N^{3/2}=2$	$N^{3/2}=0$	$N^{3/2}=2$
ε_α (MeV)	32.206	32.204	18.779	18.709

TABLE IX. Three-nucleon binding energy ε_t, and four-nucleon ground state and excited 0^+ energies for different potentials parameters with max$[L_T]$ equals zero and 4.

		max$[L_T]=4$		max$[L_T]=0$	
	ε_t	ε_α^*	ε_α	ε_α^*	ε_α
*Y*1	11.013			11.438	45.666
*Y*2	10.206			10.622	42.398
*Y*4	8.573	8.752	32.305	8.775	32.775
*Y*5	8.040	8.150	29.066	8.176	29.587
*Y*6	7.588	7.654	26.535	7.680	27.046

kernels. As discussed in Ref. 6 the two major approximations that have been used in the past to include tensor effects in the four-body problem, without including *d*-state components, are "t_{00}" and "truncated $\frac{1}{2}^+$ subamplitude." In "t_{00}" one reduces the triplet *NN t* matrix to its dominant *s*-wave component, which accounts to removing the third channel in Table III, and consequently, all *d*-wave channels in Tables IV and I. Unlike "t_{00}", the "truncated $\frac{1}{2}^+$ subamplitude" keeps all three channels of the two-nucleon system but truncates the $\frac{1}{2}^+$ (3)+1 subsystem to the two dominant *s*-wave channels. In this truncated space the three-nucleon kernel is identical to the full kernel. As for the (2)+(2) subamplitudes they are treated as in "t_{00}". Therefore, from the practical point of view, both methods are identical, except on the way they treat the (3)+1 subamplitudes. Table X clearly shows that "t_{00}" tends to overbind while "truncated $\frac{1}{2}^+$ subamplitude" tends to underbinding. In addition, approximate calculations using $[1V+C]$ lead to energy results that are more bound than those obtained with $[2V]$ formulations.[16] Similar effects regarding "t_{00}" vs "truncated $\frac{1}{2}^+$ subamplitude" are seen in the three-nucleon problem as shown in Table XI.

In Fig. 2 we plot the exact results for ε_α vs ε_t (crosses) and draw a solid line through the crosses corresponding to the potentials that have the same singlet parameters (*Y*2–*Y*6). The cross corresponding to *Y*1 lies outside such a line. This strong correlation between ε_α and ε_t was first suggested by Tjon,[17] and remains valid even in the presence of tensor force. Our calculation seems to indicate that potentials that share the same singlet interaction, but different triplet potentials, all fitted to the same triplet scattering length, deuteron binding energy, and quadrupole moment, may lead to ^{4}He and ^{3}He binding energies that correlate almost exactly through a straight line. We also find that approximate $[2V]$ results using either "t_{00}" (open circles) or "truncated $\frac{1}{2}^+$ subamplitude" (open triangles) also lie nearly on the same solid line defined by the crosses. As for $(1V+C)$ approximate results they define a new straight line (dashed) which goes through the cross corresponding to *Y*2.

IV. CONCLUSIONS

Using one-term separable potentials between pairs in the $^1S_0$ and $^3S_1$-$^3D_1$ channels, we have calculated for the first time the binding energy of ^{4}He, taking into account all *d*-wave components of the two-, three-, and four-nucleon channels. On a Vax 8550 computer it takes about 8 h of CPU time to calculate all matrix elements of the kernel at a given energy *E*, using 21 mesh points in the momentum variable, four terms in the separable representation of the $\frac{1}{2}^+$ (3)+1 subamplitude, and max$[L_T]=4$. The same calculation in the "t_{00}" approximation takes 1.5 h of CPU time. Given the computer limitations we had to face, we limited ourselves to a few simple potential models, and kept an optimum balance between the number of mesh points, the number of separable terms per subamplitude, and the number of *d*-wave components in the driving term $\mathcal{B}$. Nevertheless we expect our energy results to be at least 99.9% converged.

If one concentrates on Tables IX–XI one may draw the following conclusions. (a) Raising the percent *D*-state probability in the deuteron from 4 to 7%, introduces a 1-MeV change in the binding energy of ^{3}He but a 5.8 MeV shift in the binding energy of ^{4}He. Although the effective change is not linear with percent *D* due to saturation, as expected, ^{4}He is much more sensitive to changes in the *NN* tensor force than ^{3}He. (b) Although the agreement is not perfect and changes with percent *D*, we find that the $[2V]$ results in the "t_{00}" approximation, shown in the third column of Table X, are the ones that come closer to

TABLE X. Four-nucleon binding energies resulting from different integral equation formulas and taking different approximation methods used to include the tensor force. The exact result is extrapolated to max$[L_T]=8$ from the value obtained in Table IX for max$[L_T]=4$.

	"t_{00}"			"Truncated $\frac{1}{2}^+$"	
Potential	$[1V+C]$	$[2V]$	Exact	$[1V+C]$	$[2V]$
*Y*4	34.02	32.87	32.34	31.68	30.42
*Y*5	30.82	29.25	29.10	28.15	26.58
*Y*6	28.04	26.17	26.56	25.41	23.65

TABLE XI. Three-nucleon binding energies for different potential models and approximation methods used to include tensor-force effects.

Potential	"t_{00}"	Exact	"Truncated $\frac{1}{2}^+$
$Y4$	8.657	8.573	8.191
$Y5$	8.078	8.040	7.560
$Y6$	7.560	7.588	7.051

the energies obtained with max$[L_T]=0$, and presented in the sixth column of Table IX. (c) Adding the d-wave components to the four-nucleon kernel decreases the ground-state binding energy by roughly 1.5–2 % as may be seen by comparing columns four and six in Table IX. In ^{3}He a similar change only results in at most 1% less binding. (d) The $\bar{S}=\bar{L}=2$ $2+2$ channel gives an attractive contribution of about 0.5%, while the $\frac{3}{2}^+$ three-nucleon subamplitude in channel $S=L=2$ contributes less than 0.01% to the ground-state energy. (e) Approximate methods of treating the tensor force are not as reliable in ^{4}He as in ^{3}He for the reason pointed out in (a), and strongly, depend on the type of equations used, approximation method, and percent D in the deuteron. Nevertheless one may expect exact results to fall between $[2V]$–"t_{00}" calculations and $[1V+C]$–"truncated $\frac{1}{2}^+$ subamplitude" for percent D-state probabilities around 4% and approach $[2V]$–"t_{00}" at 7%. A similar effect takes place in ^{3}He as shown in Table XI. (f) As reported previously[11] $2+2$ channels account for about 45% of ^{4}He binding energy. (g) The Tjon line shows an almost perfect linear correlation between ε_α and ε_t for interactions that share the same singlet potential. Although $[1V+C]$ approximate results do not fall on the exact Tjon line, they are the ones that show a similar variation with percent D as the exact calculation.

As mentioned before one should not assume that a small change in the binding energy resulting from including a given $2+2$ or $3+1$ state, implies a zero or negligible contribution of that same state to the wave function. Careful study of the wave function and its structure in terms of components and their respective weights or percentages is needed before one may reach any conclusion. Work in this direction is underway.

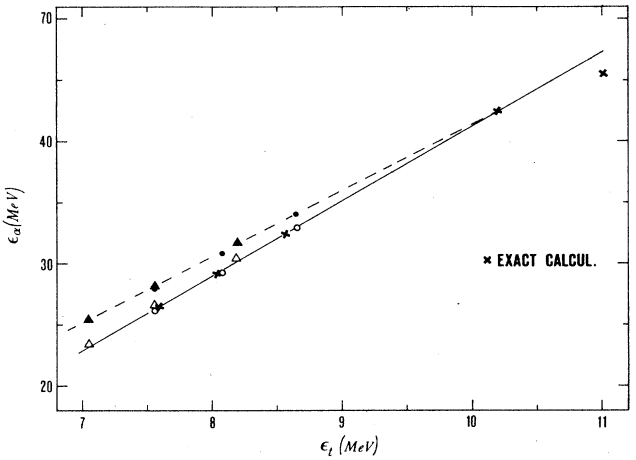

FIG. 2. Four-nucleon binding ε_α (MeV) vs the corresponding three-nucleon energy ε_t (MeV) for different potentials, approximation methods used to include the tensor force, and integral equation formulations. The crosses correspond to exact results for ε_t and ε_α. The circles correspond to "t_{00}" while the triangles correspond to "truncated $\frac{1}{2}^+$ subamplitude." Open symbols are from $[2V]$ equations while full symbols are from $[1V+C]$.

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