y scaling and hard-core potentials

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We investigate the scaling function in the asymptotic limit of inclusive scattering for cases where the interactions contain a hard core. For a particle bound in a potential we give an analytic expression for the correction to the standard impulse approximation result. For the response of a many-body system the same correction is obtained as a finite multiple-scattering series. The approach to the asymptotic limit is studied numerically in a simple model.

I. INTRODUCTION

In recent years there have been extensive discussions on the concept and application of y scaling to the response $S(q,\omega)$ of various systems. The first study for nonrelativistic systems is due to West.[1] Since then many data on inclusive electron scattering from nuclei have been analyzed using this tool.[2,3] There exists further a rich literature on inclusive scattering of neutrons on quantum liquids. In fact, y scaling is implicit in publications by Gersch and co-workers on the response for quantum fluids.[4]

For nonrelativistic systems y scaling addresses the reduced response $F=(q/m)S$ with m the mass of the constituents. It is then convenient to introduce a specific scaling variable y in terms of the momentum and energy transfers q,ω

$$y=\frac{m}{q}\left[\omega-\frac{q^2}{2m}\right] . \qquad (1.1)$$

In the so-called scaling limit

$$q\to\infty,\quad \omega\to\infty,\quad \text{while } y \text{ is fixed} , \qquad (1.2)$$

F, when written as function of y,q satisfies

$$\frac{q}{m}S(q,\omega)\equiv F(y,q)\to F_0(y)=2\pi\int_{|y|}^{\infty}dp\, pn(p) . \qquad (1.3)$$

$F_0(y)$ is called the scaling function and depends only on $n(p)$, the momentum distribution of the constituents inside the target. If Eq. (1.3) holds, $n(p)$ can be directly extracted from the data by differentiation of $F_0(y)$ with respect to y.

For sufficiently regular interactions one shows[4,5] that $F(y,q)-F_0(y)=0(V/q)$ where V describes final-state interactions (FSI) between the knocked-on particle and the remaining core constituents. The FSI due to regular interactions thus vanishes in the scaling limit. However, it was pointed out by Weinstein and Negele[6] that for interactions containing a hard core, i.e., when for $r<a$ the interaction $V(r)\to\infty$, FSI interactions do contribute to $F(y,q)$ in the scaling limit and Eq. (1.3) has to be modified.[7] Using a diagrammatic approach for the evaluation of the scaling function at large y the aforementioned authors obtained up to 60% corrections to the impulse approximation result (1.3).

Short-range repulsions are characteristic for many physical systems: Examples are given by effective nucleon-nucleon potentials in nuclei and the atom-atom interaction in liquids or gases. In fact, any finite repulsion in $V(r)$ acts like a hard core of radius a, if the incoming energy $E<V(r)$ for $r<a$. In view of the preceding it seems appropriate to analyze hard-core FSI effects on the response in detail. This we propose to do in a nonrelativistic context.

This paper is organized as follows: In Sec. II we consider the model of a single particle moving in a potential which contains a hard core. Using methods of geometrical optics we find the correct formula for the response in the scaling limit, replacing the standard impulse approximation expression (1.3). An alternative derivation of the same result using scattering theory is given in Appendix A. In Sec. III we generalize the results of Sec. II to many-body systems, which have interactions containing a hard core. In Sec. IV we investigate a simple solvable model where the potential contains an attractive as well as a repulsive component. This model is evaluated numerically in order to study the convergence to the scaling limit (details of these calculations are given in Appendix B). In Sec. V we summarize our results.

II. SCALING LIMIT FOR HARD-CORE POTENTIALS

Consider the inclusive scattering of a projectile weakly interacting with a particle of mass m. The latter is bound in a potential V which contains a hard core with radius $r=a$. The longitudinal structure function (response) is in general

$$S(q,\omega)=\sum_{n=0}^{\infty}\delta[\omega-(E_n-E_0)]|\langle n|e^{i\mathbf{q}\cdot\mathbf{r}}|0\rangle|^2 \quad (2.1a)$$

$$=-\frac{1}{\pi}\mathrm{Im}\int d^3r\, d^3r'\Phi_0^*(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}} \times g(\omega+E_0,\mathbf{r},\mathbf{r}')e^{i\mathbf{q}\cdot\mathbf{r}'}\Phi_0(\mathbf{r}') , \quad (2.1b)$$

where the sum extends over all states $|n\rangle$ with energy E_n (Ref. 8) and g is the full Green's function. Evaluating Eq. (2.1) in the scaling limit we can make use of some simplifications: First, we know from previous work[4,5] that to leading order, the regular part of the potential does not contribute to the final states $|n\rangle$. The final states $|n\rangle$ in Eq. (2.1a) are therefore just the scattering states $|\Phi_{\mathbf{k}}\rangle$ for a pure hard-core potential. Second, at short wavelengths one may exploit the concepts of geometrical optics:[9,10] The scattering wave function consists of an incident wave, a shadow-forming wave, and a reflected wave (see Fig. 1):

$$\Phi_{\mathbf{k}}=e^{i\mathbf{k}\cdot\mathbf{r}}+\Phi_{\mathbf{k}}^{\mathrm{sh}}+\Phi_{\mathbf{k}}^{\mathrm{re}} . \quad (2.2a)$$

For high momenta the shadow-forming wave cancels completely the incident wave in the region behind the illuminated object

$$\Phi_{\mathbf{k}}^{\mathrm{sh}}(\mathbf{r})=\begin{cases}-e^{i\mathbf{k}\cdot\mathbf{r}} & \text{if } \mathbf{r}\in \mathrm{sh}\equiv\{z>0,b<a\}\\ 0 & \text{else} .\end{cases} \quad (2.2b)$$

Here the z direction is parallel to the incident momentum $\mathbf{k}$ and $\mathbf{b}$ is the coordinate perpendicular to that direction. The reflected wave is, in general, a spherical wave. However, near the scatterer it can be approximated by

$$\Phi_{k}^{\mathrm{re}}(\mathbf{r})=\begin{cases}-e^{i[2\delta(b)-\mathbf{k}\cdot\mathbf{r}]} & \text{if } \mathbf{r}\in I\equiv\{z<0,b<a\}\\ 0 & \text{else} ,\end{cases} \quad (2.2c)$$

where

$$\delta(b)=-k(a^2-b^2)^{1/2} \quad (2.2d)$$

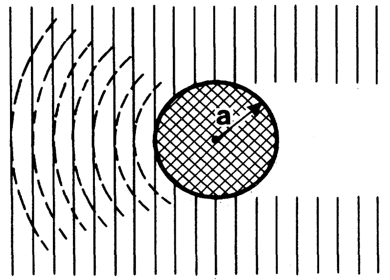

FIG. 1. Scattering wave function $\Phi_{\mathbf{k}}$ (for $\mathbf{k}\to\infty$) for a pure hard-core potential of the radius a. The incident plane wave is drawn by solid lines and the reflected wave by dashed lines.

is the phase needed to satisfy the boundary condition $\Phi_{\mathbf{k}}\equiv 0$ at $r=a$ in the illuminated region I, i.e., for $z=-(a^2-b^2)^{1/2}$.[11]

Substituting Eqs. (2.2) into the expression (2.1a) one obtains several terms for the structure function which will be evaluated separately. The first comes exclusively from the plane wave in Eq. (2.2a) and produces the usual impulse approximation (IA)

$$S_{\mathrm{IA}}(q,\omega)=\int\frac{d^3p}{(2\pi)^3}|\tilde{\Phi}_0(\mathbf{p})|^2\delta\left[\omega-\frac{q^2}{2m}-\frac{\mathbf{p}\cdot\mathbf{q}}{m}+E_0-\frac{p^2}{2m}\right], \quad (2.3)$$

where $\mathbf{p}=\mathbf{k}-\mathbf{q}$ is the struck nucleon momentum and

$$\tilde{\Phi}_0(\mathbf{p})=\int d^3r\, \Phi_0(\mathbf{r})e^{i\mathbf{p}\cdot\mathbf{r}} \quad (2.4)$$

is the bound-state wave function in momentum space. Introducing the scaling variable y according to Eq. (1.1) and neglecting terms $O(1/q)$ in the δ function, we obtain Eq. (1.3) with $n(p)=|\tilde{\Phi}_0(p)|^2/(2\pi)^3$.

Consider next the interference term between plane and shadow-forming waves

$$-2\frac{m}{q}\mathrm{Re}\int\frac{d^3p}{(2\pi)^3}\delta(y-\mathbf{p}\cdot\hat{\mathbf{q}})\int d^3r\int_{\mathrm{sh}}d^3r'\Phi_0^*(\mathbf{r})\Phi_0(\mathbf{r}')e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} ,$$

where, in general, the shadow region (sh) also depends on $\mathbf{p}$ since it is defined with respect to $\mathbf{k}=\mathbf{p}+\mathbf{q}$. However, the momentum $\mathbf{p}$ is bounded and for large momentum transfers the direction of $\mathbf{k}$ practically coincides with that of $\mathbf{q}$. One can then perform the $\mathbf{p}$ integration which gives for the interference term

$$-2\frac{m}{q}\frac{1}{2\pi}\mathrm{Re}\int_{b<a}d^2b\int_{-\infty}^{+\infty}dz\int_0^{\infty}dz'\Phi_0^*(b,z)\Phi_0(b,z')e^{iy(z-z')} . \quad (2.5a)$$

Similarly, for the term involving only the shadow-forming wave we get

$$\frac{m}{q}\frac{1}{2\pi}\int_{b<a}d^2b\int_0^{\infty}dz\int_0^{\infty}dz'\Phi_0^*(\mathbf{b},z)\Phi_0(b,z')e^{iy(z-z')} . \quad (2.5b)$$

Note that the lower limit of the z integrations can be replaced by

$$w=(a^2-b^2)^{1/2} \quad (2.6)$$

since the bound-state wave function vanishes identically inside the hard core. There is no contribution from interference terms involving the reflected wave since the phase δ in Eq. (2.2d) grows with increasing momentum transfer and leads to rapid oscillations in the integrand. However, there remains the terms coming exclusively from the reflected wave where δ is canceled:

$$\frac{m}{q}\int\frac{d^3p}{(2\pi)^3}\delta(y+\mathbf{p}\cdot\hat{\mathbf{q}})\int_I d^3r\int_I d^3r'\Phi_0^*(\mathbf{r})\Phi_0(\mathbf{r}')e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')}e^{2i[\delta(b)-\delta(b')]} .$$

Note that in the scaling limit $k=q+y+0(\mathbf{p}_\perp^2/q)$, i.e., δ becomes independent of $\mathbf{p}_\perp$. As before, the dependence of the illuminated region I on $\mathbf{p}$ may be neglected and the integration over $\mathbf{p}_\perp$ can be performed to yield $b=b'$. Thus the phase δ drops out and the contribution from the reflected wave is identical with the one from the shadow-forming wave, Eq. (2.5b).

Combining the leading hard-core FSI contributions to the scaling function gives

$$\Delta F_0(y)=-\frac{1}{\pi}\mathrm{Re}\int_{b<a}d^2b\int_{-\infty}^{0}dz\int_0^{+\infty}dz'\rho_1(\mathbf{b},z;\mathbf{b},z')e^{iy(z-z')} , \tag{2.7}$$

where

$$\rho_1(\mathbf{r};\mathbf{r}')=\Phi_0^*(\mathbf{r})\Phi_0(\mathbf{r}') \tag{2.8}$$

is the one-body density matrix. If, for simplicity, we assume an s-wave bound state and introduce

$$C_b(y)+iS_b(y)=\int_0^\infty dz\,\Phi_0(b,z)e^{iyz} , \tag{2.9}$$

we can write

$$\Delta F_0(y)=-\int_0^a db\,b[C_b^2(y)-S_b^2(y)] . \tag{2.10}$$

For later use we notice that the preceding results can also be obtained directly from the eikonal Green's function of the interacting system

$$g(E=k^2/2m,\mathbf{r},\mathbf{r}')=-i\frac{m}{k}\Theta(z-z')\times\delta^{(2)}(\mathbf{b}-\mathbf{b}')e^{ik(z-z')}\times\exp\left[-i\frac{m}{k}\int_{z'}^{z}V(\mathbf{b},z'')dz''\right] . \tag{2.11}$$

By postulating that $V(r)\to-i\infty$ inside the hard-core radius one finds that g is zero if the straight line connecting z and z' goes through the hard core. Since the contribution from the regular part of $V(r)$ vanishes in the high-energy limit, Eq. (2.11) can be rewritten as

$$g(E,\mathbf{r},\mathbf{r}')=-i\frac{m}{k}\Theta(z-z')\delta^{(2)}(\mathbf{b}-\mathbf{b}')\times[1-\Theta(a-b)\Theta(z+w)\Theta(w-z')]\times\exp[ik(z-z')] . \tag{2.12}$$

Indeed, substituting Eq. (2.12) into Eq. (2.1b) we obtain in the limit $q\to\infty$

$$S(q,\omega)=\mathrm{Re}\frac{m}{\pi q}\int d^2b\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dz'\Theta(z-z')\Phi_0^*(\mathbf{b},z)\Phi_0(\mathbf{b},z')e^{iy(z-z')}[1-\Theta(a-b)\theta(z)\Theta(-z')] . \tag{2.13}$$

Here we again used the fact that the bound-state wave function vanishes inside the hard core to replace the integration limits $\pm w$ by 0. The "1" inside the square bracket gives the impulse approximation result (by exchanging $z\leftrightarrow z'$ the remaining Θ function can be replaced by $\frac{1}{2}$) and one immediately arrives at Eq. (2.7) for the correction terms $\Delta F_0=q(S-S_{\mathrm{IA}})/m$. An alternative derivation, using the t matrix for hard-core scattering, is given in Appendix A.

Let us discuss some of the properties of the additional contribution (2.10) to the scaling function. First, it is evident that $\Delta F_0(y)$ is *even* in the variable y as is $F_0(y)$. Second, it does not change the sum rule

$$\int_0^\infty d\omega\,S(q,\omega)=\int_{-q/2}^{+\infty}dy\,F(y,q)=1 \tag{2.14}$$

which follows directly from Eq. (2.1a) and must hold for *all* values of q. In particular, for $q\to\infty$ one should have

$$\int_{-\infty}^{+\infty}dy\,\Delta F_0(y)=1-\int_{-\infty}^{+\infty}dy\,F_0(y)=0 .$$

Indeed, from Eq. (2.7) we find

$$\int_{-\infty}^{+\infty}dy\,\Delta F_0(y)=-2\,\mathrm{Re}\int_{b<a}d^2b\int_{-\infty}^{0}dz\int_0^{+\infty}dz'\Phi_0^*(\mathbf{b},z)\Phi_0(\mathbf{b},z')\delta(z-z')=0 .$$

The last identity holds because only the point $z=z'=0$, $b=a$, where the hard-core wave function is zero, can contribute to the integral. An immediate consequence of these properties is that also the energy-weighted sum rule is unchanged because

$$\int_{-\infty}^{+\infty} dy\, \Delta F_0(y) \left[\frac{q^2}{2m} + \frac{q}{m} y \right] = 0 \; .$$

Finally, we expand $\Delta F_0(y)$ for small y and find

$$\Delta F_0(y) = -2 \int_0^a db\, b \left\{ C_b^2(0) - y^2 \left[\int_0^\infty dz\, z^2 \Phi_0(b,z) \int_0^\infty dz' \Phi_0(b,z') + \left[\int_0^\infty dz\, z \Phi_0(b,z) \right]^2 \right] + O(y^4) \right\} . \tag{2.15}$$

Since an s state does not have a node, the coefficient of y^2 in Eq. (2.15) is positive. Therefore, at $y=0$, corresponding to the quasielastic peak $\omega = q^2/2m$, the scaling function is decreased compared to the impulse approximation result (1.3) whereas on both sides it gets enhanced in such a way that the sum rule (2.14) is conserved.

III. MANY-BODY CASE

In the many-body case the response function per particle is given by

$$S(q,\omega) = -\frac{1}{\pi A} \operatorname{Im} \sum_{i,j} \left\langle 0 \left| e^{-i\mathbf{q}\cdot\mathbf{r}_i} \frac{1}{\omega + E_0 - H + i\epsilon} e^{i\mathbf{q}\cdot\mathbf{r}_j} \right| 0 \right\rangle . \tag{3.1a}$$

Here H is the full A-body Hamiltonian containing the hard-core interaction between the particles. Note that due to the explicit $1/A$ factor Eq. (3.1a) is also well defined for an infinite system.

In the high-energy limit only the incoherent part contributes and assuming A identical particles we have

$$S(q,\omega) = -\frac{1}{\pi} \operatorname{Im} \int d^3r_1 d^3r_1' \ldots d^3r_A d^3r_A' \Phi_0^*(\mathbf{r}_1 \ldots \mathbf{r}_A) e^{-i\mathbf{q}\cdot\mathbf{r}_1} g(\omega + E_0, \mathbf{r}_1 \ldots \mathbf{r}_A; \mathbf{r}_1' \ldots \mathbf{r}_A') e^{i\mathbf{q}\cdot\mathbf{r}_1'} \Phi_0(\mathbf{r}_1' \ldots \mathbf{r}_A') \tag{3.1b}$$

which is the generalization of Eq. (2.1b). The geometrical approach to hard-core final-state interaction effects, which has been given in Sec. III for a particle in a potential, may be generalized straightforwardly to the case of many-body systems. This is because in the high-energy limit one can use the fixed scatterer approximation. There the knocked-out particle (1) scatters from (A-1) fixed spectator particles via a many-body potential

$$V = \sum_{j=2}^{A} v_{1j}(\mathbf{r}_1 - \mathbf{r}_j) \; . \tag{3.2}$$

Using the same arguments as in the Sec. II we find that the correction term to the IA arises from shadows and reflections generated by spectators (see Fig. 1). The only difference from the single-particle model of Sec. II is that different constituents may form overlapping shadows and produce multiple reflections (Fig. 2).

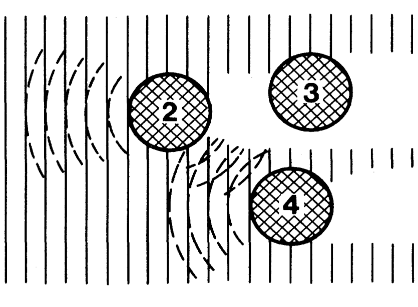

FIG. 2. Scattering wave function for the many-body case. The knocked-out particle (1) scatters from fixed spectator particles (2,3,4) via pure hard-core potentials. The incident plane wave is drawn by solid lines and the reflected waves by dashed lines. The shadows from the particles (2) and (3) are overlapping. The multiple reflection wave generated by particles (2) and (4) is also shown.

Instead of using this geometrical picture it is easier to generalize the result (2.7) by employing the appropriate many-body eikonal Green's function. One merely has to replace the single-particle potential in Eq. (2.11) by the many-body interaction (3.2) and postulate that $V_{1j} \to -i\infty$ for $|\mathbf{r}_1 - \mathbf{r}_j| < a$. Then in the high-energy limit the Green's function g is zero whenever $|\mathbf{b}_1 - \mathbf{b}_j| < a, z_1 - z_j > -w$, and $z_1' - z_j < w$. Elsewhere it is the free eikonal Green's function. Substituting this Green's function into Eq. (3.1b) and subtracting S_{IA} we find the following correction term to the scaling function per particle for the many-body case:

$$\Delta F_0(y)=\frac{1}{\pi}\,\mathrm{Re}\int d^2b_1dz_1dz_1'\Theta(z-z')\int d^3r_2d^3r_3\dots d^3r_A\Phi_0^*(\mathbf{b}_1,z_1,\mathbf{r}_2\dots\mathbf{r}_A)\times\Phi_0(\mathbf{b}_1,z_1',\mathbf{r}_2\dots\mathbf{r}_A)e^{iy(z_1-z_1')}\times\left\{\prod_{j=2}^{A}[1-\Theta(a-|\mathbf{b}_1-\mathbf{b}_j|)\Theta(z_1-z_j+w)\Theta(w-z_1'+z_j)]-1\right\}. \quad (3.3)$$

Note that we can again omit the w in the step functions because the many-body wave function vanishes exactly for relative distances smaller than the hard core. Expanding the product in the usual way we obtain

$$\Delta F_0=\sum_{n=1}^{A-1}\Delta F_0^{(n)}\,, \quad (3.4)$$

where n is the number of spectator nucleons with overlapping shadows. For $n=1$ we have

$$\Delta F_0^{(1)}(y)=-\frac{1}{\pi}(A-1)\,\mathrm{Re}\int d^3r_2\dots d^3r_A\int_{|\mathbf{b}_1-\mathbf{b}_2|<a}d^2b_1\int_{-\infty}^{z_2}dz_1\int_{z_2}^{\infty}dz_1'e^{iy(z_1-z_1')}\times\Phi_0^*(\mathbf{b}_1,z_1,\mathbf{r}_2,\dots,\mathbf{r}_A)\Phi_0(\mathbf{b}_1,z_1',\mathbf{r}_2,\dots,\mathbf{r}_A)$$

$$=-\frac{1}{\pi A}\,\mathrm{Re}\int d^3r_2\int_{|\mathbf{b}_1-\mathbf{b}_2|<a}d^2b_1\int_{-\infty}^{z_2}dz_1\int_{z_2}^{\infty}dz_1'e^{iy(z_1-z_1')}\rho_2(\mathbf{b}_1,z_1,\mathbf{r}_2;\mathbf{b}_1,z_1',\mathbf{r}_2)\,, \quad (3.5)$$

where ρ_2 is the standard two-particle density matrix. Obviously $\Delta F_0^{(1)}$ is the analogue of Eq. (2.7) for the single-particle model. In general, we have

$$\Delta F_0^{(k)}(y)=\frac{(-1)^k}{\pi A k!}\,\mathrm{Re}\int d^3r_2\dots d^3r_{k+1}\times\int_B d^2b_1\int_{-\infty}^{\min(z_2,\dots,z_{k+1})}dz_1\int_{\max(z_2,\dots,z_{k+1})}^{\infty}dz_1'e^{iy(z_1-z_1')}\times\rho_{(k+1)}(\mathbf{b}_1,z_1\mathbf{r}_2,\dots,\mathbf{r}_{k+1};\mathbf{b}_1,z_1',\mathbf{r}_2,\dots,\mathbf{r}_{k+1})\,, \quad (3.6a)$$

where the integration region B is given by

$$B=\{|\mathbf{b}_1-\mathbf{b}_2|<a,\dots,|\mathbf{b}_1-\mathbf{b}_{k+1}|<a\} \quad (3.6b)$$

and $\rho_{(k+1)}$ is the $(k+1)$-body density matrix.

Further discussions of the A-body case and applications to quantum liquids can be found in Ref. 12. See also Ref. 13 for a different approach to the hard-core problem.

IV. HARD-CORE AND ATTRACTIVE PART: A NUMERICAL EXAMPLE

In the preceding sections we have found the asymptotic limit $F_0(y)+\Delta F(y)$ of the reduced structure function $F(y,q)\equiv(q/m)S(q,\omega)$ when the interaction between the constituents has a hard-core repulsive part. However, an intriguing question is *how* $F(y,q)$ does approach the asymptotic limit. One would like to know, for instance, whether $F_0(y)$ alone is the approximate scaling function [i.e., $F(y,q)\simeq F_0(y)$] for intermediate momentum transfers $1/r_0\ll q\ll 1/a$ and scaling to $F_0(y)+\Delta F_0(y)$ occurs only for $q\gg 1/a$ where r_0 is a typical interparticle distance. This would still allow an approximate determination of the momentum distribution at intermediate momentum transfers by means of Eq. (1.3).

In principle, an analytical study of the approach to the different scaling regimes similar to the one in Ref. 5 could answer this question. This is outside the scope of the present paper. Instead we investigate this problem numerically in the simplest soluble model which has two ranges: The particle feels a hard-core potential for $r<a$ and an attractive δ function at $r=r_0$ which only acts in the s wave,

$$V_l=\begin{cases}\infty, & r<a\\ -\gamma\delta(r-r_0)\delta_{l0}, & r>a\end{cases}\,. \quad (4.1)$$

In Appendix B we give analytic expressions for bound and scattering states and details of the numerical evaluation of the response function. Our results for $F(y,q)$ are shown in Figs. 3–6. In these graphs we take as a parameter, the binding energy of the particle instead of the strength γ.

Figure 3 gives for a few parameter sets $\Delta F_0(y)/F_0(y)$. In the range $\bar{y}\equiv ya<0.7$, $\Delta F_0/F_0$ changes sign and its maximum for $\bar{y}<0.9$ does not exceed 20%. This result is somewhat at variance with the one by Weinstein and Negele[6] who obtain for infinite matter a negative correction of 40% for $\bar{y}\sim 0.9$. Extrapolation of their results to smaller $\tilde{y}$ indicates a sign change for $\Delta F_0/F_0$ at a smaller $\tilde{y}$ than we found. The difference may be due to the short range of the attractive potential in our simple model: The momentum distribution of the bound state is quite soft and aside from some oscillations decreases like p^{-6} [cf. Eq. (B8)].

Figure 4 shows the reduced structure function $F(y,q)$ for a particular set of parameters together with the expected scaling function $F_0(y)+\Delta F_0(y)$. For large $|y|$ the convergence to the asymptotic limit is slow. This is made more visible in Figs. 5(a)–5(e) which show a cut through the graph of Fig. 4 at constant (negative) y. For low values of $|y|$ the convergence to the asymptotic limit is rapid but there is practically no distinction between $F_0(y)$ and $F_0(y)+\Delta F_0(y)$. Although the effects of the additional hard-core scaling function $\Delta F_0(y)$ show up at larger values of $|y|$ the approach to the asymptotic function is slow for these kinematical conditions. In neither case is there any clear sign of an intermediate range of momentum transfers where scaling to $F_0(y)$ only occurs. This is corroborated by Figs. 6(a)–6(e) where the particle is assumed to more strongly bound ($E_0=-25$ MeV) than in the previous figure. We now observe some oscillatory behavior as function of the momentum transfer.

It is known[14] that the convergence of $F(y,q)$ to the asymptotic limit depends on the choice of the scaling variable y. For instance, in the nonasymptotic region the scaling variable

$$y \equiv y_0 = -q + \sqrt{2m\omega + E_0} \tag{4.2}$$

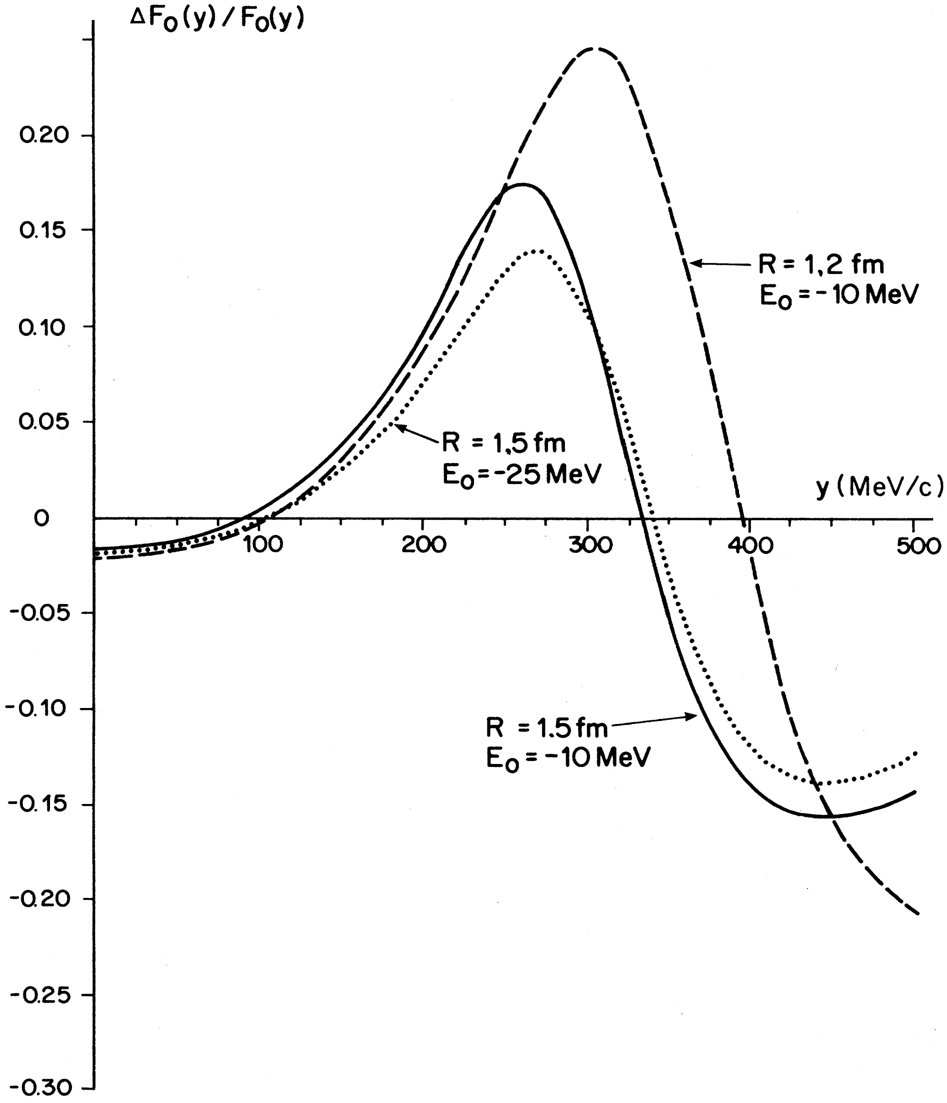

FIG. 3. Effect of the additional hard-core scaling function $\Delta F_0(y)$ relative to the impulse approximation result $F_0(y)$ for the hard-core $+\delta$-function potential model. Results are shown for different ground-state energies E_0 and radii r_0 where the attractive δ function acts. In all cases the hard-core radius was taken as $a=0.4$ fm.

obtained in the plane-wave impulse approximation, and $y \equiv y_{\rm West}$ defined by Eq. (1.1), are quite different [although $y_0 \rightarrow y_{\rm West}$ in the scaling limit (1.2)]. The optimal choice of the scaling variable y which provides the best convergence of $F(y,q)$ to its asymptotic limit depends on the dynamical model (see Refs. 5 and 14). In Fig. 6 we have investigated only two choices, (1) $y = y_{\rm West}$ and (2) $y = y_0$. It is seen that in most cases y_0 gives a much more rapid convergence to the asymptotic limit than $y_{\rm West}$. In fact, already in the region of $q \simeq 1000$ MeV/c, $F(y_0,q)$ reaches its asymptotic limit for small values of $|y_0| < 200$ MeV/c. For larger values of $|y_0|$ (200 MeV/c $< |y_0| < 400$ MeV/c), $F(y_0,q)$ deviates only by about 15% from its asymptotic limit. For a short-range potential the superiority of y_0 as a scaling variable may be expected as the ejected particle spends only a very short time inside the potential range whereas the bound particle always "feels" the attraction. For a long-range potential the time scales are comparable and we have a near cancellation of bound- and final-state interaction effects. Consequently, in this case $y_{\rm West}$ is expected to be the superior scaling variable. Figure 6 also demonstrates that for large $|y_0|$ the approach to the scaling limit is from *above*—a feature also seen in the data.[3]

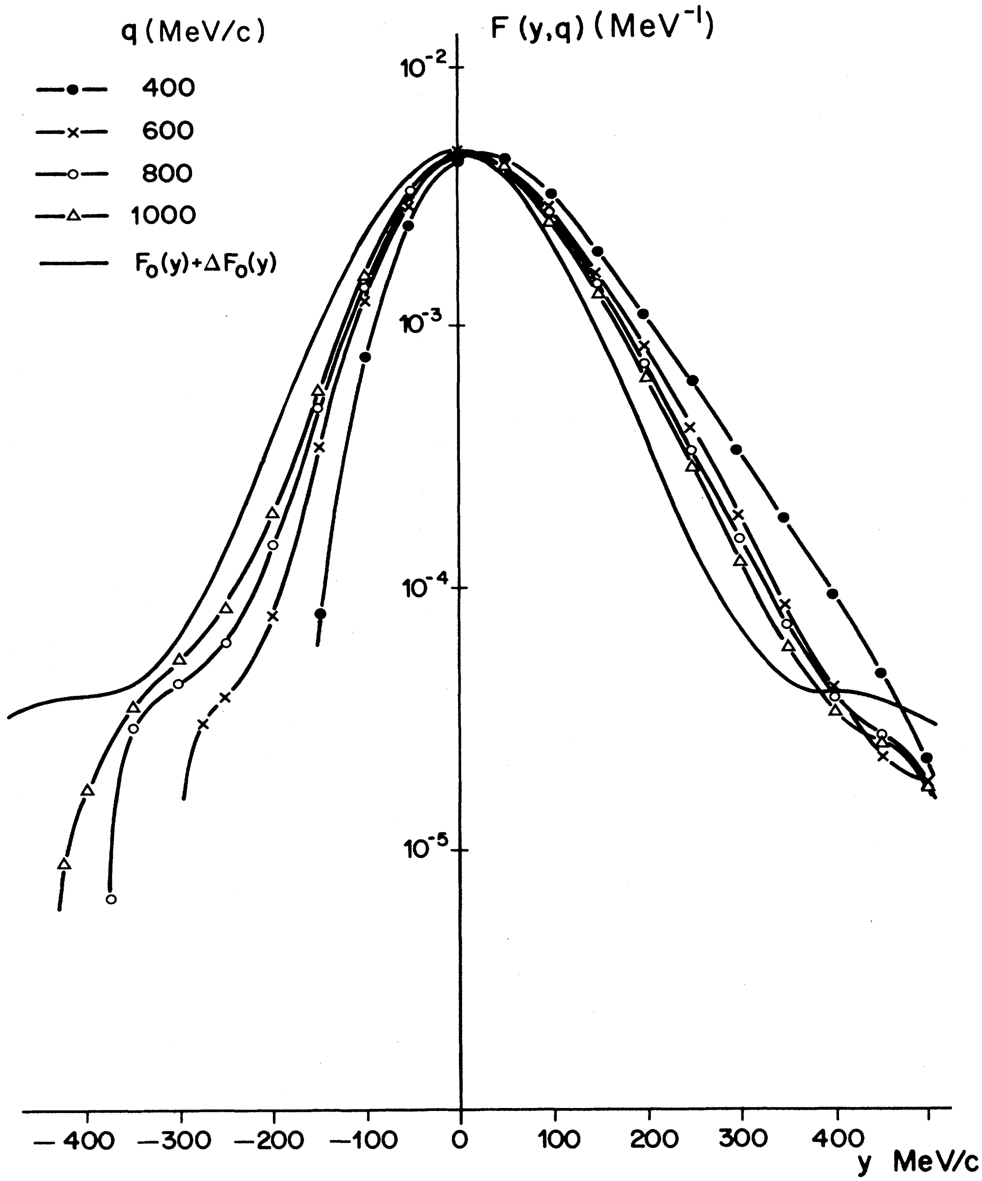

FIG. 4. Reduced structure function $F(y,q)$ at different momentum transfer q as a function of the scaling variable y ($a = 0.4$ fm, $r_0 = 1.5$ fm, $E_0 = -10$ MeV). Also shown is the asymptotic limit $F_0(y) + \Delta F_0(y)$.

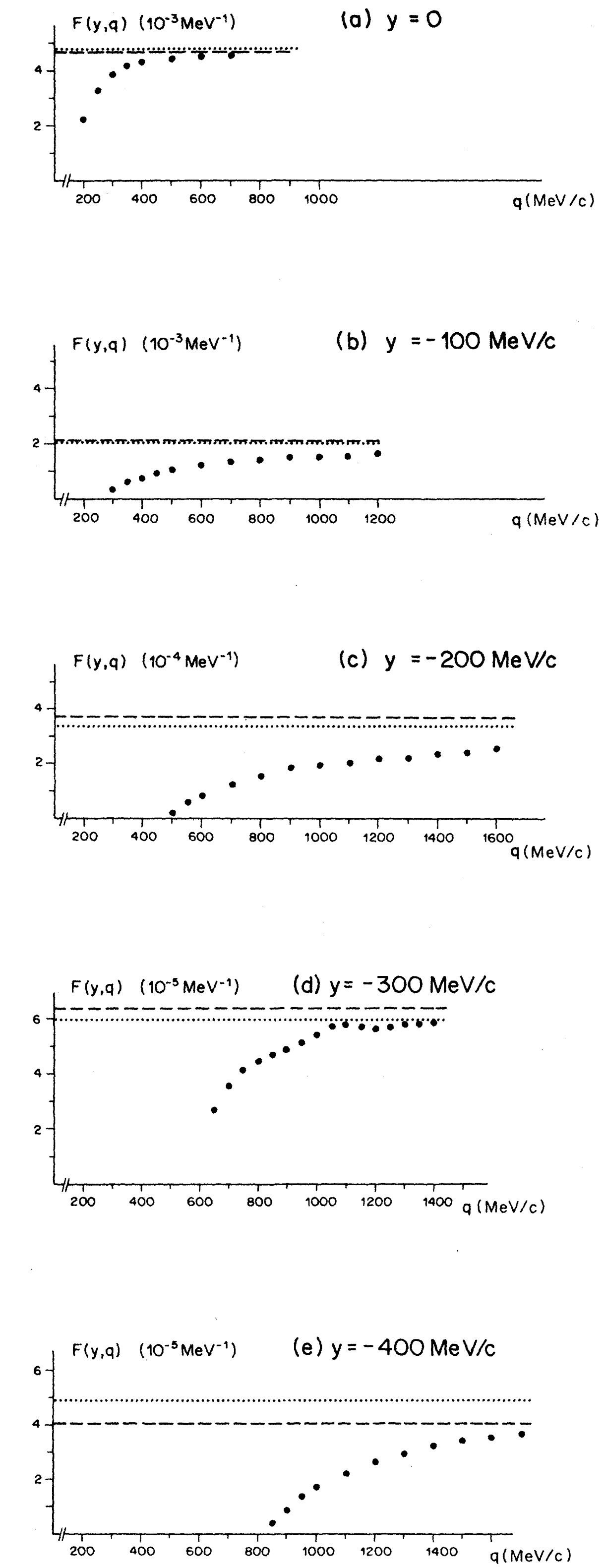

FIG. 5. Approach to scaling for a fixed value of the scaling variable y and for the same parameter set as in Fig. 4. The dotted line is the scaling limit $F_0(y)$, the dashed line $F_0(y)+\Delta F_0(y)$. (a) $y=0$, (b) $y=-100$ MeV, (c) $y=-200$ MeV, (d) $y=300$ MeV, (e) $y=-400$ MeV.

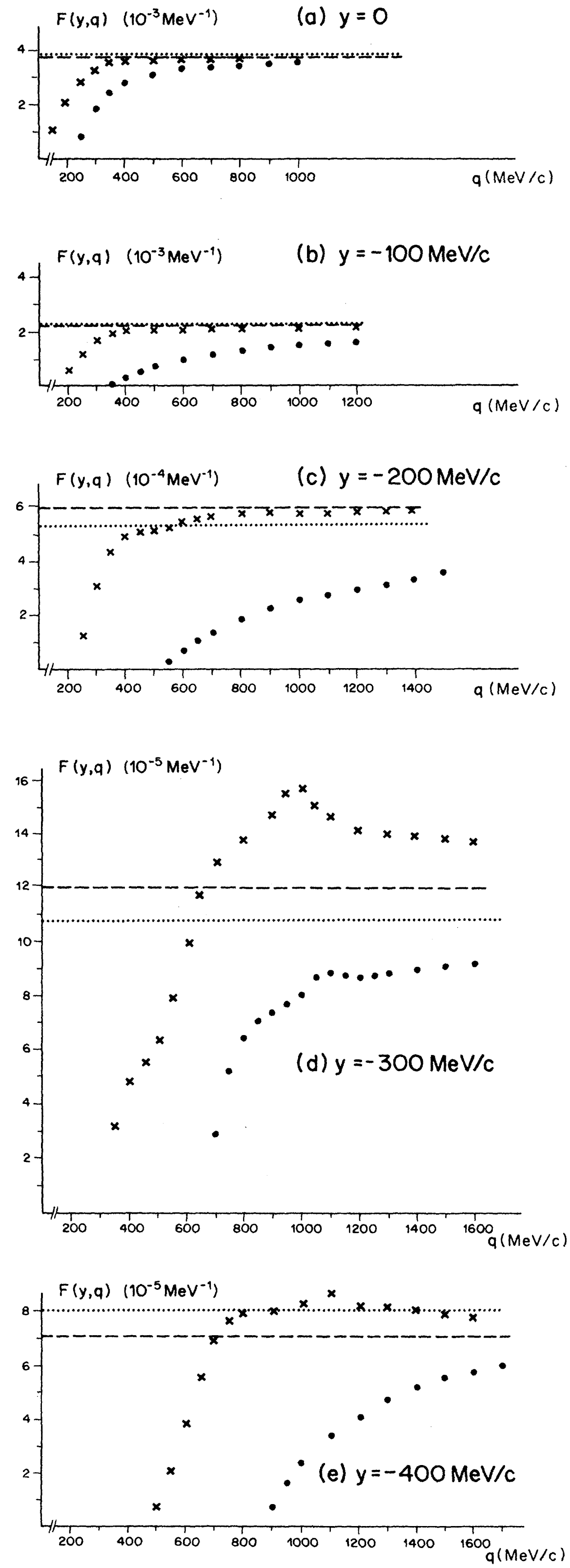

FIG. 6. Same as in Fig. 5 but for the parameter set $a=0.4$ fm, $r_0=1.5$ fm, $E_0=-25$ MeV. The numerical results are plotted by using two (asymptotically equivalent) definitions of the scaling variable $y=y_{\rm West}$ (dots) and $y=y_0$ (crosses).

V. SUMMARY

In this work we have studied the scaling function as the scaling limit (1.2) of the reduced response $F(y,q)$ measured in inclusive scattering. For nonrelativistic Hamiltonian systems, where the particles interact via a regular interaction, that limit is the well-known expression (1.3) in terms of the single-particle momentum distribution $n(p)$. If, however, that interaction contains a hard core, additional terms contribute to the standard scaling function, subsequently complicating the extraction of $n(p)$ from inclusive scattering data.

Our starting point was a model with a particle, bound in a potential. The final-state interaction (FSI) due to the regular part of the potential is known not to contribute in the scaling limit. This is not the case for the repulsive component which need not be a strict hard core, but any strongly repulsive interaction. Since in the scaling limit the energy of the outgoing particle tends to infinity, we applied concepts of geometrical optics to describe the FSI between the knocked-on particle and the potential. An exact expression in closed form, Eq. (2.7), has been given for the correction to the "standard" scaling function (1.3). This result was also obtained by algebraic methods.

Next we generalized the result to the response of a many-body system interacting through potentials with a hard-core component. In the language of optics there are now contributions of multiple reflections and overlapping shadows from different constituents, not unlike those appearing in Glauber theory. The formal expression, Eqs. (3.4)–(3.6), corresponds to those contributions and provides the sought corrections to the impulse approximation result. It is of interest to note that the truly asymptotic is the *sum* of impulse approximation and correction terms. In particular, it defies a representation as a convolution of the former with a (sharply) peaked smearing function—a form advocated by Silver.[13]

In order to investigate the approach of the structure function to the asymptotic hard-core limit, we finally considered a simple, exactly solvable model where a particle moves in a potential composed of an attractive δ function and a hard core. For increasing momentum transfer q the numerical results showed good convergence to the correct scaling function for relatively low y, but for increasing y the convergence becomes poorer. Use of a variable y_0 [Eq. (4.2)], which is equivalent to y [Eq. (1.1)] in the asymptotic limit but differs from y for finite q, considerably improves the convergence at medium y, but is still not satisfactory for large y.

Concerning actual applications for nuclei, the scaling behavior of the reduced response will, in addition, be modified by relativity, pion production, nucleon excitation, etc. Ultimately, also, new subnucleonic degrees of freedom will have to enter the description. Yet we believe that our study is relevant, even for nuclei. First, the results may be generalized for relativistic kinematics, although one has to take into account that the direct relation between the momentum distribution and the scaling function as used in Eq. (1.3) has to be modified for $A \geq 3$.[15] Second, specific effects due to a hard core (i.e., the impenetrability of nucleons) may show up at large momentum transfer and at energy transfers, small enough to allow a description solely in terms of nucleons. Indeed, it would be of interest to detect the predicted hard-core effects in inclusive scattering data, since these would speak in favor of nucleonic degrees of freedom in the relevant q,ω region.

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APPENDIX A: AN ALTERNATIVE DERIVATION OF THE HARD-CORE SCALING RESULT

Here we present another derivation of the additional contribution ΔF_0 [Eq. (2.7)] to the scaling function for hard-core potentials. It is based on Eq. (2.1b) and the following decomposition of the exact Green's function of the target:

$$g(z)=g_0(z)+g_0(z)t(z)g_0(z) \ . \tag{A1}$$

If Eq. (A1) is inserted into Eq. (2.1b), the first term—the free Green's function g_0—yields the impulse approximation whereas the second one—involving the t matrix—gives rise to

$$\Delta S=-\frac{1}{\pi}\,\mathrm{Im}\int \frac{d^3p\,d^3p'}{(2\pi)^6}\frac{\tilde{\Phi}_0^*(\mathbf{p})}{\omega+E_0-(\mathbf{p}+\mathbf{q})^2/2m+i\epsilon} \times\langle \mathbf{p}+\mathbf{q}|t(\omega+E_0)|\mathbf{p}'+\mathbf{q}\rangle \times\frac{\tilde{\Phi}_0(\mathbf{p}')}{\omega+E_0-(\mathbf{p}'+\mathbf{q})^2/2m+i\epsilon} \ . \tag{A2}$$

In the scaling limit (1.2), we find

$$\Delta F_0(y)=-\frac{1}{\pi}\,\mathrm{Im}\int \frac{d^3p\,d^3p'}{(2\pi)^6}\frac{\tilde{\Phi}_0^*(\mathbf{p})}{y-\mathbf{p}\cdot\hat{\mathbf{p}}+i\epsilon}\frac{\tilde{\Phi}_0(\mathbf{p}')}{y-\mathbf{p}'\cdot\hat{\mathbf{q}}+i\epsilon}\lim_{q\to\infty}\frac{m}{q}\left\langle \mathbf{p}+\mathbf{q}\left|t\left[\frac{q^2}{2m}+\frac{q}{m}y\right]\right|\mathbf{p}'+\mathbf{q}\right\rangle \ . \tag{A3}$$

For local bounded potentials the last factor would be $m\tilde{V}(\mathbf{p}-\mathbf{p}')/q$ and $\Delta F(y)$ would become identical with $mF_1(y)/q$, the next-to-leading-order term studied in Ref. 5. This is because the higher-order terms in

$$t\left[\frac{q^2}{2m}\right]=V+V\frac{1}{q^2/2m-H_0+i\epsilon}V+\cdots \tag{A4}$$

are suppressed at high q. However, hard-core potentials can never be considered "small" compared to the incident energy: As is well known, the scattering phase shifts in such a potential grow linearly with momentum, as does the imaginary part of the forward amplitude.[7] As a consequence, the dominant final-state interaction in the scaling limit is of the same order as the leading term $F_0(y)$.

It is tempting to use the on-shell forward scattering amplitude in Eq. (A3) but one has to be more careful since the off-shell terms are also linear in q. We thus use the explicit form of the fully off-shell hard-core t matrix[16]

$$\langle\mathbf{k}'|t(z)|\mathbf{k}\rangle=(4\pi a)^2\left[-\left[z-\frac{\mathbf{k}\cdot\mathbf{k}'}{2m}\right]\frac{j_l(Qa)}{4\pi Q}-\frac{1}{2m}\sum_{l,m}j_l(ka)j_l(k'a)\frac{\partial}{\partial a}\times\ln h_l(\sqrt{2mz}\,a)\times Y_{lm}(\hat{\mathbf{k}})Y^*_{lm}(\hat{\mathbf{k}}')\right]. \tag{A5}$$

Here $j_{l,h_l^{(+)}}$ are the spherical Bessel and Hankel functions, respectively, and $\mathbf{Q}=\mathbf{k}-\mathbf{k}'$. It is convenient to split up Eq. (A5) into on- and off-shell parts,

$$\langle\mathbf{k}'|t(z)|\mathbf{k}\rangle=-\frac{4\pi}{2m}\left[\frac{K^2}{kk'}f^{(\mathrm{on})}(z;\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')+\Delta_1+\Delta_2\right], \tag{A6}$$

where $K=\sqrt{2mz}$, and

$$f^{(\mathrm{on})}(z;\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')=\frac{1}{2iK}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l(K)}-1)P_l(\mathbf{k}\cdot\mathbf{k}') \tag{A7a}$$

is the on-shell scattering amplitude with hard-core phase shifts

$$\delta_l(K)=-\arg h_l^{(+)}(Ka)\ . \tag{A7b}$$

The off-shell parts may be written as

$$\Delta_1=a^2\left[K^2-\frac{k^2+k'^2}{2}\right]\frac{j_1(Qa)}{Q}\ , \tag{A8a}$$

$$\Delta_2=\frac{1}{kk'}\left[iK-\frac{1}{2}\frac{\partial}{\partial a}\right]\sum_{l=0}^{\infty}(2l+1)\times\frac{\hat{j}_l(ka)\hat{j}_l(k'a)-\hat{j}_l^2(Ka)}{\hat{j}_l^2(Ka)+\hat{n}_l^2(Ka)}\times P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\ , \tag{A8b}$$

where $n_l(x)$ is the spherical Neumann function[17] and $\hat{\mathbf{j}}_l(x)=xj_l(x)$, etc. Now we substitute Eq. (A6) into Eq. (A3) by taking $K=(q^2+2qy)^{1/2}\simeq q+y$ and $\mathbf{k}=\mathbf{q}+\mathbf{p}$, $\mathbf{k}'=\mathbf{q}+\mathbf{p}'$, so that $Q=|\mathbf{p}-\mathbf{p}'|$. Taking into account that partial waves up to $l_{\max}\simeq qa$ contribute, we see that Δ_1 is linear in q,

$$\Delta_1=a^2[2qy-(\mathbf{p}+\mathbf{p}')\cdot\mathbf{q}]\frac{j_1(Qa)}{Q}\ , \tag{A9}$$

whereas Δ_2 is at most $O(1)$ (since the leading term in the numerator is canceled).

We first demonstrate that Δ_1 does not contribute to the scaling function. Indeed using $\mathrm{Im}(y-\mathbf{p}\cdot\hat{\mathbf{q}}+i\epsilon)^{-1}=-\pi\delta(y-\mathbf{p}\cdot\hat{\mathbf{q}})$ we obtain

$$\Delta F_0^{\mathrm{off}}(y)=-a^2\int\frac{d^3p\,d^3p'}{(2\pi)^5}\tilde{\Phi}_0^*(\mathbf{p})\tilde{\Phi}_0(\mathbf{p}')\times[\delta(y-\mathbf{p}\cdot\hat{\mathbf{q}})+\delta(y-\mathbf{p}'\cdot\hat{\mathbf{q}})]\frac{j_1(aQ)}{Q}\ . \tag{A10}$$

Transforming into $\mathbf{r}$ space by means of

$$\frac{j_1(aQ)}{Q}=\frac{1}{4\pi a^2}\int_{x\le a}d^3x\,e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')} \tag{A11}$$

and using the fact that the bound-state wave function $\Phi_0(\mathbf{x})$ is identically zero inside the hard core, we have

$$\Delta F_0^{\mathrm{off}}(y)=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}ds\cos sy\int_{x\le a}d^3x\,\Phi_0^*(\mathbf{x})\times\Phi_0(\mathbf{x}+s\hat{\mathbf{q}})=0\ .$$

Thus we only have to consider the on-shell contribution in Eqs. (A3) and (A6). The high-energy on-shell amplitude for small scattering angles is given by[9]

$$f(\Theta)\simeq\tfrac{1}{2}ia\cot\frac{\Theta}{2}J_1(aq\sin\Theta)\simeq iaq\frac{J_1(a|\mathbf{p}_\perp-\mathbf{p}'_\perp|)}{|\mathbf{p}_\perp-\mathbf{p}'_\perp|}\ , \tag{A12}$$

where we used

$$\Theta=\arccos(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\simeq\frac{|\mathbf{p}_\perp-\mathbf{p}'_\perp|}{q}+O\left[\frac{1}{q^2}\right]. \tag{A13}$$

Using the two-dimensional analogue of Eq. (A11)

$$\frac{J_1(aQ_\perp)}{Q_\perp}=\frac{1}{2\pi a}\int_{b\le a} d^2b\, e^{i\mathbf{b}\cdot(\mathbf{p}_\perp-\mathbf{p}'_\perp)} \tag{A14}$$

we obtain from Eq. (A4)

$$\Delta F_0^{\rm on}(y)=\frac{1}{\pi}\,\mathrm{Re}\int \frac{d^3p\,d^3p'}{(2\pi)^6}\frac{\tilde{\Phi}_0^*(\mathbf{p})\tilde{\Phi}_0(\mathbf{p}')}{(y-\mathbf{p}\cdot\hat{\mathbf{q}}+i\epsilon)(y-\mathbf{p}'\cdot\hat{\mathbf{q}}+i\epsilon)} \times\int_{b\le a} d^2b\, e^{i\mathbf{b}\cdot(\mathbf{p}_\perp-\mathbf{p}'_\perp)} . \tag{A15}$$

If the denominators in Eq. (A15) are written as

$$\frac{1}{y-\mathbf{p}\cdot\hat{\mathbf{q}}+i\epsilon}=-i\int_0^\infty ds\, e^{is(y-\mathbf{p}\cdot\hat{\mathbf{q}})} ,$$

the $\mathbf{p},\mathbf{p}'$ integrations can be performed and we obtain

$$\Delta F_0(y)=-\frac{1}{\pi}\,\mathrm{Re}\int_{b\le a}\int_0^\infty ds\,ds'e^{iy(s+s')} \times\Phi_0(\mathbf{b}-s\hat{\mathbf{q}})\Phi_0(\mathbf{b}+s'\hat{\mathbf{q}}) , \tag{A16}$$

which is Eq. (2.7) if we replace $s\to -s$.

The foregoing derivation of $\Delta F_0(y)$ showed that the off-shell hard-core t matrix does not contribute for a single particle moving in a potential. An analogous approach to the many-body case would be much more involved since the off-shell contribution does not vanish in those multiple-scattering terms which arise from overlapping shadows.

APPENDIX B: INCLUSIVE SCATTERING FOR THE HARD-CORE +δ-FUNCTION POTENTIAL

We calculate the structure function for the model (4.1) by using the partial-wave expansion[18]

$$S(q,\omega)=8m\,\Theta(\omega+E_0)k\sum_{l=0}^{\infty}(2l+1)R_l^2 . \tag{B1}$$

Here the radial integral is given by

$$R_l=\int_0^\infty dr\, r^2\Phi_0(r)j_l(qr)\frac{y_l(k,r)}{kr} , \tag{B2}$$

where the scattering wave function obeys the boundary condition

$$y_l(k,r)\to \sin\left[kr-l\frac{\pi}{2}+\delta_l(k)\right] \quad \text{for } r\to\infty . \tag{B3}$$

The momentum k of the scattered wave is

$$k=\sqrt{2m(\omega+E_0)} . \tag{B4}$$

We first discuss the bound-state problem in our model: The dimensionless strength $\gamma>0$ determines the binding energy of the s state from the eigenvalue equation

$$\coth[\alpha(r_0-a)]=-1+\frac{2m\gamma}{\alpha} , \tag{B5}$$

where $\alpha=\sqrt{-2mE_0}$. There exists only a solution if $2m\gamma>(r_0-a)^{-1}$. The bound-state function is

$$\Phi_0(r)=\frac{C}{r}\Theta(r-a)\begin{cases}\dfrac{\sinh\alpha(r-a)}{\sinh\alpha(r_0-a)} & a\le r\le r_0\\ \exp[-\alpha(r-r_0)] & r\ge r_0 ,\end{cases} \tag{B6}$$

where the normalization C is determined from $4\pi\int_a^\infty dr\, r^2\Phi_0^2(r)=1$. We find

$$C^2=\frac{\alpha^2}{2\pi[A-(r_0-a)B^2]} , \tag{B7}$$

where $A=2m\gamma$ and $B=\alpha/\sinh\alpha(r_0-a)$. Note that from Eq. (B5) these constants are related by $B^2=A^2-2A\alpha$. The momentum distribution of the bound particle is

$$n(p)=\frac{2C^2}{\pi p^2}\left[\frac{A\sin pr_0-B\sin pa}{\alpha^2+p^2}\right]^2 . \tag{B8}$$

Since the δ function is assumed to act only in the s state, the scattering wave functions for $l>0$ are just pure hard-core scattering states,

$$y_l(k,r)=\Theta(r-a)[\cos\delta_l(k)\hat{j}_l(kr)+\sin\delta_l(k)\hat{n}_l(kr)] , \tag{B9}$$

where the phase shifts are defined in Eq. (A7b). For $l=0$ we find

$$y_0(k,r)=\Theta(r-a)\begin{cases}\dfrac{\sin(kr_0+\delta_0)}{\sin k(r_0-a)}\sin k(r-a) & a\le r\le r_0\\ \sin(kr+\delta_0) & r\ge r_0\end{cases} \tag{B10}$$

with

$$\tan(kr_0+\delta_0)=\frac{1}{\cot k(r_0-a)-2m\gamma/k} . \tag{B11}$$

We evaluate the radial integral (B2) by using the integral relation

$$\hat{h}_l^{(+)}(kr)\hat{j}_l(qr)=\tfrac{1}{2}r\int_{k-q}^{k+q} ds\, e^{irs}P_l\left[\frac{q^2+k^2-s^2}{2kq}\right] , \tag{B12}$$

where $P_l(x)$ is the lth-order Legendre poynomial and $\hat{h}_l^{(+)}=\hat{n}_l+i\hat{j}_l$. For $l\neq 0$ we obtain

$$R_l=-\frac{1}{2kq}\,\mathrm{Re}\int_{k-q}^{k+q} ds\, P_l\left[\frac{q^2+k^2-s^2}{2kq}\right]e^{i\delta_l(k)}\epsilon_0'(s) , \tag{B13}$$

where

$$\epsilon_0(s)=C_0(s)+iS_0(s) \tag{B14}$$

and $C_0(s),S_0(s)$ are defined in Eq. (2.9). For the bound-state wave function (B6) we find

$$\epsilon_0(s)=\frac{C}{\alpha}\{\tfrac{1}{2}Bg^{(-)}(\alpha,s)+\alpha e^{\alpha r_0}E_1[r_0(\alpha-is)]\} , \tag{B15}$$

where $E_1(z)$ is the exponential integral.[19] In addition, we have defined

$$g^{(\pm)}(\alpha,s)=e^{-\alpha a}\{E_1[-a(\alpha+is)]-E_1[-r_0(\alpha+is)]\}\pm e^{\alpha a}\{E_1[a(\alpha-is)]-E_1[r_0(\alpha-is)]\}\ . \quad \text{(B16)}$$

For $l=0$ the radial integral is

$$R_0=\frac{1}{2kq}\frac{C}{\alpha}\,\mathrm{Re}\left[\frac{1}{2}B\frac{\sin(kr_0+\delta_0)}{\sin k(r_0-a)}e^{-ika}\times[g^{(-)}(\alpha,k-q)-g^{(-)}(\alpha,k+q)]+\alpha e^{\alpha r_0+i\delta_0}(E_1\{r_0[\alpha-i(k-q)]\}-E_1\{r_0[\alpha-i(k+q)]\})\right]\ . \quad \text{(B17)}$$

To speed up convergence the partial-wave sum is evaluated in the following way: We write

$$F(y,q)=8kq\sum_{l=0}^{\infty}(2l+1)R_l^2=8kq\sum_{l=0}^{\infty}(2l+1)R_l^{(0)^2}+8kq\sum_{l=0}^{\infty}(2l+1)[R_l^2-R_l^{(0)^2}]\ , \quad \text{(B18)}$$

where

$$R_l^{(0)}=-\frac{1}{2kq}\int_{k-q}^{k+q}ds\,P_l\left[\frac{q^2+k^2-s^2}{2kq}\right]C_0'(s) \quad \text{(B19)}$$

is obtained from Eq. (B13) by setting $\delta_l(k)=0$. The first term in Eq. (B18) may be summed by the completeness relation of the Legendre polynomials, and gives

$$8kq\sum_{l=0}^{\infty}(2l+1)R_l^{(0)^2}=4\int_{|k-q|}^{k+q}ds\frac{1}{s}C'^2_0(s)=2\pi\int_{|k-q|}^{k+q}ds\,sn(s)\ . \quad \text{(B20)}$$

This just the IA result and approaches (1.3) in the scaling limit.

We have numerically evaluated the radial integrals (B13) and (B19) as well as Eq. (B20). The exponential integral was calculated either by its power-series expansion for small z or by its representation as continued fraction for large z (Ref. 19 , p. 229). As a check of the numerical procedure we evaluated the sum rule Eq. (2.11) which also requires the elastic form factor

$$f(q)=\frac{4\pi}{q}\frac{C^2}{\alpha^2}\,\mathrm{Im}\{\tfrac{1}{4}B^2[g^{(+)}(2\alpha,q)-g^{(+)}(0,q)]+\alpha^2e^{2\alpha r_0}E_1[r_0(2\alpha-iq)]\}\ . \quad \text{(B21)}$$

When adding $f^2(q)$ we obtained, for instance, for the parameter set $a=0.4$ fm, $r_0=1.5$ fm, $E_0=-10$ MeV, and a fixed upper limit $Y=1200$ MeV/c,

$$\int_{-q/2}^{Y}F(y,q)dy=0.9965,0.9924,0.9935\ ,\quad 0.9964,0.9978,0.9984\ .$$

The numbers on the right correspond to $q=100$, (100), 600 MeV/c.

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$$\Phi^s(\mathbf{r})=\frac{a^2}{2m}\int d\Omega'\left[e^{i\mathbf{k}\cdot\mathbf{r}'}\frac{\partial}{\partial r'}g_0(\mathbf{r},\mathbf{r}')+g_0(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial r'}\Phi^s(\mathbf{r}')\right]_{r'=a}$$

if the eikonal Green's function with propagation direction along sgn$(z')\mathbf{k}$ is used for g_0.

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