Description of Elastic Scattering Processes with the Use of Nonlocal Separable Potentials

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This paper is devoted to the study of elastic scattering processes with the use of nonlocal separable interactions. An explicit and general expression for the transition amplitude has been given which includes Coulomb effects and possible excitation of the particles involved. As a first application of the formalism it is shown that a striking improvement of Mongan's results can be obtained in p-p elastic scattering just by including Coulomb effects at low energy. We have then studied the nucleon-α elastic scattering below 50 MeV and the α-α elastic scattering below 100 MeV, and the results obtained are in very good agreement with experimental data.

I. INTRODUCTION

Nonlocal separable potentials have been extensively used in the description of two-body interactions, and the main reason for such a choice is their extreme convenience and simplicity. They provide, for instance, a simple analytical form of the transition amplitude in elastic scattering phenomena, and for this reason have been used in the description of the nucleon-nucleon interaction.[1-5] In connection with this, we have tried to see if nonlocal separable potentials could describe the more complicated n-α, p-α, or α-α interactions. In these cases, Coulomb effects or the internal structure of the α particle may play a prominent role. Of course, the chosen nonlocal separable potentials must fulfill the usual conditions of time-reversal and rotational invariance, Hermiticity, and correct threshold behavior. We have then shown that Coulomb effects could be included in the process just by expressing the nonlocal separable potential in a "Coulomb representation" rather than in the usual "impulse representation." In other words, an integral transformation leads to a new form factor which took into account Coulomb effects. On the other hand, the internal structure of the α particle involved has been described by considering the different possible states of the α particle and essentially its lowest inelastic threshold.

In the second section an exact and analytical expression of the transition amplitude in any elastic scattering process is established. The third part is devoted to the choice of the potential shapes even when the Coulomb field is present. In Sec. IV we show how the discrepancy appearing at low energy in the p-p elastic scattering between the experimental data and the calculated phase shifts can be removed even when the form factors and parameters introduced by Mongan[1] are used.

In Sec. V, our formalism has been applied to the description of nucleon-α elastic scattering below 50 MeV, while Sec. VI is devoted to an α-α elastic scattering study below 100 MeV. In all particular cases considered here, form factors and parameters have been determined through a phase-shift expansion of the scattering amplitude.

II. TRANSITION AMPLITUDE

A. General Expression

We consider the scattering of a structureless particle on a target, the eventual structure of which is described by a Hamiltonian h_α. Its eigenstates and eigenvalues will be defined by the relation

$$h_\alpha |\varphi_{\alpha i}\rangle = \epsilon_{\alpha i} |\varphi_{\alpha i}\rangle . \tag{1}$$

The total Hamiltonian H may be decomposed in three different ways

$$H = h_\alpha + H_0 + V_C + V_N = H_{\alpha C} + V_N = h_\alpha + H_C + V_N . \tag{2}$$

The operators V_C and V_N denote, respectively, the Coulomb and nuclear interactions, while H_0 is the usual kinetic energy operator, the eigenstates of which are denoted by $|\vec{K}\rangle$:

$$H_0 |\vec{K}\rangle = (\hbar^2 K^2 / 2m^*) |\vec{K}\rangle , \quad \frac{1}{m^*} = \frac{1}{m_{\text{incident}}} + \frac{1}{m_{\text{target}}} . \tag{3}$$

The Hamiltonian H_C describes pure Coulomb scattering:

$$H_C = H_0 + V_C , \quad H_C |\chi_K^\pm\rangle = \epsilon_K |\chi_K^\pm\rangle , \tag{4}$$

and we introduce the Hamiltonian $H_{\alpha C} = h_\alpha + H_C$ with its eigenstates

$$H_{\alpha C} |\varphi_{\alpha i}, \chi_K^\pm\rangle = \epsilon_{\alpha i K} |\varphi_{\alpha i}, \chi_K^\pm\rangle \tag{5}$$

$$\epsilon_{\alpha i K} = \epsilon_{\alpha i} + \epsilon_K . \tag{6}$$

The transition amplitude is now expressed as

$$T = T_C + T_N = \sum_{S_I\sigma_I S_F\sigma_F} \langle S_F\sigma_F, \varphi_{\alpha F}, \vec{K}_F | V_C | S_I\sigma_I, \varphi_{\alpha I}, \chi_I^+\rangle + \langle S_F\sigma_F, \varphi_{\alpha F}, \chi_F^- | V_N | \psi_I^+\rangle , \tag{7}$$

where $S_I\sigma_I$ and $S_F\sigma_F$ denote the initial and final spin states, while $|\psi_I^+\rangle$ is solution of the total Hamiltonian H, solution of the following equation

$$|\psi_I^+\rangle = \sum_{S_I\sigma_I} |S_I\sigma_I, \varphi_{\alpha I}, \chi_I^+\rangle + \frac{1}{E - H_{\alpha C} + i\epsilon} V_N |\psi_I^+\rangle . \tag{8}$$

The transition amplitude T_N is now projected onto the common eigenstates $|\vec{K}, \varphi_{\alpha i}\rangle$ of H_0 and h_α which form a complete basis

$$T_N = \sum_{S_F\sigma_F} \mathbf{S}_{ij} \iint \langle S_F\sigma_F, \varphi_{\alpha F}, \chi_F^- | \vec{K}, \varphi_{\alpha i}\rangle d\vec{K} \langle \vec{K}, \varphi_{\alpha i} | V_N | \vec{K}', \varphi_{\alpha j}\rangle d\vec{K}' \langle \vec{K}', \varphi_{\alpha j} | \psi_I^+\rangle . \tag{9}$$

If the internal Hamiltonian h_α of the target has a purely discrete spectrum, $\mathbf{S}_{ij}$ will represent a discrete summation, while for a continuous spectrum it will denote an integration on the eigenstates $|\varphi_{\alpha i}\rangle$ and $|\varphi_{\alpha j}\rangle$.

The nuclear potential V_N is chosen to be nonlocal separable with p terms, and the decomposition of its matrix elements on a $\mathcal{Y}_{(LS)J\mu}$ basis leads to the following:

$$\langle \vec{K}, \varphi_{\alpha i} | V_N | \vec{K}', \varphi_{\alpha j}\rangle = \sum_{p l l' S J\mu} i^{l'-l} C_{ll'p}^{JS} g_{lip}^{JS}(K) g_{l'jp}^{JS}(K') i^l \mathcal{Y}_{(lS)J\mu}(\hat{K}) i^{-l'} \mathcal{Y}_{(l'S)J\mu}^*(\hat{K}') . \tag{10}$$

The $\mathcal{Y}_{(lS)J\mu}(\hat{K})$ are defined as usual by the relation

$$\mathcal{Y}_{(lS)J\mu}(\hat{K}) = \sum_{m\sigma} \langle lmS\sigma | J\mu\rangle Y_{lm}(\hat{K}) | s\sigma\rangle , \tag{11}$$

where $\vec{J} = \vec{l} + \vec{S}$ and $\vec{S}$ is the total spin of the interacting particles, $\vec{S} = \vec{S}_1 + \vec{S}_2$. The i^l factor ensures the time-reversal invariance of the spherical harmonics.

We use the above decomposition (10) in expression (9) of the transition amplitude and thus obtain

$$T_N = \mathbf{S}_{ij} \sum_{ll'pSJ\mu} C_{ll'p}^{JS} (I_F^-)_{lSip}^{J\mu} (J_I^+)_{l'Sjp}^{J\mu} . \tag{12}$$

We have set

$$(I_F^-)_{lSip}^{J\mu} = \sum_{S_F\sigma_F} \int \langle S_F\sigma_F, \varphi_{\alpha F}, \chi_F^- | \vec{K}, \varphi_{\alpha i}\rangle g_{lip}^{JS}(K) \mathcal{Y}_{(lS)J\mu}(\hat{K}) d\vec{K} , \tag{13}$$

$$(J_I^+)_{lSjp}^{J\mu} = \int g_{l'jp}^{JS}(K') \mathcal{Y}_{(l'S)J\mu}^*(\hat{K}') \langle \vec{K}', \varphi_{\alpha j} | \psi_I^+\rangle d\vec{K}' . \tag{14}$$

B. Evaluation of $\langle K', \varphi_{\alpha j} | \psi_I^+\rangle$

Using the definition (8) of $|\psi_I^+\rangle$ we get

$$\langle \vec{K}', \varphi_{\alpha j} | \psi_I^+\rangle = \sum_{S_I\sigma_I} \langle \vec{K}', \varphi_{\alpha j} | S_I\sigma_I, \varphi_{\alpha I}, \chi_I^+\rangle + \langle \vec{K}', \varphi_{\alpha j} | \frac{1}{E - H_{\alpha C} + i\epsilon} V_N | \psi_I^+\rangle . \tag{15}$$

The purely repulsive Coulomb potential has no bound state and thus the scattering states of the Hamiltonian H_C form a complete basis[6] as well as the eigenstates $|\chi_K^+, \varphi_{\alpha k}\rangle$ of the Hamiltonian $H_{\alpha C}$ on which the above relation can be projected

$$\langle K', \varphi_{\alpha j} | \psi_I^+\rangle = \sum_{S_I\sigma_I} \langle \vec{K}', \varphi_{\alpha j} | S_I\sigma_I, \varphi_{\alpha I}, \chi_I^+\rangle + \frac{2m^*}{\hbar^2} \mathbf{S}_k \int \langle \vec{K}', \varphi_{\alpha j} | \chi_K^+, \varphi_{\alpha k}\rangle \frac{1}{K_E^2 - K^2 - K_{\alpha k}^2 + i\epsilon} \langle \chi_K^+, \varphi_{\alpha k} | V_N | \psi_I^+\rangle d\vec{K} . \tag{16}$$

We have set

$$K_E^2 = (2m^*/\hbar^2)E \tag{17}$$

$$K^2 + K_{\alpha k}^2 = (2m^*/\hbar^2)\epsilon_{\alpha k K} . \tag{18}$$

We introduce a new projection on the $|\vec{K}, \varphi_{\alpha k}\rangle$ basis to make the nonlocal matrix element of the nuclear interaction appear,

$$\langle \chi_K^+, \varphi_{\alpha k} | V_N | \psi_I^+ \rangle = \mathop{\mathrm{S}}_{k'k''} \int \langle \chi_K^+, \varphi_{\alpha k} | \vec{K}, \varphi_{\alpha k'} \rangle d\vec{K}' \langle \vec{K}', \varphi_{\alpha k'} | V_N | \vec{K}'', \varphi_{k''} \rangle d\vec{K}'' \langle \vec{K}'', \varphi_{\alpha k''} | \psi_I^+ \rangle . \tag{19}$$

With the use of relation (10) we get the following:

$$\langle \chi_K^+, \varphi_{\alpha k} | V_N | \psi_I^+ \rangle = \mathop{\mathrm{S}}_{k'k''} \sum_{\substack{LL'S'q \\ J'\mu'}} C_{LL'q}^{J'S'} \int \langle \chi_K^+, \varphi_{\alpha k} | \vec{K}', \varphi_{\alpha k'} \rangle g_{L'k'q}^{J'S'}(K') \mathcal{Y}_{(LS')J'\mu'}(\hat{K}') d\vec{K}'$$
$$\times \int g_{L'k''q}^{J'S'}(K'') \mathcal{Y}^*_{(L'S')J'\mu'}(\hat{K}'') \langle \vec{K}'', \varphi_{\alpha k''} | \psi_I^+ \rangle d\vec{K}'' . \tag{20}$$

Let us use the orthogonality properties of the $|\varphi_{\alpha i}\rangle$ eigenstates and the definition (14) of $(J_I^+)_{L'S'k''q}^{J'\mu'}$ to write $\langle \vec{K}', \varphi_{\alpha j} | \psi_I^+ \rangle$ in the form

$$\langle \vec{K}', \varphi_{\alpha j} | \psi_I^+ \rangle = \sum_{S_I \sigma_I} \langle \vec{K}', \varphi_{\alpha j} | S_I \sigma_I, \varphi_{\alpha I}, \chi_I^+ \rangle + \frac{2m^*}{\hbar^2} \mathop{\mathrm{S}}_{k''} \sum_{\substack{LL'S'q \\ J'\mu'}} C_{LL'q}^{J'S'} (J_I^+)_{L'S'k''q}^{J'\mu'}$$
$$\times \int \langle \chi_K^+ | \vec{K}'' \rangle \langle \vec{K}' | \chi_K^+ \rangle g_{Ljq}^{J'S'}(\hat{K}'') \mathcal{Y}_{(LS')J'\mu'}(\hat{K}'') \frac{1}{K_E{}^2 - K^2 - K_{\alpha j}{}^2 + i\epsilon} d\vec{K} d\vec{K}'' . \tag{21}$$

We can express $\langle \vec{K}' | \chi_K^\pm \rangle$ in the configuration space

$$\langle \vec{K}' | \chi_K^\pm \rangle = \int \langle \vec{K}' | \vec{r} \rangle d\vec{r} \langle \vec{r} | \chi_K^\pm \rangle . \tag{22}$$

We decompose $\langle \vec{K}' | \vec{r} \rangle$ and $\langle \vec{r} | \chi_K^\pm \rangle$ on a spherical basis

$$\begin{aligned} \langle \vec{r} | \chi_K^\pm \rangle &= (2\pi)^{-3/2} \frac{4\pi}{K^2} \sum_{\lambda\nu} i^\lambda e^{\pm i \sigma_\lambda} F_\lambda(Kr) Y^*_{\lambda\nu}(\hat{K}) Y_{\lambda\nu}(\hat{r}) , \\ \langle \vec{K}' | \vec{r} \rangle &= 4\pi (2\pi)^{-3/2} \sum_{\lambda'\nu'} i^{-\lambda'} j_{\lambda'}(K'r) Y_{\lambda'\nu'}(\hat{K}') Y^*_{\lambda'\nu'}(\hat{r}) . \end{aligned} \tag{23}$$

The Coulomb phase shift σ_λ is defined by

$$\sigma_\lambda = \mathrm{Arg}\Gamma(\lambda + 1 - i\eta) , \tag{24}$$

where $\eta = m^* Z_1 Z_2 / \hbar^2 K$ is the Coulomb parameter. If we introduce the integral transform $F_\lambda(KK')$ of $F_\lambda(Kr)$

$$F_\lambda(KK') = \int \frac{F_\lambda(Kr)}{Kr} j_\lambda(K'r) r^2 dr , \tag{25}$$

we get

$$\langle \vec{K}' | \chi_K^\pm \rangle = \frac{2}{\pi} \sum_{\lambda\nu} e^{\pm i \sigma_\lambda} F_\lambda(KK') Y_{\lambda\nu}(\hat{K}') Y^*_{\lambda\nu}(\hat{K}) \tag{26}$$

and thus the following expression for $\langle \vec{K}', \varphi_{\alpha j} | \psi_I^+ \rangle$:

$$\langle \vec{K}', \varphi_{\alpha j} | \psi_I^+ \rangle = \sum_{S_I \sigma_I} \langle \vec{K}', \varphi_{\alpha j} | S_I \sigma_I, \varphi_{\alpha I}, \chi_I^+ \rangle + \frac{2m^*}{\hbar^2} \mathop{\mathrm{S}}_{k''} \sum_{\substack{LL'S'q \\ J'\mu'}} C_{LL'q}^{J'S'} \left(\frac{2}{\pi}\right)^2 (J_I^+)_{L'S'k''q}^{J'\mu'}$$
$$\times \int F_L(KK') F_L(KK'') g_{Ljq}^{J'S'}(K'') \mathcal{Y}_{(LS')J'\mu'}(\hat{K}') \frac{1}{K_E{}^2 - K^2 - K_{\alpha j}{}^2 + i\epsilon} K^2 dK K''^2 dK'' . \tag{27}$$

C. Evaluation of $(J_I^+)_{l'Sjp}^{J\mu}$

The orthonormality properties of $\mathcal{Y}_{(LS)J\mu}$ and the definitions (13) and (14) of $(I_I^+)_{l'Sjp}^{J\mu *}$ and $(J_I^+)_{l'Sjp}^{J\mu}$ lead to the following expression for $(J_I^+)_{l'Sjp}^{J\mu}$:

$$(J_I^+)_{l'Sjp}^{J\mu} = (I_I^+)_{l'Sjp}^{J\mu *} + \frac{2m^*}{\hbar^2} \sum_{qL'} C_{l'L'q}^{JS} G_{l'jpq}^{JS} \mathop{\mathrm{S}}_{k''} (J_I^+)_{L'Sk''q}^{J\mu} . \tag{28}$$

We have set

$$G^{JS}_{l'jpq}=\int\frac{h^{JS}_{l'jp}(K)h^{JS}_{l'jq}(K)}{K_E^{\,2}-K^2-K_{\alpha j}^{\,2}+i\epsilon}K^2dK\,, \tag{29}$$

$$h^{JS}_{l'jp}(K)=\frac{2}{\pi}\int F_{l'}(KK')g^{JS}_{l'jp}(K')K'^2dK'\,. \tag{30}$$

For the sake of simplicity we can sum over j and define

$$(J_I^+)^{J\mu}_{l'Sp}=\mathop{\mathsf{S}}_{j}(J_I^+)^{J\mu}_{l'Sjp}\,,$$

$$(I_I^+)^{J\mu\,*}_{l'Sp}=\mathop{\mathsf{S}}_{j}(I_I^+)^{J\mu\,*}_{l'Sjp}\,, \tag{31}$$

$$G^{JS}_{l'pq}=\mathop{\mathsf{S}}_{j}G^{JS}_{l'jpq}\,.$$

The transition amplitude T_N expressed in (12) takes now the simpler form

$$T_N=\sum_{\substack{ll'Sp\\J\mu}}C^{JS}_{ll'p}(I_F^-)^{J\mu}_{lSp}(J_I^+)^{J\mu}_{l'Sp}\,. \tag{32}$$

D. Evaluation of $(I_I^{\pm})^{J\mu}_{lSp}$

We use the orthonormality of the eigenfunctions $\varphi_{\alpha j}$ to simplify the expression (13) defining $(I_F^-)^{J\mu}_{lSp}$

$$(I_F^-)^{J\mu}_{lSp}=\sum_{S_F\sigma_F}\int\langle S_F\sigma_F,\chi_F^-|\vec{\mathbf{K}}\rangle g^{JS}_{lFp}(K)\mathcal{Y}_{(lS)J\mu}(\hat{K})d\vec{\mathbf{K}}$$

$$=\sum_{\substack{S_F\sigma_F\\m\sigma}}\int\langle\chi_F^-|\vec{\mathbf{K}}\rangle g^{JS}_{lFp}(K)\langle lmS\sigma|J\mu\rangle$$

$$\times\delta_{SS_F}\delta_{\sigma\sigma_F}Y_{lm}(\hat{K})d\vec{\mathbf{K}}\,. \tag{33}$$

The decomposition (26) of $\langle\chi_F^-|\vec{\mathbf{K}}\rangle$ leads now to the following:

$$(I_F^-)^{J\mu}_{lSp}=\frac{2}{\pi}\sum_{\sigma_F m\lambda\nu}e^{i\sigma_\lambda}\langle lmS\sigma_F|J\mu\rangle\int F_\lambda(K_FK)$$

$$\times Y_{\lambda\nu}(\hat{K}_F)Y^*_{\lambda\nu}(\hat{K})g^{JS}_{lFp}(K)Y_{lm}(\hat{K})d\vec{\mathbf{K}}\,, \tag{34}$$

$$(I_F^-)^{J\mu}_{lSp}=e^{i\sigma_l}\sum_{\sigma_F m}Y_{lm}(\hat{K}_F)\langle lmS\sigma_F|J\mu\rangle h^{JS}_{lFp}(K_F)\,. \tag{35}$$

The same procedure should lead to the following result:

$$(I_I^+)^{J\mu\,*}_{l'Sp}=e^{i\sigma_{l'}}\sum_{\sigma_I m'}Y^*_{l'm'}(\hat{K}_I)\langle l'm'S\sigma_I|J\mu\rangle h^{JS}_{l'Ip}(K_I)\,. \tag{36}$$

These expressions substituted in (28) and (32) give the desired transition amplitude. As we shall see later in some applications, we can thus get an analytical and exact expression for T_N.

E. Expression for G^{JS}_{lpq}

The factor G^{JS}_{ljpq} has been defined as in (29)

$$G^{JS}_{ljpq}=\int\frac{h^{JS}_{ljp}(K)h^{JS}_{ljq}(K)}{K_E^{\,2}-K^2-K_{\alpha j}^{\,2}+i\epsilon}K^2dK\,.$$

We can separate its real and imaginary parts by setting

$$G^{JS}_{ljpq}=\frac{\pi}{4}(A^{JS}_{ljpq}-iB^{JS}_{ljpq})\,. \tag{37}$$

If now we use the well-known property

$$\frac{1}{x-x_0+i\epsilon}\mathop{=}_{\epsilon\to0}\mathrm{P}\frac{1}{x-x_0}-i\pi\delta(x-x_0)$$

it appears that

$$A^{JS}_{ljpq}=\frac{4}{\pi}\mathrm{P}\int_0^\infty\frac{h^{JS}_{ljp}(K)h^{JS}_{ljq}(K)}{K_E^{\,2}-K^2-K_{\alpha j}^{\,2}}K^2dK \tag{38}$$

$$B^{JS}_{ljpq}=2K_Eh^{JS}_{ljp}(K_E)h^{JS}_{ljq}(K_E)\,. \tag{39}$$

The imaginary part is expressed directly with the form factors, while the real part will be often evaluated numerically.

III. FORM FACTORS

The choice of the interaction is often the main problem of any theory. We know that the scattering potentials must fulfill certain conditions such as Hermiticity, time-reversal invariance, etc., but a large indeterminacy on their form remains. Generally speaking these interactions are chosen in the configuration space, as it is easier to imagine their physical representation. Nevertheless, it is absolutely equivalent to infer directly an interaction in another representation. In fact, it is just the matter of an integral transformation to obtain the interaction in the more suitable representation. For convenience, we shall thus start by choosing the nonlocal form factors in the configuration space and then construct more elaborate form factors in the $\{K\}$ momentum space or $\{\chi_K^\pm\}$ "Coulomb space."

A. Form Factors in Different Representations

The matrix elements of the nonlocal separable potential V_N have been defined in the momentum space by

$$\langle\vec{\mathbf{K}},\varphi_{\alpha i}|V_N|\vec{\mathbf{K}}',\varphi_{\alpha j}\rangle=\sum_{\substack{ll'Sp\\J\mu}}C^{JS}_{ll'p}g^{JS}_{lip}(K)g^{JS}_{l'jq}(K')$$

$$\times\mathcal{Y}_{(lS)J\mu}(\hat{K})\mathcal{Y}^*_{(l'S)J\mu}(\hat{K}')\,. \tag{40}$$

It can be easily shown (Appendix A) that an analog expansion can be obtained in the configuration space or in the "Coulomb space," i.e.,

$$\langle \vec{r}, \varphi_{\alpha i} | V_N | \vec{r}', \varphi_{\alpha j}\rangle = \sum_{\substack{ll'Sp\\J\mu}} C^{JS}_{ll'p} \mathcal{V}^{JS}_{lip}(r)\mathcal{V}^{JS}_{l'jp}(r') \times \mathcal{Y}_{(lS)J\mu}(\hat{r})\mathcal{Y}^*_{(l'S)J\mu}(\hat{r}'), \quad (41)$$

$$\langle \chi^+_K, \varphi_{\alpha i} | V_N | \chi^+_{K'}, \varphi_{\alpha j}\rangle = \sum_{\substack{ll'Sp\\J\mu}} C^{JS}_{ll'p} h^{JS}_{lip}(K) h^{JS}_{l'jp}(K) \times \mathcal{Y}_{(lS)J\mu}(\hat{K})\mathcal{Y}^*_{(l'S)J\mu}(\hat{K}'). \quad (42)$$

The different form factors obtained are linked by integral transformations

$$g^{JS}_{ljp}(K) = \left(\frac{2}{\pi}\right)^{1/2} \int j_l(Kr)\mathcal{V}^{JS}_{ljp}(r) r^2 dr = \frac{2}{\pi}\int F_l(KK') h^{JS}_{ljp}(K') K'^2 dK', \quad (43)$$

$$\mathcal{V}^{JS}_{ljp}(r) = \left(\frac{2}{\pi}\right)^{1/2} \int j_l(Kr) g^{JS}_{ljp}(K) K^2 dK = \left(\frac{2}{\pi}\right)^{1/2} \int \frac{F_l(Kr)}{Kr} h^{JS}_{ljp}(K) K^2 dK, \quad (44)$$

$$h^{JS}_{ljp}(K) = \left(\frac{2}{\pi}\right)^{1/2} \int \frac{K_l(Kr)}{Kr} \mathcal{V}^{JS}_{ljp}(r) r^2 dr = \frac{2}{\pi}\int F_l(KK') g^{JS}_{ljp}(K') K'^2 dK'. \quad (45)$$

B. Determination of $g^{JS}_{ljp}(K)$ and $h^{JS}_{ljp}(K)$

Let us express the regular Coulomb wave function in terms of spherical Bessel functions[7]

$$\frac{F_l(Kr)}{Kr} = (2l+1)!!\,C_l(\eta) \sum_{s=l}^{\infty} b_s j_s(Kr), \quad (46)$$

where

$$C_l(\eta) = \frac{[(1+\eta^2)(4+\eta^2)\cdots(l^2+\eta^2)]^{1/2}}{(2l+1)!} 2^l C_0(\eta), \qquad C_0(\eta) = \frac{2\pi\eta}{e^{2\pi\eta}-1}. \quad (47)$$

The b_s coefficients are defined by the recursion relation

$$b_l = 1, \quad b_{l+1} = \frac{2l+3}{l+1}\eta, \quad \ldots, \quad b_s = \frac{2s+1}{s(s+1)-l(l+1)} \times \left[2\eta b_{s-1} - \frac{(s-1)(s-2)-l(l+1)}{2s-3} b_{s-2}\right], \quad s > l+1. \quad (48)$$

We thus obtain the form factor $h^{JS}_{ljp}(K)$ in the serial form

$$h^{JS}_{ljp}(K) = \left(\frac{2}{\pi}\right)^{1/2} (2l+1)!!\,C_l(\eta) \times \sum_{s=1}^{\infty} b_s \int_0^\infty j_s(Kr)\mathcal{V}^{JS}_{ljp}(r) r^2 dr, \quad (49)$$

while $g^{JS}_{ljp}(K)$ is obtained by evaluating the integral

$$g^{JS}_{ljp}(K) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty j_l(Kr)\mathcal{V}^{JS}_{ljp}(r) r^2 dr. \quad (50)$$

We can, for instance, take a Yukawa form factor in the configuration space

$$\mathcal{V}^{JS}_{ljp}(r) = \frac{e^{-\beta^{JS}_{ljp} r}}{r}. \quad (51)$$

The corresponding form factors $g^{JS}_{ljp}(K)$ and h^{JS}_{ljp} become

$$g^{JS}_{ljp}(K) = \left(\frac{2}{\pi}\right)^{1/2} \frac{(l+1)!}{(2l+1)!!} \frac{K^l}{[K^2+(\beta^{JS}_{ljp})^2]^{(l+2)/2}} \times F\left(l+2, \tfrac{1}{2}l; l+\tfrac{3}{2}; \frac{K^2}{K^2+(\beta^{JS}_{ljp})^2}\right), \quad (52)$$

$$h^{JS}_{ljp}(K) = \left(\frac{2}{\pi}\right)^{1/2} (2l+1)!!\,C_l(\eta) \times \sum_{s=l}^{\infty} b_s \frac{K^s}{[K^2+(\beta^{JS}_{ljp})^2]^{(s+2)/2}} \frac{(s+1)!}{(2s+1)!!} \times F\left(s+2, \frac{s}{2}; \frac{s+3}{2}; \frac{K^2}{K^2+(\beta^{JS}_{ljp})^2}\right). \quad (53)$$

The $F(a, b; c; Z)$ function is an hypergeometric function. As these form factors are in general rather complicated, we shall use them just as an indication in the choice of the more convenient form factor. In the simple case where only the $l=0$ contribution is considered, i.e., $g^{JS}_{ljp}(K) = g_{0p}(K) = 1/(K^2+\beta_{jp}^2)$, we can express $h_{0p}(K)$ either in the serial form

$$h_{0p}(K) = C_0(\eta) \sum_{s=0}^{\infty} b_s \frac{K^s}{(K^2+\beta_{jp}^2)^{(s+2)/2}} \frac{(s+1)!}{(2s+1)!!} \times F\left(s+2, \tfrac{1}{2}s; s+\tfrac{3}{2}; \frac{K^2}{K^2+\beta_{jp}^2}\right) \quad (54)$$

or in an analytical form[8]

$$h_{0p}(K) = C_0(\eta) g_{0p}(K) \exp\left(2\eta \tan^{-1}\frac{K}{\beta_{jp}}\right). \quad (55)$$

The next section will be devoted to a straightforward application of these expressions.

IV. PROTON-PROTON ELASTIC SCATTERING AT LOW ENERGY

In this particular case great simplification occurs, since the target has no internal structure.

For the singlet states ($S=0$, $J=l=l'$) the matrix element of the nonlocal separable potential is written in the simple form

$$\langle\vec{K}|V_N|\vec{K}'\rangle=\sum_{lmp}C_{lp}g_{lp}(K)g_{lp}(K')Y_{lm}(\hat{K})Y^*_{lm}(\hat{K}'). \tag{56}$$

The transition amplitude is then evaluated through its simpler value

$$T_N=\sum_{lmp}C_{lp}(I_F^-)_{lmp}(J_I^+)_{lmp},$$

and we have now

$$(J_I^+)_{lmp}=(I_I^+)^*_{lmp}+\frac{2m^*}{\hbar^2}\sum_q C_{lq}G_{lpq}(J_I^+)_{lmq} \tag{57}$$

and

$$\begin{aligned}(I_I^+)^*_{lmp}&=e^{i\sigma_l}Y^*_{lm}(\hat{K}_I)h_{lp}(K_I),\\ (I_F^-)_{lmp}&=e^{i\sigma_l}Y_{lm}(\hat{K}_F)h_{lp}(K_F),\\ G_{lpq}&=\int\frac{h_{lp}(K)h_{lq}(K)}{K_E^2-K^2+i\epsilon}K^2dK.\end{aligned} \tag{58}$$

It has been shown[1] that the nucleon-nucleon elastic scattering can be described by a two-term nonlocal separable potential ($p,q\le 2$) if we introduce the following functions:

$$\begin{aligned}\Delta_1&=h_{l1}(K_I)\left(1-\frac{2m^*}{\hbar^2}C_{l2}G_{l22}\right)+\frac{2m^*}{\hbar^2}C_{l2}G_{l12}h_{l2}(K_I)\\ \Delta_2&=h_{l2}(K_I)\left(1-\frac{2m^*}{\hbar^2}C_{l1}G_{l11}\right)+\frac{2m^*}{\hbar^2}C_{l1}G_{l21}h_{l1}(K_I)\\ \Delta&=\left(1-\frac{2m^*}{\hbar^2}C_{l1}G_{l11}\right)\left(1-\frac{2m^*}{\hbar^2}C_{l2}G_{l22}\right)-\left(\frac{2m^*}{\hbar^2}\right)^2C_{l1}C_{l2}G_{l12}G_{l21}.\end{aligned} \tag{59}$$

The transition amplitude becomes expressed as

$$T_N=\sum_{lm}\frac{e^{2i\sigma_l}}{\Delta}Y^*_{lm}(\hat{K}_I)Y_{lm}(\hat{K}_F)[C_{l1}h_{l1}(K_F)\Delta_1+C_{l2}h_{l2}(K_F)\Delta_2]. \tag{60}$$

The scattering amplitude expanded in partial waves gives, by comparison with the above expression, the phase shift δ_l

$$f_N(\theta)=\sum_l f_{Nl}(\theta)=-\frac{4\pi^2m^*}{\hbar^2}T_N. \tag{61}$$

We thus obtain

$$\begin{aligned}f_{Nl}(\theta)&=\frac{2l+1}{2iK_I}e^{2i\sigma_l}(e^{2i\delta_l}-1)P_l(\cos\theta)\\ &=\frac{2(2l+1)}{\Delta}e^{2i\sigma_l}[\lambda_{l1}^{-1}h_{l1}(K_F)\Delta_1+\lambda_{l2}^{-1}h_{l2}(K_F)\Delta_2]P_l(\cos\theta).\end{aligned} \tag{62}$$

For simplicity's sake, we have set

$$\lambda_{lp}^{-1}=-(2m^*/\hbar^2)(\pi/4)C_{lp}. \tag{63}$$

It is then easy to extract the phase shift δ_l (see Appendix B)

$$\tan\delta_l=2K_I\frac{h_{l1}^2(K_I)+\gamma_l h_{l2}^2(K_I)+\dfrac{\gamma_l}{\lambda_{l1}}A_{l11}h_{l2}^2(K_I)+A_{l22}h_{l1}^2(K_I)-2A_{l12}h_{l1}(K_I)h_{l2}(K_I)}{\lambda_{l1}+A_{l11}+\gamma_l A_{l22}+\dfrac{\gamma_l}{\lambda_{l1}}(A_{l11}A_{l22}-A_{l12}^2)}. \tag{64}$$

In this expression we have used the separation of G_{lpq} into its imaginary and real parts

$$G_{lpq}=\tfrac{1}{4}\pi(A_{lpq}-iB_{lpq}) \tag{65}$$

and set

$$\gamma_l=\lambda_{l1}/\lambda_{l2}. \tag{66}$$

Nucleon-nucleon elastic scattering has been previously studied with the use of nonlocal separable potentials up to 400 MeV by Mongan.[1] With only four free parameters, the fit obtained was excellent between 20 and 400 MeV. Nevertheless, at low energy ($\lesssim$20 MeV) an important discrepancy appeared because the Coulomb effects were not taken into account.

As only the $^1S_0$ phase shift is significant in the

TABLE I. p-p scattering. Parameter values.

	Mongan's parameters	Proposed parameters
C_R (MeV/fm)$^{1/2}$	302	305 ($\gamma_0 = 7.953 \times 10^{-3}$)
C_A (MeV/fm)$^{1/2}$	27.33	27.20 ($\lambda_0 = 5.724 \times 10^{-4}$)
a_R (fm^{-1})	6.157	6.157
a_A (fm^{-1})	1.786	1.786

p-p elastic scattering, Mongan used the nonlocal separable potential

$$V_0(KK') = g_{01}(K)g_{01}(K') - g_{02}(K)g_{02}(K') , \tag{67}$$

where

$$g_{01}(K) = C_R/(K^2 + a_R^2) ,$$
$$g_{02}(K) = C_A/(K^2 + a_A^2) . \tag{68}$$

In that simple case we can use the $h_{0p}(K)$ defined in the previous section in (54) or (55) to include Coulomb effects. The linkage between Mongan's parameters and ours is easily obtained

$$(1/\lambda_0) = -(2m^*/\hbar^2)(\pi/4)C_R^2 ,$$
$$\gamma_0 = -(C_A/C_R)^2 . \tag{69}$$

We first evaluated the δ phase shift without Coulomb effects, using Mongan's form factors and parameters, to show that a discrepancy appeared below 20 MeV. We then replaced the $g_{0p}(K)$ form factors by the $h_{0p}(K)$, keeping the same parameters. Figure 1[9,10] shows the striking improvement

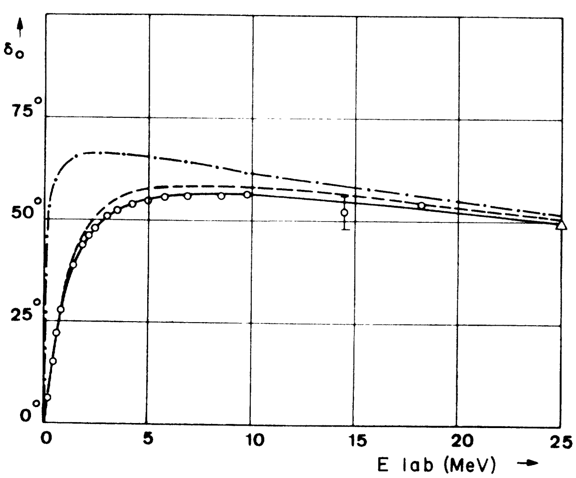

FIG. 1. δ_0 phase shifts in the p-p elastic scattering: (–·–·–·), Mongan's form factors and parameters; (------), PBE form factors and Mongan's parameters; (———), PBE form factors and proposed parameters. The experimental data are from Hulthén and Sugawara (Ref. 9) (○); and Noyes *et al.* (Ref. 10) (△).

thus obtained. Finally, by a slight modification (less than 1%) of two parameters, we obtained an excellent fit below 20 MeV (Table I and Fig. 1).

It can be seen from the expression for η that an increase in the incident energy gives a smaller η, and $C_0(\eta)$ trends to unity as $h_{0p}(K)$ trends to $g_{0p}(K)$ [(47) and (55)]. It thus appears that the new $h_{0p}(K)$ form factor can be used at any energy in p-p elastic scattering and gives an improvement over Mongan's results.

V. NUCLEON-α ELASTIC SCATTERING

In this case the total spin S is equal to $\frac{1}{2}$ and parity conservation gives $l = l'$. The general expressions (32), (35), and (36) simplify into

$$(J_I^+)_{lp}^{J\mu} = (I_I^+)_{lp}^{J\mu*} + \frac{2m^*}{\hbar^2}\sum_q C_{lq}^J G_{lpq}^J (J_I^+)_{lq}^{J\mu} ,$$
$$(I_I^+)_{lp}^{J\mu*} = e^{i\sigma_l} \sum_{\sigma_I m'} Y_{lm'}^*(\hat{K}_I)\langle lm'\tfrac{1}{2}\sigma_I|J\mu\rangle h_{lIp}^J(K_I) , \tag{70}$$
$$(I_F^-)_{lp}^{J\mu} = e^{i\sigma_l} \sum_{\sigma_F m} Y_{lm}(\hat{K}_F)\langle lm\tfrac{1}{2}\sigma_F|J\mu\rangle h_{lFp}^J(K_F) .$$

In the study of n-α and p-α elastic scattering, it appears that the phase shifts do not vanish,[11-18] and a one-term nonlocal separable potential is sufficient to describe the experimental data.[19] As $p = q = 1$, the above equations are greatly simplified and we get, for instance,

$$(J_I^+)_l^{J\mu} = \frac{(I_I^+)_l^{J\mu*}}{1 - (2m^*/\hbar^2)C_l^J G_l^J} . \tag{71}$$

The transition amplitude T_N may thus be written

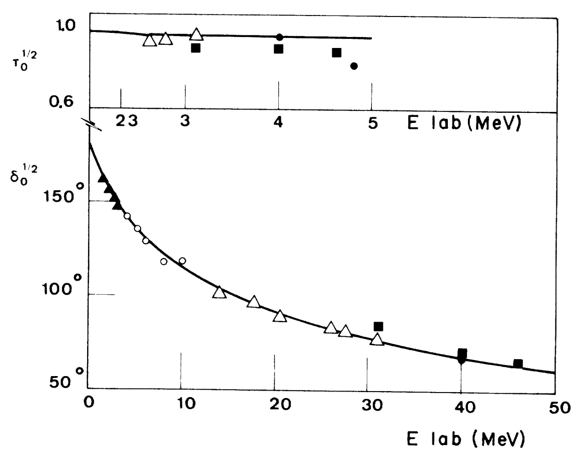

FIG. 2. p-α scattering. Real and imaginary parts of phase shifts for $l = 0$. The experimental data are from Brown, Haeberli, and Trächslin (Ref. 11) (▲); Barnard, Jones, and Weil (Ref. 12) (○); Weitkamp and Haeberli (Ref. 13) (△); Thompson, Epstein, and Sawada (Ref. 14) (■); and Davies *et al.* (Ref. 16) (●).

TABLE II. p-α scattering. Parameter values.

δ_l^J	λ_l^J (fm^3)	β_{lj}^J (fm^{-1})	β_{le}^J (fm^{-1})	ρ_l^J
$\delta_0^{1/2}$	8.13×10^{-2}	0.83	0.83	4.0×10^{-3}
$\delta_1^{3/2}$	6.02×10^{-2}	1.25	0.5	6.0×10^{-3}
$\delta_1^{1/2}$	$1.6\ \times10^{-1}$	1.0	0.8	5.0×10^{-2}
$\delta_2^{5/2}$	$6.0\ \times10^{-2}$	1.2	1.2	1.5
$\delta_2^{3/2}$	$4.0\ \times10^{-2}$	1.4	0.7	0.2
$\delta_3^{5/2}$	$1.1\ \times10^{-2}$	1.4	1.0	1.0
$\delta_3^{7/2}$	$1.1\ \times10^{-2}$	1.4	1.0	1.0
$\delta_4^{9/2}$	$5.6\ \times10^{-3}$	1.3	0.9	1.0
$\delta_4^{7/2}$	2.34×10^{-3}	1.3	0.6	2.0

TABLE III. n-α scattering. Parameter values.

δ_l^J	λ_l^J (fm^3)	β_{lf}^J (fm^{-1})	β_{le}^J (fm^{-1})	ρ_l^J
$\delta_0^{1/2}$	$7.7\ \times10^{-2}$	0.87	0.87	4.0×10^{-3}
$\delta_1^{3/2}$	$6.2\ \times10^{-2}$	1.25	0.5	6.0×10^{-3}
$\delta_1^{1/2}$	$1.7\ \times10^{-1}$	1.0	0.8	5.0×10^{-2}
$\delta_2^{5/2}$	$6.4\ \times10^{-2}$	1.2	1.2	1.5
$\delta_2^{3/2}$	$4.0\ \times10^{-2}$	1.4	0.7	0.2
$\delta_3^{7/2}$	$1.0\ \times10^{-2}$	1.4	1.0	1.0
$\delta_3^{5/2}$	$1.0\ \times10^{-2}$	1.4	1.0	1.0
$\delta_4^{9/2}$	$5.8\ \times10^{-3}$	1.3	0.9	1.0
$\delta_4^{7/2}$	2.34×10^{-3}	1.3	0.6	2.0

in the simple form

$$T_N=\sum_{lJ\mu} C_l^J \frac{(I_I^+)_l^{J\mu}(I_F^-)_l^{J\mu}}{1-(2m^*/\hbar^2)C_l^J G_l^J}\,; \tag{72}$$

substituted into the scattering amplitude (61) it gives

$$f_N(\theta)=8\pi\sum_{lJ\mu}\frac{(I_I^+)_l^{J\mu *}(I_F^-)_l^{J\mu}}{\lambda_l^J+(4/\pi)G_l^J} \tag{73}$$

with

$$(\lambda_l^J)^{-1}=-\frac{2m^*}{\hbar^2}\,\frac{\pi}{4}C_l^J\,. \tag{74}$$

Let us now develop the scattering amplitude for a given value of the magnetic moments σ_I and σ_F, when the quantum axis has been chosen along the incident vector $\vec{K}_I$

$$\begin{aligned} f_N^{\sigma_I\sigma_F}(\theta)=4\sqrt{\pi}\sum_{lJ}(2l+1)&\frac{e^{2i\sigma_l}}{\lambda_l^J+(4/\pi)G_l^J}\\ &\times\langle l0\tfrac{1}{2}\sigma_I|J\sigma_I\rangle\langle lm'\tfrac{1}{2}\sigma_F|J\sigma_I\rangle\\ &\times Y_{lm'}(\theta)h_{lI}^J(K_I)h_{lF}^J(K_F)\,. \end{aligned} \tag{75}$$

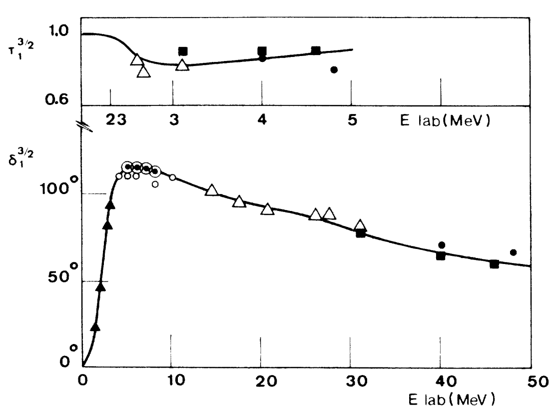

FIG. 3. p-α scattering. Real and imaginary parts of phase shifts for $l=1$ and $J=\frac{1}{2}$. The symbols are as in Fig. 2. The (⊙) data are from Satchler *et al.* (Ref. 15).

We use now the facts that $J=l\pm\frac{1}{2}$ and that $K_I=K_F=K_E$ in an elastic scattering process to define scattering amplitudes with or without spin flip

$$f_N^{\uparrow\uparrow}(\theta)=f_N^{\downarrow\downarrow}(\theta)=2\sum_l e^{2i\sigma_l}P_l(\cos\theta)\times\left[\frac{(l+1)h_{lI}^+(K_I)h_{lF}^+(K_I)}{\lambda_l^+ +(4/\pi)G_l^+}+\frac{l h_{lI}^-(K_I)h_{lF}^-(K_I)}{\lambda_l^- +(4/\pi)G_l^-}\right],$$

$$f_N^{\uparrow\downarrow}(\theta)=-\sum_l 2e^{2i\sigma_l}P_l^1(\theta)e^{i\varphi}\times\left[\frac{h_{lI}^+(K_I)h_{lF}^+(K_I)}{\lambda_l^+ +(4/\pi)G_l^+}-\frac{h_{lI}^-(K_I)h_{lF}^-(K_I)}{\lambda_l^- +(4/\pi)G_l^-}\right], \tag{76}$$

$$f_N^{\downarrow\uparrow}(\theta)=\sum_l 2e^{2i\sigma_l}P_l^1(\theta)e^{-i\varphi}\times\left[\frac{h_{lI}^+(K_I)h_{lF}^+(K_I)}{\lambda_l^+ +(4/\pi)G_l^+}-\frac{h_{lI}^-(K_I)h_{lF}^-(K_I)}{\lambda_l^- +(4/\pi)G_l^-}\right].$$

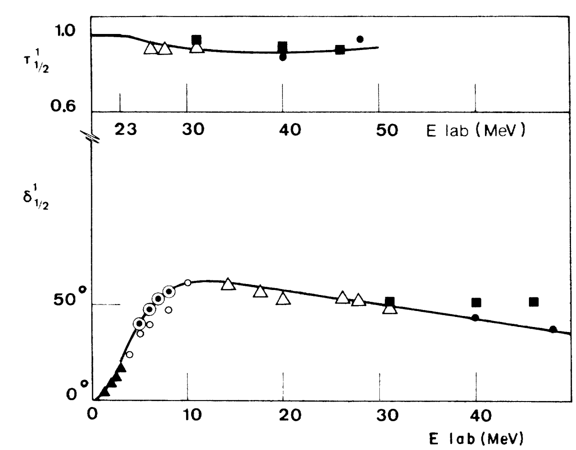

FIG. 4. p-α scattering. Real and imaginary parts of phase shifts for $l=1$ and $J=\frac{3}{2}$. The symbols are as in Figs. 2 and 3.

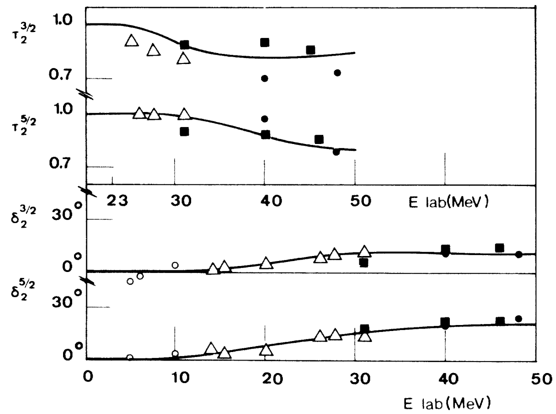

FIG. 5. p-α scattering. Real and imaginary parts of phase shifts for $l=2$. The symbols are as in Fig. 2.

The corresponding phase shifts $\delta_l^\pm$ are obtained by comparison of these expressions with the usual expansion[20] in partial waves of $f_N(\theta)$.

$$f_N^{\uparrow\uparrow}(\theta)=\frac{1}{2iK_I}\sum_l e^{2i\sigma_l}P_l(\cos\theta)$$
$$\times[(l+1)(e^{2i\delta_l^+}-1)+l(e^{2i\delta_l^-}-1)], \qquad (77)$$

$$f_N^{\uparrow\downarrow}(\theta)=-\frac{1}{2iK_I}\sum_l e^{2i\sigma_l}P_l^1(\theta)e^{i\varphi}[e^{2i\delta_l^+}-e^{2i\delta_l^-}].$$

We can now easily obtain the following expression for the phase shifts

$$\frac{e^{2i\delta_l^\pm}-1}{2iK_I}=\frac{2h_{lF}^\pm(K_I)h_{lI}^\pm(K_I)}{\lambda_l^\pm+(4/\pi)G_l^\pm}. \qquad (78)$$

The differential cross section or the polarization of the scattered particles will be obtained by evaluating the well-known expressions[20]

$$\frac{d\sigma}{d\Omega}=(|u|^2+|v|^2),$$
$$P(\theta)=-\frac{2\mathrm{Re}(uv^*)}{|u|^2+|v|^2}, \qquad (79)$$

where

$$u=f_c(\theta)+f_N^{\uparrow\uparrow}(\theta),$$
$$v=f_N^{\uparrow\downarrow}(\theta), \qquad (80)$$

with

$$\varphi=\pi/2.$$

As previously noted, we shall pass from the p-α elastic scattering case to the n-α case just by replacing $h_{ij}^\pm(K_I)$ by $g_{ij}^\pm(K_I)$. In the elastic scattering the final state F is identical to the initial state of the target: It is the fundamental state f of this target. Nevertheless, the internal structure of the target may appear through the possible excitation of its inelastic states. For the α particle the first inelastic state appears around 40 MeV in the laboratory system. It may thus play an important role above 40 MeV. We shall therefore use (31) to write

$$G_l^\pm=\sum_j G_{lj}^\pm=G_{le}^\pm+G_{lf}^\pm, \qquad (81)$$

where f denotes the fundamental and e the first excited state of the target. Then

$$\frac{e^{2i\delta_l^\pm}-1}{2iK_I}=\frac{2[h_{lf}^\pm(K_I)]^2}{\lambda_l^\pm+(4/\pi)(G_{lf}^\pm+G_{le}^\pm)}. \qquad (82)$$

From this expression, it is easy to obtain the imaginary and real parts of the phase shifts. The results are similar to the α-α case (cf. next section). To describe the p-α elastic scattering we

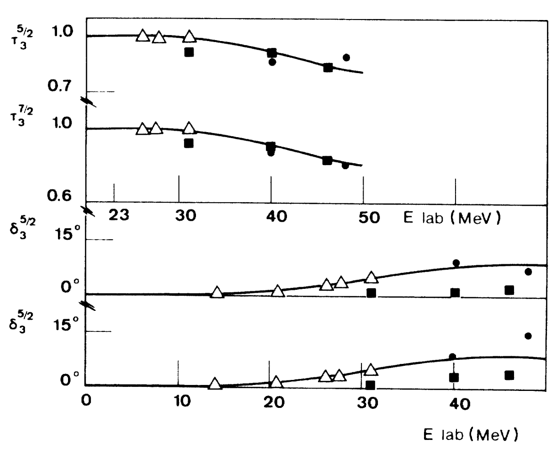

FIG. 6. p-α scattering. Real and imaginary parts of phase shifts for $l=3$. The symbols are as in Fig. 2.

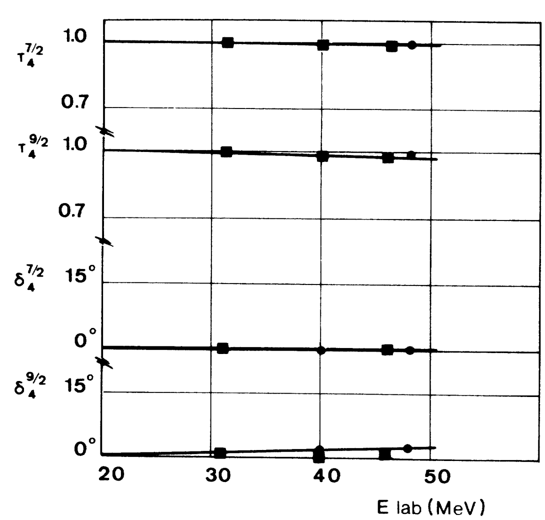

FIG. 7. p-α scattering. Real and imaginary parts of phase shifts for $l=4$. The symbols are as in Fig. 2.

have thus chosen two form factors:

$$h^{\pm}_{lf}(K) = (2l+1)\,!!C_l(\eta)\sum_{s=l}^{\infty} b_s \frac{K^s}{[K^2+(\beta^{\pm}_{lf})^2]^{(s+2)/2}}\,\frac{(s+1)!}{(2s+1)!!} \qquad (83)$$

and

$$h^{\pm}_{le}(K) = \rho^{\pm}_l(2l+1)!!C_l(\eta) \times \sum_{s=l}^{\infty} b_s \frac{K^s}{[K^2+(\beta^{\pm}_{le})^2]^{(s+2)/2}}\,\frac{(s+1)!}{(2s+1)!!}\,. \qquad (84)$$

These form factors are nearly the Coulomb transform of a Yukawa potential (53), except that the hypergeometrical function has been set equal to unity for the sake of simplicity. When the n-α elastic scattering is studied, we set $\eta=0$, and noting that $(2l+1)!!C_l(0)=1$ and $b_s=\delta_{sl}$, we get

$$g^{\pm}_{lf} = \frac{(l+1)!}{(2l+1)!!}\,\frac{K^l}{[K^2+(\beta^{\pm}_{lf})^2]^{(l+2)/2}}\,,$$
$$g^{\pm}_{le} = \rho^{\pm}_l\,\frac{(l+1)!}{(2l+1)!!}\,\frac{K^l}{[K^2+(\beta^{\pm}_{le})^2]^{(l+2)/2}}\,. \qquad (85)$$

The $\lambda^{\pm}_l$, $\beta^{\pm}_l$, and $\rho^{\pm}_l$ parameters used are given (Table II) for the p-α elastic scattering and (Table

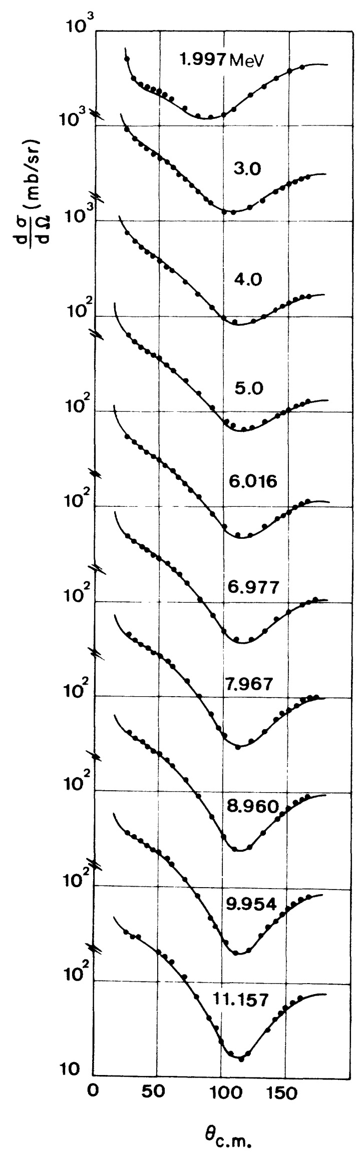

FIG. 8. p-α scattering. Theoretical cross sections. The experimental data are from Barnard, Jones and Weil (Ref. 12).

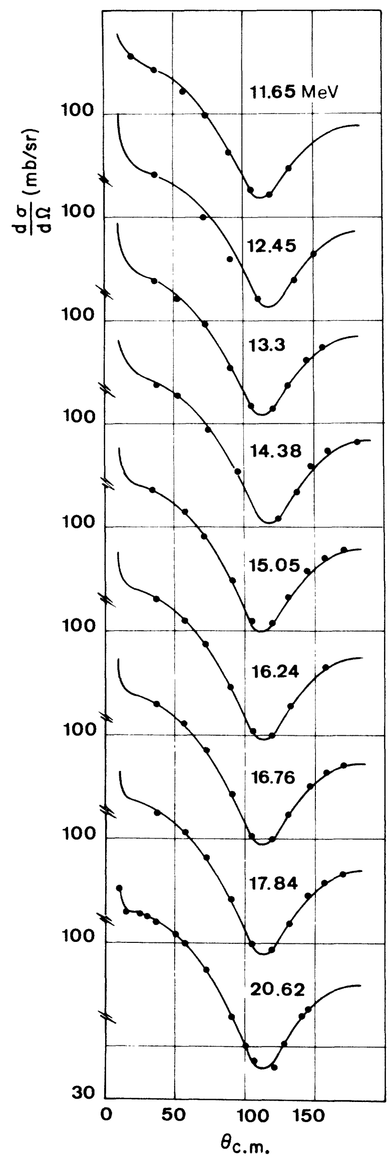

FIG. 9. p-α scattering. Theoretical cross sections. The experimental data are from Brockman (11.65–17.84 MeV) (Ref. 23) and Allison and Smythe (20.62 MeV) (Ref. 24).

III) for the n-α elastic scattering. Corresponding phase shifts are given in Figs. 2–7. We note here that a $\pi/2$ value of the phase shift (for $\delta_0^{1/2}$ and $\delta_1^{3/2}$, for instance) gives immediately the corresponding $\lambda_l^{\pm}$ parameters. On the other hand, the experimental $\delta_2^{3/2}$ shows a sharp resonance around 22 MeV. This compound-nucleus resonance cannot be found by our formalism which postulates a direct process for elastic scattering, but it can be approached through the usual method.[21] This problem has been extensively studied by Hoop and Barschall[18] and Darriulat *et al.*[22]

The validity of the set of parameters determined has been tested by the evaluation of the polarization together with the angular distribution of the scattered particle (Figs. 8–18).[11–13, 16–18, 21, 23–37] Finally the nearly identical values of $\lambda_l^{\pm}$, $\beta_l^{\pm}$, and $\rho_l^{\pm}$ for the n-α and p-α elastic scattering indicates that the nonlocal separable potential in the configuration space is nearly the same for p-α and n-α interactions. Using a method proposed by Coz, Arnold, and MacKellar,[38] we have evaluated the equivalent local potential for an S wave and a one-term nonlocal separable potential. For $l=0$ the $g_0(K)$ form factor defined in (52) becomes

$$g_0(K)=\frac{1}{K^2+\beta^2} \tag{86}$$

and leads therefore to the result shown in Fig. 19. For more complicated form factors and nonlocal potentials, the evaluation of the equivalent local potential has not yet been achieved.

VI. α-α ELASTIC SCATTERING

This particular case is the simplest one, be-

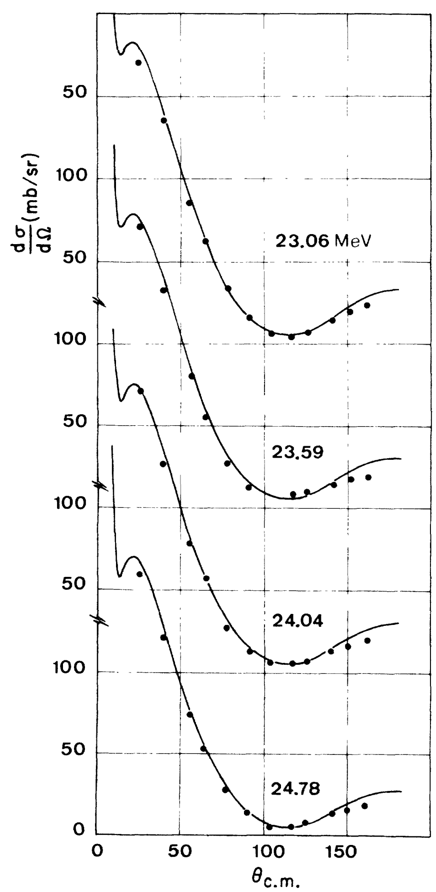

FIG. 10. p-α scattering. Theoretical cross sections. The experimental data are from Darriulat *et al.* (Ref. 22).

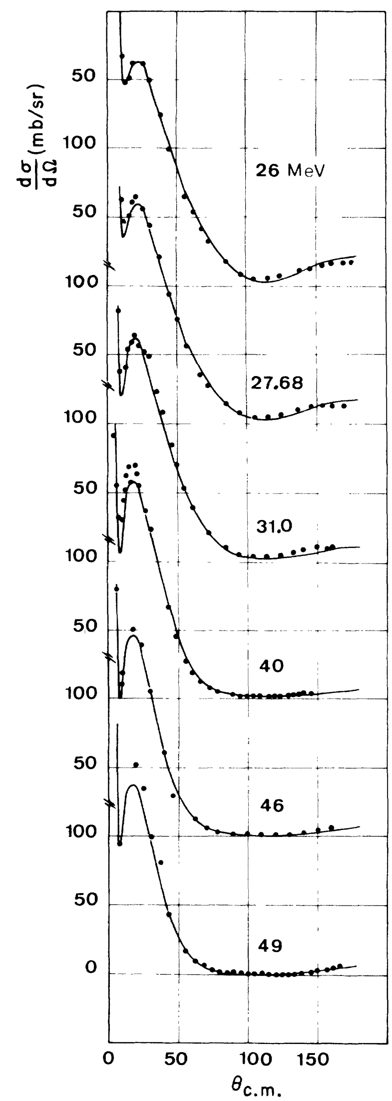

FIG. 11. p-α scattering. Theoretical cross section. The experimental data are from Allison and Smythe (26 and 27.68 MeV) (Ref. 24); Bunch, Forster, and Kim (31 MeV) (Ref. 25); Brussel and Williams (40 MeV) (Ref. 26); Bunker *et al.* (46 MeV) (Ref. 27); and Davies *et al.* (49 MeV) (Ref. 16).

cause of the zero value of the spin of particles involved. The transition amplitude becomes

$$T_N=\sum_{lmp} C_{lp}(I_F^-)_{lmp}(J_I^+)_{lmp}, \tag{87}$$

where

$$\begin{aligned}(J_I^+)_{lmp}&=(I_I^+)^*_{lmp}+\frac{2m^*}{\hbar}\sum_q C_{lq}G_{lpq}(J_I^+)_{lmq},\\ (I_I^+)^*_{lmp}&=e^{i\sigma_l}Y^*_{lm}(\hat K_I)h_{lIp}(K_I),\\ (I_F^-)_{lmp}&=e^{i\sigma_l}Y_{lm}(\hat K_F)h_{lFp}(K_F),\\ G_{lpq}&=\underset{i}{\mathsf{S}}\,G_{lipq}=\underset{i}{\mathsf{S}}\int\frac{h_{lip}(K)h_{liq}(K)}{K_E^{\,2}-K_{\alpha i}^{\,2}-K^2+i\epsilon}K^2dK.\end{aligned} \tag{88}$$

Limiting ourselves to a two-term nonlocal separable potential ($p, q\le 2$) with identical form factors $h_{li1}=h_{li2}=h_{li}$ we obtain the transition amplitude by the procedure used in the preceding section

$$T_N=\sum_{lm} e^{2i\sigma_l}(C_{l1}+C_{l2})Y^*_{lm}(\hat K_I)Y_{lm}(\hat K_F)\times\frac{h_{lI}(K_I)h_{lf}(K_F)}{1-(2m^*/\hbar^2)(C_{l1}+C_{l2})G_l}. \tag{89}$$

The one-term nonlocal potential case is obtained by setting $C_{l2}=0$ in the above expression. The phase-shift expression (82) is still valuable, i.e.,

$$\frac{e^{2i\delta_l}-1}{2iK_I}=\frac{2[h_{lf}(K_I)]^2}{\lambda_l+(4/\pi)G_l}. \tag{90}$$

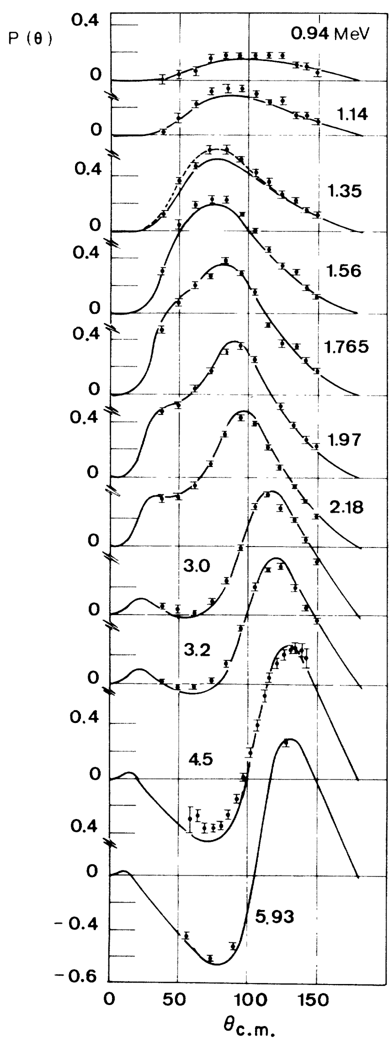

FIG. 12. p-α scattering. Theoretical polarizations. The experimental data are from Brown and Trächslin (0.94–3.2 MeV) (Ref. 11); Drigo *et al.* (4.5 MeV) (Ref. 28); and Brown, Haeberli, and Saladin (5.93 MeV) (Ref. 29). The dashed curve has been calculated for $E=1.4$ MeV.

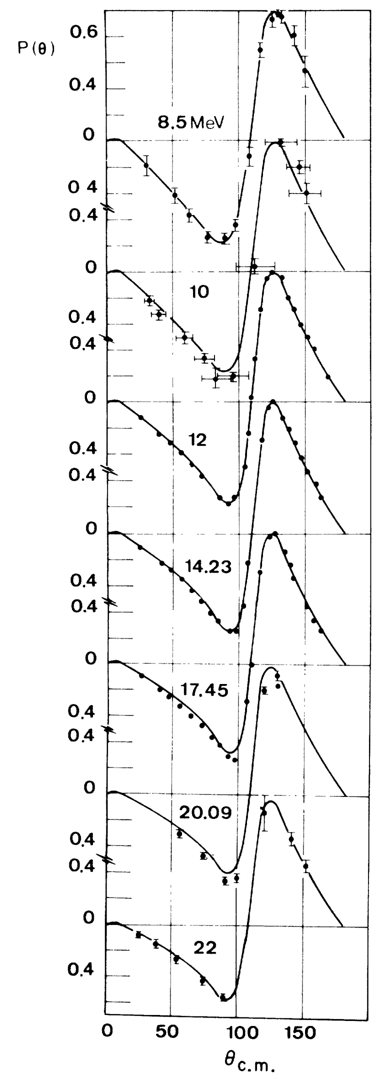

FIG. 13. p-α scattering. Theoretical polarizations. The experimental data are from Rosen *et al.* (8.5 and 10 MeV) (Ref. 30); Busser *et al.* (12 MeV) (Ref. 31); Garreta, Sura, and Tarrats (14.23 and 17.45 MeV) (Ref. 32); Weitkamp and Haeberli (20.1 MeV) (Ref. 13); and Craddock *et al.* (22 MeV) (Ref. 33).

Here too, the possible excitation of the α target is taken into account by introducing the first excited state of the α particle, i.e.,

$$G_l = \sum_{j=1}^{2} G_{lj} = G_{lf} + G_{le}, \tag{91}$$

where

$$G_{lj} = \frac{\pi}{4}(A_{lj} - iB_{lj}) = \int \frac{[h_{lj}(K)]^2}{K_I^2 - K^2 - K_{\alpha j}^2 + i\epsilon} K^2 dK. \tag{92}$$

The ground state of the α particle is defined by $K_{\alpha f}^2 = 0$ while the first excited state E_α gives

$$K_{\alpha e}^2 = (2m^*/\hbar^2)E_\alpha. \tag{93}$$

Let us first consider the incident energy E_I smaller than the excited E_α. We immediately get

$$G_{lf} = \int \frac{[h_{lf}(K_I)]^2}{K_I^2 - K^2 + i\epsilon} K^2 dK. \tag{94}$$

The imaginary and real parts of G_{lf} have thus the values

$$B_{lf} = 2K_I[h_{lf}(K_I)]^2, \qquad A_{lf} = \frac{4}{\pi} \vec{\mathrm{P}} \int \frac{[h_{lf}(K)]^2}{K_I^2 - K^2} K^2 dK. \tag{95}$$

On the other hand

$$G_{le} = \int \frac{[h_{le}(K)]^2}{K_I^2 - K^2 - K_{\alpha e}^2 + i\epsilon} K^2 dK \tag{96}$$

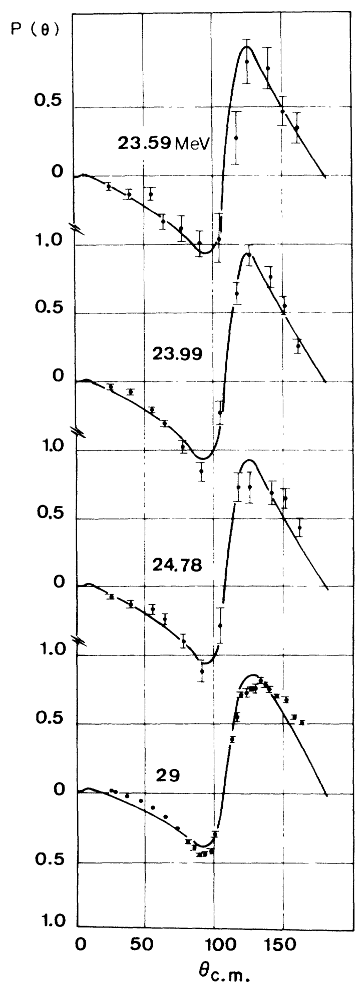

FIG. 14. p-α scattering. Theoretical polarizations. The experimental data are from Darriulat *et al.* (23.59–24.78 MeV) (Ref. 22) and Craddock *et al.* (29 MeV) (Ref. 33).

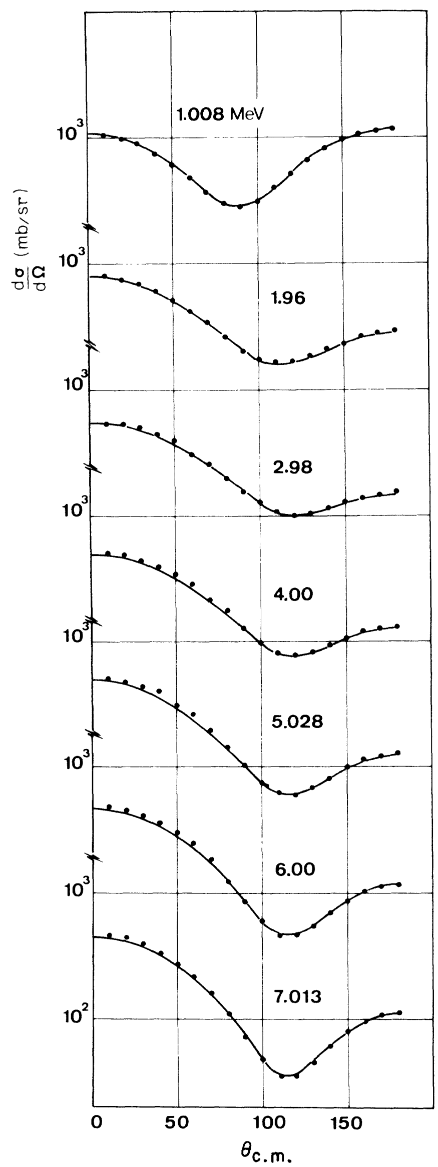

FIG. 15. n-α scattering. Theoretical cross sections. The experimental data are from Morgan and Walter (Ref. 17).

takes a purely real value, since $K_I^2 - K_{\alpha e}^2 < 0$. It follows that

$$A_{le} = -\frac{4}{\pi}\int \frac{[h_{le}(K)]^2}{|K_{\alpha e}^2 - K_I^2| + K^2} K^2 dK\,,\qquad B_{le} = 0\,. \tag{97}$$

The use of (95) and (97) leads to purely real phase shift (see Appendix C)

$$E_I < E_\alpha \Rightarrow \tan\delta_l = \frac{B_{lf}}{\lambda_l + A_{lf} + A_{le}}\,. \tag{98}$$

Let us now consider E_I greater than E_α. When setting

$$K_I'^2 = |K_{\alpha e}^2 - K_I^2| \tag{99}$$

we obtain

$$A_{le} = \frac{4}{\pi}\vec{\mathrm{P}}\int \frac{[h_{le}(K)]^2}{K_I'^2 - K^2} K^2 dK\,,\qquad B_{le} = 2K_I'[h_{le}(K_I')]^2\,, \tag{100}$$

while A_{lf} and B_{lf} are still given by (95). The phase shifts take here a slightly more complicated form (see Appendix C):

$$\delta_l = \delta_{lx} + i\delta_{ly}\,,\qquad e^{-2\delta_{ly}} = \tau_l\,,\qquad E_I > E_\alpha \Rightarrow \tan\delta_{lx} = \frac{B_{lf} + B_{le}}{\lambda_l + A_{lf} + A_{le}} + \frac{\tau_l - 1}{2K_I R_l}\,. \tag{101}$$

The parameters τ_l and R_l introduced here have

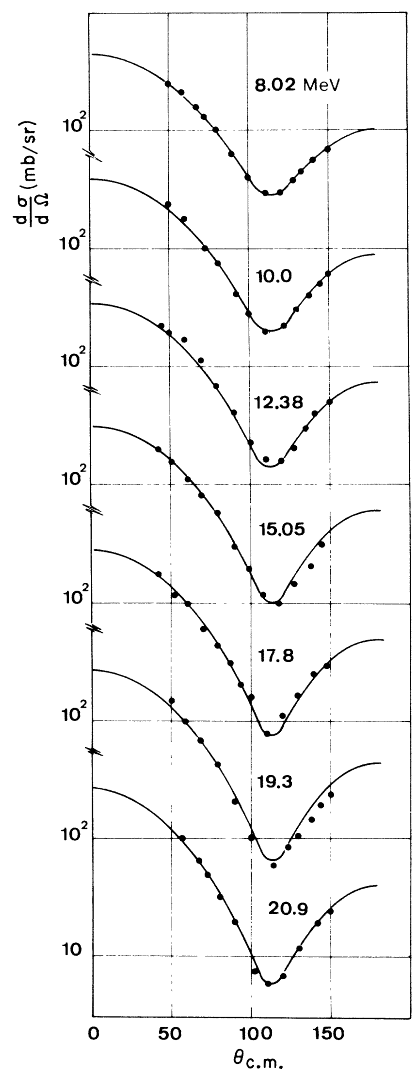

FIG. 16. n-α scattering. Theoretical cross sections. The experimental data are from Hoop and Barschall (Ref. 18).

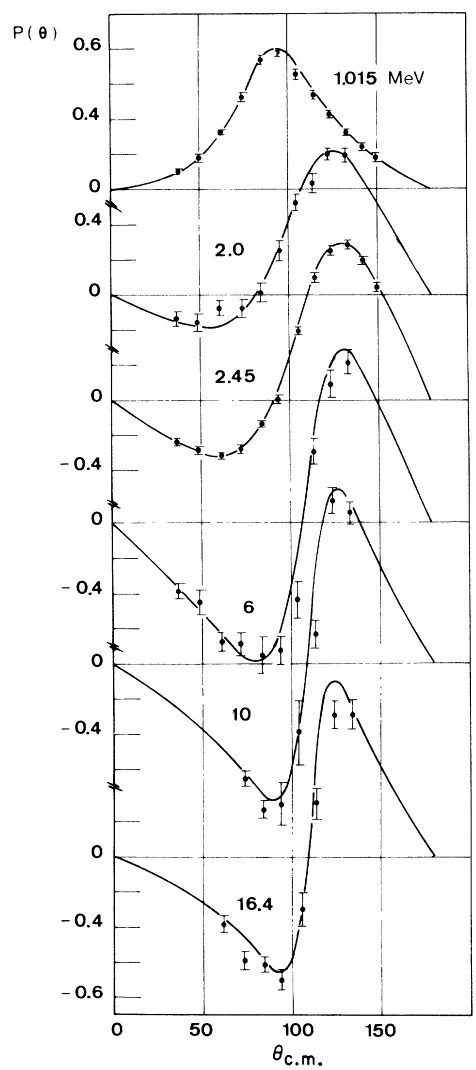

FIG. 17. n-α scattering. Theoretical polarizations. The experimental data are from Sawers *et al.* (1.015 and 2.45 MeV) (Ref. 34) and May, Walter, and Barschall (2.6–16.4 MeV) (Ref. 35).

TABLE IV. α-α scattering. Parameter values. When two values are given, the first is for $h_{lf}(K)$ and the second for $h_{le}(K)$.

	l					
	0	2	4	6	8	10
$\beta_{lf\,(\text{or } e)}$ (fm)$^{-1}$	0.43	0.98	1.57	1.7 0.5	1.65	1.70
λ_l (fm)^3	0.926	0.115	5.04×10^{-4}	2.25×10^{-5}	4.47×10^{-6}	1.94×10^{-7}
ρ_l^2	0.06	3.4	1.36	4.0	5.0	4.0
$d_{lf\,(\text{or } e)}$ (fm)$^{-1}$	0	0.38	0	0 −0.8	0	0

the values

$$\tau_l^2 = 1 - \frac{16K_lK_l'h_{lf}^2(K_l)h_{le}^2(K_l')}{(\lambda_l + A_{lf} + A_{le})^2 + (B_{lf} + B_{le})^2},$$
$$R_l = \frac{2h_{lf}^2(K_l)(\lambda_l + A_{lf} + A_{le})}{(\lambda_l + A_{lf} + A_{le})^2 + (B_{lf} + B_{le})^2}. \qquad (102)$$

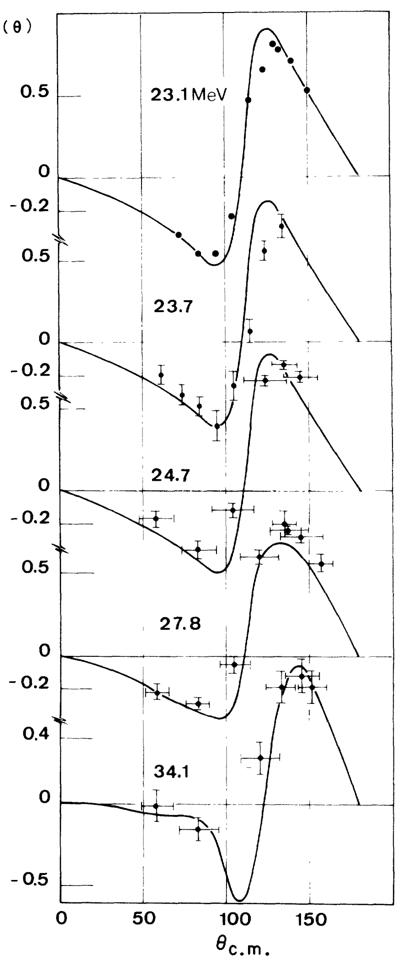

FIG. 18. n-α scattering. Theoretical polarizations. The experimental data are from Perkins and Glashausser (23.1 MeV) (Ref. 36); May, Walter, and Barschall (23.7 MeV) (Ref. 35); and Arifkanov *et al.* (24.7–34.1 MeV) (Ref. 37).

We must stress the fact that below the energy E_α of the first excited state of the α particle, the imaginary part of the phase shift vanishes independently of any choice of form factor. This corresponds to the experimental situation, since below 20 MeV in the c.m. the phase shifts appear to be purely real. As previously done, the form factors have been chosen to give, for $l=0$, a Yukawa form factor, i.e.,

$$h_{lf}(K) = (2l+1)!!C_l(\eta) \times \sum_{s=l}^{\infty} b_s \frac{K^s}{[(K-d_{lf})^2 + \beta_{lf}^2]^{(s+2)/2}} \frac{(s+1)!}{(2s+1)!!},$$
$$h_{le}(K) = \rho_l(2l+1)!!C_l(\eta) \times \sum_{s=l}^{\infty} b_s \frac{K^s}{[(K-d_{le})^2 + \beta_{le}^2]^{(s+2)/2}} \frac{(s+1)!}{(2s+1)!!}. \qquad (103)$$

The δ_0 phase shift vanishes at 19.5 MeV (lab system) and has thus been fitted with a two-identical-

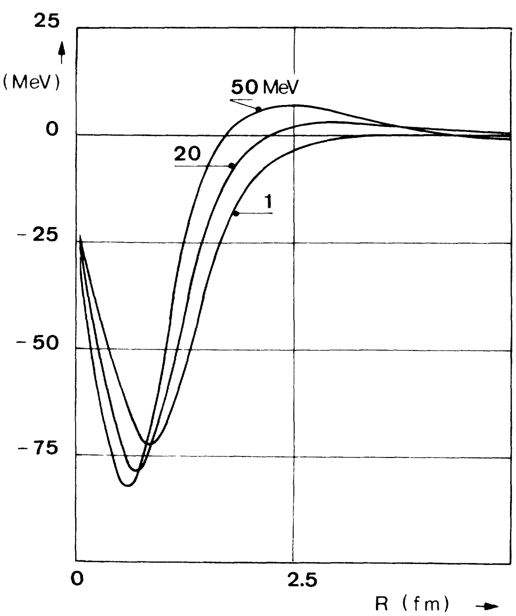

FIG. 19. n-α scattering. Local equivalent potential for $l=0$.

terms nonlocal separable potential. The C_{01} and C_{02} are then related through

$$\gamma_0 = -C_{02}/C_{01} = K_I^2/K_0^2\,, \tag{104}$$

where

$$K_I^2 = 2m^*/\hbar^2 E_I K_0^2 = (2m^*/\hbar^2)E_0 = (0.97)^2 \text{ fm}^{-2}\,. \tag{105}$$

Around 40 MeV a rapid variation in δ_2 and δ_4 corresponding to a compound-nucleus resonance is not explained by our formalism but may be reached by a one-level Breit-Wigner formula.[39] The parameters used are listed in Table IV and the corresponding phase shifts are shown in Fig. 20 and 21.[40]

VII. CONCLUSION

The transition amplitude of an elastic scattering phenomenon is formally easy to obtain with the use of nonlocal separable potentials. In practical applications three kinds of difficulties may appear.

First, the nuclear-interaction matrix element is expanded in a serial form of separable terms and this series has to be truncated. Usually at most only two-term nonlocal potentials are considered. The second problem arises from the determination of the form factors. Finally, the introduction of the internal structure of the target is easily tractable, only when few excitation states are taken into account.

We have shown that merely good results can be obtained in p-p elastic scattering even at low energy just by introducing Coulomb effects in a new form factor, the Coulomb transform of the usually used $g(K)$ form factor. We have then evaluated theoretically the phase shifts in nucleon-nucleon elastic scattering, and very good results were obtained for the angular distribution as well as for the polarization of the scattered particle.

Finally, for α-α elastic scattering we have shown that simple consideration of the first inelastic state of the target could explain the realness of the phase shifts below 30 MeV and leads to an excellent fit of the phase shift up to 100 MeV. Indeed, the elastic scattering description has not been exhaustively treated here, but the use of nonlocal potential seems to be very promising.

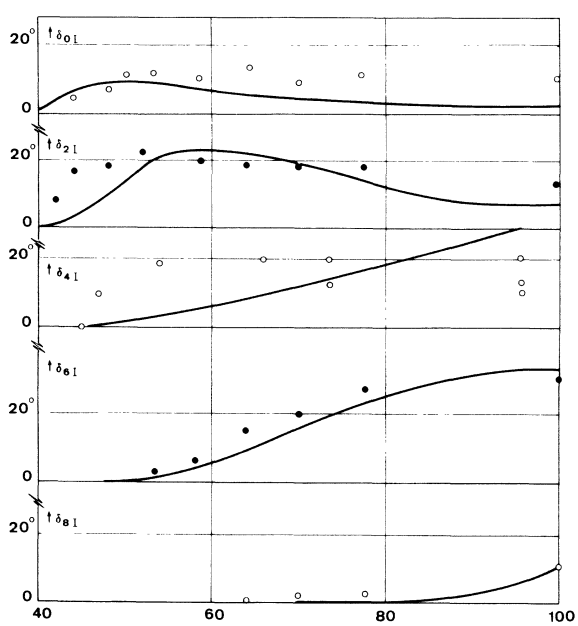

FIG. 20. α-α scattering. Imaginary part of phase shifts. The experimental data are from Darriulat (Ref. 40).

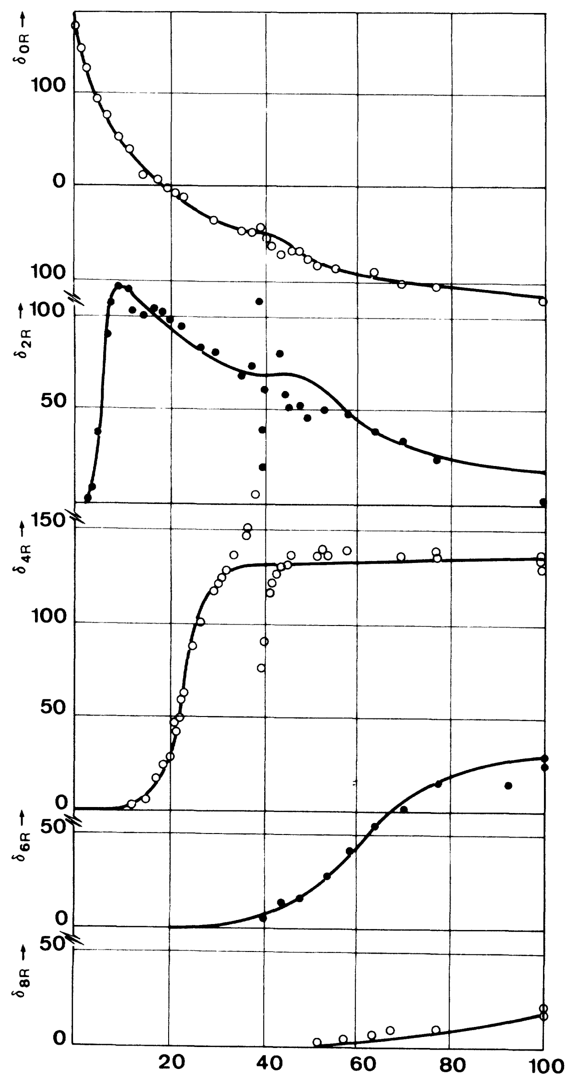

FIG. 21. α-α scattering. Real part of phase shifts. The experimental data are from Darriulat (Ref. 40).

APPENDIX A. NONLOCAL SEPARABLE POTENTIAL IN DIFFERENT REPRESENTATIONS

We set

$$\langle \vec{K}, \varphi_{\alpha i} | V_N | \vec{K}', \varphi_{\alpha j} \rangle = \sum_{\substack{l l' S p \\ J\mu}} C^{JS}_{ll'p} g^{JS}_{lip}(K) g^{JS}_{l'jp}(K') \mathcal{Y}_{(lS)J\mu}(\hat{K}) \mathcal{Y}_{(l'S)J\mu}(\hat{K}') \tag{A1}$$

and we want now the expression for $\langle \chi_K^+, \varphi_{\alpha i} | V_N | \chi_{K'}^+, \varphi_{\alpha j} \rangle$; i.e., the matrix element of the nonlocal separable potential when Coulomb effects are involved:

$$\langle \chi_K^+, \varphi_{\alpha i} | V_N | \chi_{K'}^+, \varphi_{\alpha j} \rangle = \mathop{\mathbf{S}}_{kk'} \int \langle \chi_K^+, \varphi_{\alpha i} | \vec{K}'', \varphi_{\alpha k} \rangle \langle \vec{K}'', \varphi_{\alpha k} | V_N | \vec{K}''', \varphi_{\alpha k'} \rangle \langle \vec{K}''', \varphi_{\alpha k'} | \chi_{K'}^+, \varphi_{\alpha j} \rangle d\vec{K}'' d\vec{K}''' .$$

We use the fact that

$$\langle \varphi_{\alpha i} | \varphi_{\alpha k} \rangle = \delta_{ik}, \quad \langle \varphi_{\alpha k'} | \varphi_{\alpha j} \rangle = \delta_{jk'} \tag{A2}$$

to obtain

$$\langle \chi_K^+, \varphi_{\alpha i} | V_N | \chi_{K'}^+, \varphi_{\alpha j} \rangle = \int \langle \chi_K^+ | \vec{K}'' \rangle d\vec{K}'' \langle \vec{K}'', \varphi_{\alpha i} | V_N | \vec{K}''', \varphi_{\alpha j} \rangle d\vec{K}''' \langle \vec{K}''' | \chi_{K'}^+ \rangle . \tag{A3}$$

We use the matrix-element value of the potential in the $\{K\}$ representation (A1) and expression (22) of $\langle \chi_K^+ | \vec{K}'' \rangle$ to obtain

$$\langle \chi_K^+, \varphi_{\alpha i} | V_N | \chi_{K'}^+, \varphi_{\alpha j} \rangle = \left(\frac{2}{\pi}\right)^2 \sum_{\lambda j \lambda' j'} \sum_{\substack{l l' S p \\ J\mu}} C^{JS}_{ll'p} e^{i(\sigma_\lambda - \sigma_{\lambda'})} \int F_\lambda(KK'') Y_{\lambda\nu}(\hat{K}) Y^*_{\lambda\nu}(\hat{K}'') g^{JS}_{lip}(K''') \mathcal{Y}_{(lS)J\mu}(\hat{K}'')$$
$$\times \mathcal{Y}^*_{(l'S)J\mu}(\hat{K}''') F_\lambda(KK''') Y_{\lambda'\nu'}(\hat{K}''') Y^*_{\lambda'\nu'}(\hat{K}') d\vec{K}'' d\vec{K}''' . \tag{A4}$$

The orthonormalization of the spherical harmonics leads to the result

$$\langle \chi_K^+, \varphi_{\alpha i} | V_N | \chi_{K'}^+, \varphi_{\alpha j} \rangle = \left(\frac{2}{\pi}\right)^2 \sum_{\substack{l l' S p \\ J\mu}} C^{JS}_{ll'p} \left[\int F_l(KK'') g^{JS}_{lip}(K'') K''^2 dK'' \right]$$
$$\times \left[\int F_{l'}(KK''') g^{J}_{ljp}(K''') K'''^2 dK''' \right] \mathcal{Y}_{(lS)J\mu}(\hat{K}'') \mathcal{Y}_{(l'S)J\mu}(\hat{K}''') .$$

We now set

$$h^{JS}_{lip}(K) = \frac{2}{\pi} \int F_l(KK'') g^{JS}_{lip}(K'') K''^2 dK'' \tag{A5}$$

to find the formula used to give the matrix element of the nonlocal separable potential

$$\langle \chi_K^+, \varphi_{\alpha i} | V_N | \chi_{K'}^+, \varphi_{\alpha j} \rangle = \sum_{\substack{l l' S p \\ J\mu}} C^{JS}_{ll'p} h^{JS}_{lip}(K) h^{JS}_{ljp}(K') \mathcal{Y}_{(lS)J\mu}(\hat{K}) \mathcal{Y}^*_{(l'S)J\mu}(\hat{K}') . \tag{A6}$$

APPENDIX B. SINGLET PHASE SHIFT IN p-p ELASTIC SCATTERING

By using the formula (62), we get for the phase shift δ_l the following:

$$\frac{e^{2i\delta_l} - 1}{2iK_l} = 2\left[\frac{1}{\lambda_{l1}} h_{l1}(K_l) \frac{\Delta_1}{\Delta} + \frac{1}{\lambda_{l2}} h_{l2}(K_l) \frac{\Delta_2}{\Delta_1}\right]. \tag{B1}$$

It can be shown that the phase shift δ_l is always real and we set

$$\Delta = \Delta_x + i\Delta_y, \quad \Delta_1 = \Delta_{1x} + i\Delta_{1y}, \quad \Delta_2 = \Delta_{2x} + i\Delta_{2y}. \tag{B2}$$

We thus obtain the following relation:

$$\frac{\sin 2\delta_l}{2K} + \frac{1 - \cos 2\delta_l}{2K} = \frac{2(\Delta_x - i\Delta_y)}{|\Delta|^2}\left[\frac{1}{\lambda_{l1}} h_{l1}(K_l)(\Delta_{1x} + i\Delta_{1y}) + \frac{1}{\lambda_{l2}} h_{l2}(K_l)(\Delta_{2x} + i\Delta_{2y})\right]. \tag{B3}$$

We can now express $\tan\delta_l$

$$\tan\delta_l = \frac{(1/\lambda_{l1})h_{l1}(K_l)(\Delta_x\Delta_{1y} - \Delta_y\Delta_{1x}) + (1/\lambda_{l2})h_{l2}(K_l)(\Delta_x\Delta_{2y} - \Delta_y\Delta_{2x})}{(1/\lambda_{l1})h_{l1}(K_l)(\Delta_x\Delta_{1x} + \Delta_y\Delta_{1y}) + (1/\lambda_{l2})h_{l2}(K_l)(\Delta_x\Delta_{2x} + \Delta_y\Delta_{2y})}. \quad \text{(B4)}$$

The imaginary parts of Δ_1 and Δ_2, i.e., Δ_{1y} and Δ_{2y}, may be extracted from the definitions (59) of Δ_1 and Δ_2 and from the decomposition (65) of G_{lpq} into its real and imaginary parts. It thus appears that

$$\Delta_{1y} = 0, \quad \Delta_{2y} = 0. \quad \text{(B5)}$$

The phase shift is then given by the simple expression

$$\tan\delta_l = -(\Delta_y/\Delta_x). \quad \text{(B6)}$$

If we remember now that

$$\Delta = \Delta_x + i\Delta_y = \left(1 + \frac{4}{\pi}\frac{1}{\lambda_{l1}}G_{l11}\right)\left(1 + \frac{4}{\pi}\frac{1}{\lambda_{l2}}G_{l22}\right) - \left(\frac{4}{\pi}\right)^2 \frac{1}{\lambda_{l1}}\frac{1}{\lambda_{l2}}G_{l21}G_{l12} \quad \text{(B7)}$$

and

$$G_{lpq} = \tfrac{1}{4}\pi(A_{lpq} - iB_{lpq}),$$

we obtain

$$\Delta_x = 1 + \frac{1}{\lambda_{l1}}A_{l11} + \frac{1}{\lambda_{l2}}A_{l22} + \frac{1}{\lambda_{l1}\lambda_{l2}}(A_{l11}A_{l22} - A_{l12}{}^2),$$
$$\Delta_y = -\left(1 + \frac{1}{\lambda_{l1}}A_{l11}\right)\frac{1}{\lambda_{l2}}B_{l22} - \left(1 + \frac{1}{\lambda_{l2}}A_{l22}\right)\frac{1}{\lambda_{l1}}B_{l11} + \frac{2}{\lambda_{l1}\lambda_{l2}}A_{l12}B_{l12}. \quad \text{(B8)}$$

This gives expression (64) when we set $\gamma_l = \lambda_{l1}/\lambda_{l2}$ and use the value (B6) for $\tan\delta_l$.

APPENDIX C. REAL AND IMAGINARY PARTS OF PHASE SHIFT WITH ONE-TERM NONLOCAL SEPARABLE POTENTIAL

The relation (90),

$$\frac{e^{2i\delta_l} - 1}{2iK_l} = \frac{2[h_{lf}(K_l)]^2}{\lambda_l + (4/\pi)G_l},$$

may be written, when using the fact that $\delta_l = \delta_{lx} + i\delta_{ly}$ and $\tau_l = e^{-2\delta_{ly}}$,

$$\frac{\tau_l \sin 2\delta_{lx}}{2K_l} = \frac{2[h_{lf}(K_l)]^2(\lambda_l + A_{lf} + A_{le})}{(\lambda_l + A_{lf} + A_{le})^2 + (B_{lf} + B_{le})^2} = R. \quad \text{(C1)}$$

$$\frac{1 - \tau_l \cos 2\delta_{lx}}{2K_l} = \frac{2[h_{lf}(K_l)]^2(B_{lf} + B_{le})^2}{(\lambda_l + A_{lf} + A_{le})^2 + (B_{lf} + B_{le})^2} = S. \quad \text{(C2)}$$

It follows immediately that

$$\tau_l{}^2 - 1 = 4K_l{}^2(R^2 + S^2) - 4K_lS \quad \text{(C3)}$$

and the value (C1) and (C2) of the parameters R and S lead to the relation

$$\tau_l{}^2 - 1 = \frac{8K_l[h_{lf}(K_l)]^2}{(\lambda_l + A_{lf} + A_{le})^2 + (B_{lf} + B_{le})^2} \times \{2K_l[h_{lf}(K_l)]^2 - (B_{lf} + B_{le})\}. \quad \text{(C4)}$$

Let us now apply these formulas to different cases.

First Case: $E < E_\alpha$

We have seen that

$$B_{le} = 0,$$
$$B_{lf} = 2K_l[h_{lf}(K_l)]^2. \quad \text{(C5)}$$

It follows then

$$\tau_l{}^2 - 1 = 0. \quad \text{(C6)}$$

The imaginary part vanishes and

$$\tan\delta_{lx} = S/R = \frac{B_{lf}}{\lambda_l + A_{lf} + A_{le}}. \quad \text{(C7)}$$

Second Case: $E > E_\alpha$

In this case B_{le} takes the value

$$B_{le} = 2K_l'[h_{le}(K_l')]^2, \quad \text{(C8)}$$

and

$$\tau_l{}^2 - 1 = \frac{16K_lK_l'[h_{lf}(K_l)]^2[h_{le}(K_l')]^2}{(\lambda_l + A_{lf} + A_{le})^2 + (B_{lf} + B_{le})^2}. \quad \text{(C9)}$$

We can note here that $0 \leq \tau_l \leq 1$. The real part takes now the value

$$\tan\delta_{lx} = \frac{S}{R} + \frac{\tau_l{}^{-1}}{2K_lR} = \frac{B_{lf} + B_{le}}{\lambda_l + A_{lf} + A_{le}} + \frac{\tau_l{}^{-1}}{2K_lR}. \quad \text{(C10)}$$

[1]T. R. Mongan, Phys. Rev. 178, 1597 (1969).
[2]Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).
[3]A. N. Mitra, Phys. Rev. 123, 1892 (1961).
[4]F. Tabakin, Ann. Phys. (N.Y.) 30, 51 (1964).
[5]T. Hamman, G. Oberlechner, G. Trapp, and J. Yoccoz, J. Phys. (Paris) 28, 755 (1967).
[6]F. Villars, in *Fundamentals in Nuclear Theory,* edited by A. De-Shalit and C. Villi (International Atomic Energy Agency, Vienna, Austria, 1967).
[7]*Handbook of Mathematical Functions,* edited by M. Abramovitz and I. A. Stegun (Dover Publications, New York, 1965).
[8]D. R. Harrington, Phys. Rev. 139, 691 (1965).
[9]L. Hulthén and M. Sugawara, *Encyclopedia of Physics,* edited by S. Flügge (Springer-Verlag, Berlin, Germany, 1957), Vol. XXIV.
[10]H. P. Noyes, D. S. Bailey, R. H. Arndt, and M. H. MacGregor, Phys. Rev. 139, B1380 (1965).
[11]L. Brown and W. Trachslin, Nucl. Phys. A90, 334 (1967); L. Brown, W. Haeberli, and W. Trächslin, Nucl. Phys. A90, 339 (1967).
[12]A. C. L. Barnard, C. M. Jones, and J. L. Weil, Nucl. Phys. 50, 604 (1964).
[13]W. G. Weitkamp and W. Haeberli, Nucl. Phys. 83, 46 (1966).
[14]G. E. Thompson, M. P. Epstein, and T. Sawada, Nucl. Phys. A142, 571 (1970).
[15]G. R. Satchler, L. W. Owen, A. J. Elwyn, G. L. Morgan, and R. L. Walter, Nucl. Phys. A112, 1 (1968).
[16]B. W. Davies, M. K. Craddock, R. C. Hanna, Z. J. Moroz, and L. P. Robertson, Nucl. Phys. A97, 241 (1967).
[17]G. L. Morgan and R. L. Walter, Phys. Rev. 168, 1114 (1968).
[18]B. Hoop and H. H. Barschall. Nucl. Phys. 83, 65 (1966).
[19]A. Martin, Nuovo Cimento 7, 607 (1958).
[20]C. Bloch, Cours sur la Théorie des Réactions Nucléaires, Saclay, 1955.
[21]A. M. Lane and R. G. Thomas, Rev. Mod. Phys. 30, 257 (1958).
[22]P. Darriulat, D. Garreta, A. Tarrats, and J. Testoni, Nucl. Phys. A108, 316 (1968).
[23]K. W. Brockman, Phys. Rev. 108, 1000 (1957).
[24]P. W. Allison and R. Smythe, Nucl. Phys. A121, 97 (1968).
[25]S. M. Bunch, H. H. Forster, and C. C. Kim, Nucl. Phys. 53, 241 (1964).
[26]M. K. Brussel and J. H. Williams, Phys. Rev. 106, 286 (1957).
[27]S. N. Bunker, J. M. Cameron, M. B. Epstein, G. Paic, J. R. Richardson, J. G. Rogers, P. Thomas, and J. W. Verba, Nucl. Phys. A133, 537 (1969).
[28]L. Drigo, C. Manduchi, G. E. Nardelli, M. R. Russo-Manduchi, and G. Zannoni, Nucl. Phys. 60, 441 (1964).
[29]R. I. Brown, W. Haeberli, and J. X. Saladin, Nucl. Phys. 47, 212 (1963).
[30]L. Rosen, J. E. Brolley, M. L. Gursky, and L. Stewart, Phys. Rev. 124, 199 (1961); L. Rosen and J. E. Brolley, Phys. Rev. 117, 1454 (1957).
[31]F. W. Busser, F. Neibergall, G. Sohngen, and J. Christiansen, Nucl. Phys. 88, 593 (1966).
[32]D. Garreta, J. Sura, and A. Tarrats, Nucl. Phys. A132, 46 (1969).
[33]M. K. Craddock, R. C. Hanna, L. P. Robertson, and B. W. Davies, Phys. Letters 5, 335 (1963).
[34]J. R. Sawers, G. L. Morgan, L. A. Schaller, and R. L. Walter, Phys. Rev. 168, 1102 (1968).
[35]T. H. May, R. L. Walter, and H. H. Barschall, Nucl. Phys. 45, 17 (1963).
[36]R. B. Perkins and G. Glaushausser, Nucl. Phys. 60, 433 (1964).
[37]U. R. Arifkhanov, N. A. Vlasov, V. V. Davydov, and L. N. Samoilov, Yadern. Fiz. 2, 239 (1965) [transl.: Soviet J. Nucl. Phys. 2, 170 (1966).
[38]M. Coz, L. G. Arnold, and A. D. MacKellar, Ann. Phys. (N.Y.) 59, 219 (1970).
[39]J. Pigeon, Ph.D. thesis, University of Lyon, France, 1970 (unpublished).
[40]P. Darriulat, Ph.D. thesis, University of Paris, 1965 (unpublished).