Extrapolation to the Limit of Singular-Core Interactions in the Three-Body Problem*

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Several aspects of the three-body problem with singular-core interactions given by the boundary-condition model (BCM), are studied. The kernel of the Faddeev equations for these interactions is shown to have an infinite Schmidt norm even if the two-body interactions are confined to a single partial wave. This does not necessarily imply that the Faddeev kernel is noncompact. A proof of the compactness (or noncompactness) of the kernel has not been found. We consider a family of two-body interactions, with square repulsions of strength V for particle separation $r<r_0$, which become the BCM in the limit $V_0 \to \infty$. For finite V_0, the Faddeev kernel has a finite Schmidt norm (and is hence compact) and standard numerical matrix texhniques may be used for solving the three-body equations. Simplified calculations of the triton binding energy, using two-body s-wave interactions of this type, are carried out for a number of choices of V_0. The two-body potentials for $r>r_0$ are not varied. It is found that the three-particle binding energy has a simple dependence on V_0. The value of the binding energy extrapolated to the limit $V_0=\infty$ is found to be in excellent agreement with the result of a previous calculation based on BCM two-particle interactions and numerical methods predicated on the assumed validity of standard matrix-inversion techniques. Some implications of these results for more realistic calculations on three-body systems with two-body singular-core interactions are discussed.

I. INTRODUCTION

In a recent series of papers,[1,2] we have derived and studied numerically the completely off-shell two-body t matrix for the case of singular-core interactions given by the boundary-condition model (BCM).[3] We have used this t matrix, for the special case of a hard-core repulsion, in a simplified calculation of the triton binding energy[4] based on the Faddeev equations.[5] A similar calculation, with different numerical procedures, was done by Fuda.[6]

Alternative derivations of the general BCM two-body t matrix have subsequently been given by Fuda[7] and Brayshaw.[8] The latter[8] showed that the pure BCM t matrix $t_{l,\mathrm{BCM}}(k'|q|k)$ given in Kim and Tubis (KT)[1] was unique if one assumes: (a) that it is analytic in the complex q^2 plane except for the unitarity cut and possible bound-state poles for negative q^2, and (b) that

$$\lim_{|q^2|\to\infty}\left|\frac{t_{l,\mathrm{BCM}}(k'|q|k)}{q^2}\right|<\infty .$$

The Faddeev equations, which have two-body off-shell t matrices as "driving terms," appear to be the only formalism now available for deducing the three-body implications of general two-body BCM interactions. (With the variational method, one can handle only the specialized case of two-body interactions with hard-core singular behavior.)

Brayshaw[9] has recently shown that the Faddeev equations do not have a unique solution when two-body interactions have BCM singular-core behavior in *all* partial waves. He was able to derive modified equations which have a unique solution with correct spacial and unitarity properties.

Brayshaw's analysis does not directly apply to the calculations of KT,[4] and Fuda,[6] where only two-body s-wave interactions are assumed. It does, however, emphasize the need to carefully examine the properties of the Faddeev equations in the case of two-body singular-core interactions.

In Sec. II, we show that the kernel of the Faddeev equations, with the two-body interaction given by the BCM, has an infinite Schmidt norm,[10] even if the two-body interactions are confined to a single partial wave. This does not necessarily mean that the kernel is noncompact,[10] and we have not as yet found a proof of the compactness (or noncompactness) of the kernel for this case.

It seems reasonable to expect that for BCM two-body interactions, three-body parameters, such as the binding energy, should be practically obtainable as extrapolated values of corresponding parameters which are determined for an appropriate sequence "soft-core" two-body interactions. More precisely, consider a local two-particle potential $V(r)$, with r the particle separation, and $V(r<r_0)=V_0$, $V(r>r_0)=\tilde{V}(r)$. For a fixed "outside" potential $\tilde{V}(r)$ and sufficiently large V_0, we expect to find three-body parameters to have a simple variation with V_0 which allows an accurate determination of these parameters in the limit $V_0\to\infty$.

In Sec. III, we give the complete off-shell t matrix for a soft-core interaction which becomes the

Herzfeld potential in the limit of infinite core repulsion. In Sec. IV, the simplified triton binding-energy calculation of KT[4] is repeated with this soft-core interaction. The matrix-inversion method used in solving the Faddeev equations is fully justified here because of the finite Schmidt norm and hence compactness of the Faddeev kernel.[10] We show in Sec. V that the binding energy does indeed have a simple dependence on V_0, and find the value extrapolated to the limit of infinite core repulsion to be in excellent agreement with that found in KT.[4] (The calculation of KT[4] was based on two-body BCM interactions and matrix-inversion techniques which are valid in the case of a compact Faddeev kernel.)

Some implications of these results for more realistic calculations on three-body systems with two-body singular-core interactions are given in Sec. VI.

II. SCHMIDT NORM OF THE FADDEEV KERNEL FOR THE CASE OF TWO-BODY BCM INTERACTIONS

The considerations of this paper are limited to bound-state problems. We therefore assume negative values of S, the total three-body center-of-mass energy. For simplicity, we discuss the Faddeev equation for a total angular momentum $J=0$ state of three spinless bosons which interact via a two-body s-wave potential. Our notation is that of Kim.[11] The homogeneous from of Eq. (1) of this reference may be written as

$$\left[\frac{pq\psi_S(p,q)}{p^2+q^2-S}\right]=\int_0^\infty dq_2\int_{(1/\sqrt{3})|2q-q_2|}^{(1/\sqrt{3})|2q+q_2|} dp_2 K_S(p,q;p_2,q_2)\times\left[\frac{p_2q_2\psi_S(p_2,q_2)}{p_2{}^2+q_2{}^2-S}\right], \tag{2.1}$$

where

$$K_S(p,q;p_2,q_2)=\frac{8}{\sqrt{3}\pi}\,\frac{p}{p^2+q^2-S}\,t(p|\sqrt{S-q^2}\,|p_1), \tag{2.2}$$

$$(p_1{}^2=p_2{}^2+q_2{}^2-q^2),$$

and $t(p|\sqrt{S-q^2}\,|p_1)$ is the completely off-shell s-wave two-body t matrix which is normalized so that on the energy shell

$$t(p|p|p)=\frac{e^{i\delta_0(p)}\sin\delta_0(p)}{p}, \tag{2.3}$$

where $\delta_0(p)$ is the real s-wave phase shift. For the pure BCM, with parameters f and r_0, we have

$$t_{0,\,\mathrm{BCM}}(p|q|p_1)=\frac{q^2-p_1{}^2}{2pp_1}\left[\frac{\sin(p-p_1)r_0}{p-p_1}-\frac{\sin(p+p_1)r_0}{p+p_1}\right]+\left(\frac{f}{r_0}-iq\right)^{-1}\left(\frac{f\sin pr_0}{pr_0}-\cos pr_0\right)\times\left(-\cos p_1r_0+i\frac{q}{p_1}\sin p_1r_0\right). \tag{2.4}$$

Bound-state eigenfunctions correspond to solutions of (2.1). The bound-state momentum-space wave functions, $\varphi_{B,S_0}(p,q)$, are given by

$$\varphi_{B,S_0}(p,q)=\frac{\text{const}}{p^2+q^2-S_0}\psi_{S_0}(p,q), \tag{2.5}$$

with

$$\int_0^\infty\int_0^\infty|\varphi_{B,S_0}(p,q)|^2p^2q^2dpdq=1. \tag{2.6}$$

A sufficient condition for the kernel $K_S(pq;p_2q_2)$ to be compact with respect to the Hilbert space of functions

$$\varphi(p,q)=\frac{pq\psi(p,q)}{p^2+q^2+\alpha^2},\quad(\alpha^2>0), \tag{2.7}$$

with

$$\int_0^\infty\int_0^\infty|\varphi(p,q)|^2dpdq<\infty, \tag{2.8}$$

is that the Schmidt norm of K_S,

$$\|K_S\|^2=\int dq\int dq\int dq_2\int_{(1/\sqrt{3})|2q-q_2|}^{(1/\sqrt{3})|2q+q_2|} dp_2|K_S(p,q;p_2,q_2)|^2, \tag{2.9}$$

is finite.[10]

It can be shown, however, that $\|K_S\|^2=\infty$ when K_S is given by (2.2) and (2.4). Consider, for example, the norm of the contribution to K_S, which comes from the second term on the right-hand side of (2.4). It contains the integral,

$$I=\int_0^\infty dq_2\int_{(1/\sqrt{3})|2q-q_2|}^{(1/\sqrt{3})|2q+q_2|}dp_2\left(\cos p_1r_0+\frac{\sqrt{q^2-S}}{p_1}\sin p_1r_0\right)^2$$
$$=\int_0^\infty dq_2\int_{(1/\sqrt{3})|2q_2-q|}^{(1/\sqrt{3})|2q_2+q|}\frac{p_1dp_1}{p_2}\times\left(\cos p_1r_0+\frac{\sqrt{q^2-S}}{P_1}\sin p_1r_0\right)^2, \tag{2.10}$$

which satisfies the inequality

$$I>3\int_0^\infty\frac{dq_2}{(2q+q_2)(2q_2+q)}\int_{(1/\sqrt{3})|2q_2-q|}^{(1/\sqrt{3})|2q_2+q|}p_1{}^2dp_1\times\left(\cos p_1r_0+\frac{\sqrt{q^2-S}}{p_1}\sin p_1r_0\right)^2, \tag{2.11}$$

where the p_1 integration may be done analytically. We find that

$$I > \int_{q/2}^{\infty} \frac{dq_2}{(2q+q_2)(2q_2+q)} \times \left[\frac{4}{\sqrt{3}} q_2^2 q + \frac{2q_2^2}{r_0} \cos\frac{4r_0 q_2}{\sqrt{3}} \sin\frac{2r_0 q}{\sqrt{3}}\right] + \text{convergent integrals} = \infty . \qquad (2.12)$$

By using similar arguments, it is easy to show that the norm of the total K_S is infinite. The addition of a reasonable two-body interaction for $r > r_0$ will not change the situation.

III. OFF-ENERGY-SHELL t MATRIX FOR SOFT-CORE INTERACTIONS

In this section, we give an explicit expression for the complete off-shell t matrix in the case of a square-well from for the core repulsion and the interaction outside the core. In the limit of infinite core repulsion, this interaction becomes the Herzfeld potential. It will be used in Secs. IV and V to do a "soft-core" version of the binding-energy calculation of KT.[4] We will then determine whether or not the calculation of KT gives the "physical" result (the one obtained by extrapolating the soft-core results to the hard-core limit).

For the more general case of a soft-core BCM model with an arbitrary "outside" interaction, we would use the model for the core region which was introduced in Ref. 1, with U_0 and U_1 finite and

$$r_0(\sqrt{U_0} - r_0 U_1) = f . \qquad (3.1)$$

In extrapolating to the BCM limit, we would fix r_1 and f and let $U_0 \to \infty$. Expressions for the t matrix for this case may be easily derived. However, they are very complicated and will not be presented in this paper.

For each partial wave, the parameters of the two-body interaction are as follows:

$$V_l(r) = (\hbar^2/2\mu)U_0 > 0 , \quad r < r_0 ; \qquad (3.2a)$$

$$V_l(r) = (\hbar^2/2\mu)U_1 < 0 , \quad r_0 < r < r_1 ; \qquad (3.2b)$$

$$V_l(r) = (\hbar^2/2\mu)U_2 = 0 , \quad r > r_1 ; \qquad (3.2c)$$

where μ is the reduced mass of the two nucleons, r_0 is the soft-core radius, r_1 is the outer radius of the "outside" interaction, and the U_i's are constants proportional to the interaction strength.

Since the details of deriving the off-shell t matrix are outlined in Ref. 1, we give here only the final expression,

$$\begin{aligned} t_l(k'|q|k) = &-\frac{(A_0U_0 - A_1U_1)r_0^2\chi_l(k'k|r_0) - A_1U_1r_1^2\chi_l(k'k|r_1)}{k'^2 - k^2} \\ &- \frac{U_0}{k'^2 - \alpha_0^2}\chi_l(k'\alpha_0|r_0)\Big\{(A_1 - A_0)\frac{D_l(1)}{D_l}\frac{d}{dr'}[r'j_l(kr')]\Big|_{r'=r_0} \\ &\qquad + B_l^+(1)\frac{d}{dr'}[r'h_l^{(+)}(\alpha_1 r')]\Big|_{r'=r_0} + B_l^-(1)\frac{d}{dr'}[r'h_l^{(-)}(\alpha_1 r')]\Big|_{r'=r_0}\Big\} \\ &- \Big\{B_l^+(1)\frac{d}{dr'}[r'j_l(\alpha_0 r')]\Big|_{r'=r_0} - B_l^+(2)r_0 j_l(\alpha_0 r_0)\Big\} U_1\frac{[r_1^2\chi_l(k'\alpha_1^+|r_1) - r_0^2\chi_l(k'\alpha_1^+|r_0)]}{k'^2 - \alpha_1^2} \\ &- \Big\{B_l^-(1)\frac{d}{dr'}[r'j_l(\alpha_0 r')]\Big|_{r'=r_0} - B_l^-(2)r_0 j_l(\alpha_0 r_0)\Big\} U_1\frac{[r_1^2\chi_l(k'\alpha_1^-|r_1) - r_0^2\chi_l(k'\alpha_1^-|r_0)]}{k'^2 - \alpha_1^2} \end{aligned} \qquad (3.3)$$

with

$$D_l = D_l(1)\frac{d}{dr'}[r'j_l(\alpha_0 r')]\Big|_{r'=r_0} + r_0 j_l(\alpha_0 r_0)D_l(2) , \qquad (3.4)$$

$$D_l(1) = -\alpha_1^2[r_0h_l^{(+)}(\alpha_1 r_0)\chi_l(\alpha_1^-\alpha_2^+|r_1) - r_0h_l^{(-)}(\alpha_1 r_0)\chi_l(\alpha_1^+\alpha_2^+|r_1)] , \qquad (3.5)$$

$$D_l(2) = -\alpha_1^2\Big\{-\frac{d}{dr'}[r'h_l^{(+)}(\alpha_1 r')]\Big|_{r'=r_0}\chi_l(\alpha_1^-\alpha_2^+|r_1) + \frac{d}{dr'}[r'h_l^{(-)}(\alpha_1 r')]\Big|_{r'=r_0}\chi_l(\alpha_1^+\alpha_2^+|r_1)\Big\} , \qquad (3.6)$$

$$B_l^+(1) = \frac{-\alpha_1^2}{D_l}[-(A_1 - A_0)r_0 j_l(kr_0)\chi_l(\alpha_1^-\alpha_2^+|r_1) + (A_1 - 1)r_0h_l^{(-)}(\alpha_1 r_0)\chi_l(k\alpha_2^+|r_1)] , \qquad (3.7)$$

$$B_l^-(1) = \frac{\alpha_1^2}{D_l}[-(A_1 - A_0)r_0 j_l(kr_0)\chi_l(\alpha_1^+\alpha_2^+|r_1) + (A_1 - 1)r_0h_l^{(+)}(\alpha_1 r_0)\chi(k\alpha_2^+|r_1)] , \qquad (3.8)$$

$$B_l^+(2)=\frac{-\alpha_1^2}{D_l}\left\{-(A_1-A_0)\frac{d}{dr'}[r'j_l(kr')]\Big|_{r'=r_0}\chi_l(\alpha_1^-\alpha_2^+|r_1)+(A_1-1)\frac{d}{dr'}[r'h_l^{(-)}(\alpha_1 r')]\Big|_{r'=r_0}\chi_l(k\alpha_2^+|r_1)\right\}, \quad (3.9)$$

$$B_l^-(2)=\frac{\alpha_1^2}{D_l}\left\{-(A_1-A_0)\frac{d}{dr'}[r'j_l(kr')]\Big|_{r'=r_0}\chi_l(\alpha_1^+\alpha_2^+|r_1)+(A_1-1)\frac{d}{dr'}[r'h_l^{(+)}(\alpha_1 r')]\Big|_{r'=r_0}\chi_l(k\alpha_2^+|r_1)\right\}. \quad (3.10)$$

The χ_l's are Wronskians defined by

$$\chi_l(k\alpha_2^+|r_1)=\begin{vmatrix} r_1 j_l(kr_1) & r_1 h_l^{(+)}(\alpha_2 r_1) \\ (d/dr)[rj_l(kr)]|_{r=r_1} & (d/dr)[rh_l^{(+)}(\alpha_2 r)]|_{r=r_1}\end{vmatrix}, \quad (3.11)$$

and similar expressions for other χ_l's. The $h_l^{(\pm)}(x)$ are the lth-order spherical Hankel functions of the first (second) kind, the $A_i (i=0,1)$ are given by

$$A_i=(q^2-k^2)/(q^2-k^2-U_i), \quad (3.12)$$

and the α_i $(i=0, 1, 2)$ by

$$\alpha_i=(q^2-U_i)^{1/2}. \quad (3.13)$$

In the hard-core limit ($U_0\to\infty$), we have $A_0=0$, $A_0U_0=-(q^2-k^2)$, $U_0/(k'^2-\alpha_0^2)=1$, and

$$\left|\frac{d}{dr}[rj_l(\alpha_0 r)]\Big|_{r=r_0}\right| \gg |r_0 j_l(\alpha_0 r_0)| \quad (3.14)$$

for $|\alpha_0|\to\infty$ and $\mathrm{Im}\alpha_0>0$, so that (3.3) reduces to (4.11) of Ref. 1.

For the special case of $l=0$ and $q^2<0$, which we will utilize in the next section, (3.3) can be simplified to:

$$\begin{aligned} t_0(k'|q|k) = &-\frac{q^2-k^2}{q^2-k^2-U_0}U_0\frac{r_0^2}{k'^2-k^2}\chi_0(k'k|r_0)-\frac{q^2-k^2}{q^2-k^2-U_1}U_1\frac{r_1^2\chi_0(k'k|r_1)-r_0^2\chi_0(k'k|r_0)}{k'^2-k^2} \\ &-\frac{U_0}{k'^2-q^2+U_0}\left(\frac{\sin k'r}{k'}-\frac{\tanh\gamma_0}{\gamma_0}\cos k'r_0\right) \\ &\times\left[\left(\frac{q^2-k^2}{q^2-k^2-U_1}-\frac{q^2-k^2}{q^2-k^2-U_0}\right)\frac{D_0(1)\cosh\gamma_0\cos kr_0}{D_0}+B_0^+(1)e^{i\alpha_1 r_0}+B_0^-(1)e^{-i\alpha_1 r_0}\right] \\ &-\left(B_0^+(1)-B_0^+(2)\frac{\tanh\gamma_0}{\gamma_0}\right)U_1\frac{r_1^2\chi_0(k'\alpha_1^+|r_1)-r_0^2\chi_0(k'\alpha_1^+|r_0)}{k'^2-\alpha_1^2} \\ &-\left(B_0^-(1)-B_0^-(2)\frac{\tanh\gamma_0}{\gamma_0}\right)U_1\frac{r_1^2\chi_0(k'\alpha_1^-|r_1)-r_0^2\chi_0(k'\alpha_1^-|r_0)}{k'^2-\alpha_1^2}, \end{aligned} \quad (3.15)$$

where

$$\gamma_0=\sqrt{|q^2-U_0|}=|\alpha_0|. \quad (3.16)$$

IV. THREE-BODY BOUND-STATE CALCULATION WITH SOFT-CORE INTERACTIONS

In this section, we discuss the numerical solution of (2.1) when the two-body t matrix is given by (3.15). Factors such as $U_0/(q^2-k^2-U_0)$ in $t_0(k'|q|k)$, which tend to constants in the hard-core limit, give enough inverse powers of momenta to yield a finite Schmidt norm for the kernel of (2.1). The kernel is thus compact and the usual numerical quadrature techniques for solving (2.1) are fully justified.

The numerical methods for solving (2.1) are discussed in detail by KT,[4] and we will only give a brief description of them here. In (2.1) we discretize the p and q variables with sets of N_p and N_q points, respectively. For the q_2 integration, we use N_q-point Gaussian quadrature. For the p_2 integration, we use the approximate-product technique in order to avoid the difficulties associated with the variable integration limits. This involves an expansion of $\psi(p,q)$ in a set of linearly independent functions $f_m(p)$ for fixed q, namely

$$\psi(p,q_j)\cong\sum_{m=1}^{N_p}b_m(q_j)f_m(p), \quad (4.1)$$

where the q_j's are fixed Gaussian-quadrature points. In the calculation of the triton binding energy, with two-body s-wave Herzfeld potentials, in KT,[4] the following forms were used for the $f_m(p)$:

$$f_m(p) = \frac{1}{\alpha + p}\left(\frac{p}{\alpha + p}\right)^m, \quad m = 0, 1, 2, \ldots, N_p - 1, \tag{4.2}$$

and

$$f_m(p) = \frac{\sin pr_0}{p} + \frac{1}{\alpha + p}\left(\frac{p}{\alpha + p}\right)^m, \quad m = 1, 2, \ldots, N_p - 1. \tag{4.3}$$

Both of these choices for $f_m(p)$ give results of 8.13 MeV for the triton binding energy with $N_p = 8$ and $N_q = 10$.

We present here another possible choice for the $f_m(p)$ based upon an infinite separable expansion of the t matrix. In (3.15), we note that the t matrix has the separable expansion

$$t_0(k'|q|k) = \sum_m C_m g_m(k') h_m(k, q) \tag{4.4}$$

if the integral parts can be expanded in the form $\sum_m C_m g_m(k') h_m(k, q)$. To derive such an expansion for the integrals, we first note that

$$\int_0^{r_0} dr\, r\, j_l(k'r) r\, j_l(kr) = \frac{r_0^2}{k'^2 - k^2} \chi_l(k'k|r_0). \tag{4.5}$$

Using the definition for the χ_l's given in (3.11), it is straightforward to show that

$$\frac{r_0^2}{k'^2 - k^2}[\chi_l(k'k|r_0) - \chi_{l+2}(k'k|r_0)] = \frac{(2l+3)r_0}{k'k} j_{l+1}(k'r_0) j_{l+1}(kr_0), \tag{4.6}$$

with help of various recursion relations involving the spherical Bessel functions and their derivatives. Combining (4.5) and (4.6), we obtain the separable expansion[12]:

$$\int_0^{r_0} dr\, r\, j_l(k'r) r\, j_l(kr) = \frac{r_0}{k'k} \sum_{m=1}^{\infty} [2(l+2m) - 1] \times j_{l+2m-1}(k'r_0) j_{l+2m-1}(kr_0). \tag{4.7}$$

Similarly, we obtain a separable expansion for the integral

$$\int_0^{r_0} dr\, r\, j_l(k'r) r n_l(kr) = \frac{r_0}{k'k} \sum_{m=1}^{\infty} [2(l+2m) - 1] \times j_{l+2m-1}(k'r_0) n_{l+2m-1}(kr_0), \tag{4.8}$$

where $n_l(kr)$ is the spherical Neumann function. In obtaining (4.8), use is made of the relation

$$\int_0^{r_0} dr\, r\, j_l(k'r) r n_l(kr) = \frac{r_0^2}{k'^2 - k^2} W_l(k'k|r_0), \tag{4.9}$$

where the Wronskian W_l is given by

$$W_l(k'k|r_0) = \begin{vmatrix} r_0 j_l(k'r_0) & r_0 n_l(kr_0) \\ (d/dr)[r j_l(k'r)]|_{r=r_0} & (d/dr)[r n_l(kr)]|_{r=r_0} \end{vmatrix}. \tag{4.10}$$

Combining (4.7) and (4.9), we obtain

$$\int_0^{r_0} dr\, r\, j_l(k'r) r h_l^{(\pm)}(\alpha r) = \frac{r_0}{k'k} \sum_{m=1}^{\infty} [2(l+2m) - 1] \times j_{l+2m-1}(k'r_0) h_{l+2m-1}^{(\pm)}(\alpha r_0), \tag{4.11}$$

where $h_l^{(\pm)}(\alpha r) = j_l(\alpha r) \pm i n_l(\alpha r)$.

A similar separable expansion can be derived for the integral

$$\int_{r_0}^{\infty} dr\, r\, j_l(k'r) V(r) r\, j_l(kr), \tag{4.12}$$

by use of recursion relations and integration by parts. The above integral appears in the inhomogeneous term of the Lippmann-Schwinger equation for the two-body t matrix of the general BCM.[1]

The t matrix (3.15) may thus be expanded as

$$t_0(k'|q|k) = \sum_{i=1}^{2} \sum_{m=0}^{\infty} a_m(i) j_m(k'r_i) h_m(k, r_i, q). \tag{4.13}$$

(4.13) suggests an expansion of $\psi(p, q)$, for fixed q, by the following series of linearly independent functions:

$$\psi(p, q_j) = a_0 j_0(pr_0) + b_0 j_0(pr_1) + \sum_{n=1}^{N_0} a_n j_{2n-1}(pr_0) + \sum_{n=1}^{N_1} b_n j_{2n-1}(pr_1) \tag{4.14}$$

with the restriction $N_0 + N_1 = N_p - 2$.

With the above choice of linearly independent functions for $f_m(p)$, (2.1) is converted into an $N\times N$ matrix equation with $N=N_q\times N_p$. The triton binding energy is calculated by finding the value of S for which the determinant of the matrix equation vanishes.

V. EXTRAPOLATION OF BINDING-ENERGY-CALCULATION RESULTS TO THE LIMIT OF INFINITE CORE REPULSION

In this section we extrapolate the results of the triton binding-energy calculations for soft-core two-body interactions to the hard-core limit and check to see if the extrapolated binding energy is in reasonable agreement with the value 8.13 MeV obtained in KT.[4] We fix all the parameters of the s-wave Herzfeld potential except $(\hbar^2/2\mu)U_0$ (the soft-core strength) at the values used in KT[4] [$r_0=0.4$ F, $r_1=1.737$ F and $(\hbar^2/2\mu)U_1=-63.85$ MeV].

Using the numerical method outlined in Sec. IV, the triton binding energies were calculated as a function of U_0. The $f_m(p)$ given in (4.14) were used. Binding energies were calculated for $U_0=65$, 100, 200, 300, 10^3, and 10^4 F^{-2}. The results are summarized in Table I. They were obtained with $N_q=12$, $N_p=8$, and $N_0=N_1=3$. A calculation with the same $f_m(p)$, for infinite core repulsion, gives 8.06-MeV binding energy in excellent agreement with the previously obtained value of 8.13 MeV.

The accuracy of these results was tested by increasing N_p, N_q, N_0, and N_1. The choices (N_p, N_q, N_0, N_1) = (12, 8, 3, 3), (16, 8, 3, 3), (10, 12, 5, 3) all yielded binding energies differing from each other by less than 1%. From these considerations, we estimate our error to be less than about 2%.

To extrapolate our results to the limit of infinite core repulsion ($U_0\to\infty$), we assume a simple relation between the calculated triton binding energy [$E_0=-(\hbar^2/m)S_0$] and U_0:

$$(S_\infty - S_0) = bU_0^{-a}, \quad a>0, \quad S_\infty - S_0>0. \tag{5.1}$$

a and b are constants and $-(\hbar^2/M)S_\infty$ is the binding energy for $U_0=\infty$. We may express (5.1) as

$$\ln\frac{1}{S_\infty - S_0} = a\ln U_0 - \ln b. \tag{5.2}$$

Thus, if our assumed dependence of S_0 on U_0 is correct, the plot of $\ln(1/S_\infty - S_0)$ vs U_0 should be a straight line with slope a and intercept $-\ln b$. Plots of (5.2), for $E_0(\infty)=-(\hbar^2/M)S_\infty=7.0$, 8.25, and 8.7 MeV are shown in Fig. 1. The choice $E_0(\infty)=8.25$ MeV appears to be a good estimate of $E_0(\infty)$. The dashed line represents a straight line of (5.2) with $E_0(\infty)=8.25$ MeV, $a=0.58$ and $b=3.76$. The extrapolated binding energy thus differs by 1–2% from the results of previous calculations for the case $U_0=\infty$ that were based on matrix-inversion techniques which are valid for a compact Faddeev kernel.

TABLE I. Calculated triton binding energies E_0 [$=-(\hbar^2/M)S_0$] as a function of the soft-core strength U_0 (in units of F^{-2}) with $\hbar^2/M=41.47$ MeV F^2.

U_0 (F^{-2})	S_0 (F^{-2})	E_0 (MeV)
65	−0.5277	21.88
1.0×10^2	−0.4624	19.18
2.0×10^2	−0.3729	15.46
3.0×10^2	−0.3378	14.01
1.0×10^3	−0.2671	11.08
1.0×10^4	−0.2173	9.01

VI. SUMMARY AND CONCLUSIONS

The infinite Schmidt norm of the Faddeev kernel in the three-body problem with singular core interactions, makes it difficult to rigorously determine the validity of standard numerical quadrature and matrix-inversion techniques for solving the integral equations. In this paper, we have described calculations which seem to indicate that these techinques are valid for the case of singular core interactions in a finite number of partial wave states. A reasonable extrapolation of the results of three-body calculations with soft-core interactions (which give a finite Schmidt norm for the Faddeev kernel) to the hard-core limit, yields essentially the same results as the calculation of KT (based on hard-core interactions and standard numerical

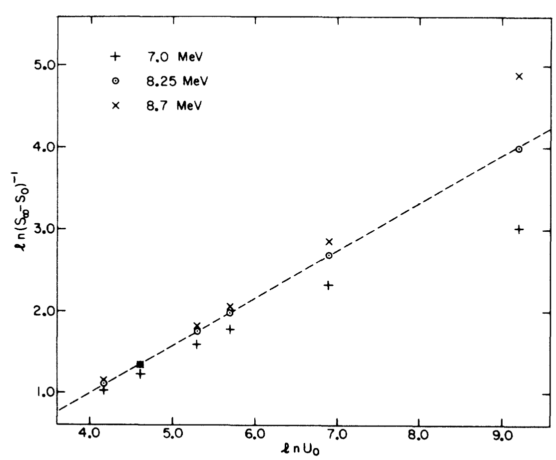

FIG. 1. Plots of $\ln(1/S_\infty - S_0)$ versus $\ln U_0$ with three different choices for $E_0(\infty)$ [$\equiv -(\hbar^2/M)S_\infty$]: 7.0, 8.25, and 8.7 MeV, with $(\hbar^2/M)=41.47$ MeV F^2. The dashed line corresponds to $E_0(\infty)=8.25$ MeV. U_0, S_0, and S_∞ are in units of F^{-2}.

quadrature and matrix-inversion techinques).

It is easy to construct examples of integral equations, with kernels having infinite Schmidt norms, which may be solved by methods similar to those used in this paper. A rather trivial one is the homogeneous eigenvalue equation

$$f(x)=\lambda\int_1^\infty dy\,K(x|y)f(y)\,, \tag{6.1}$$

with

$$K(x|y)=H(x)G(y)\,, \tag{6.2}$$

$$H(x)=\sqrt{x}/x^2\,, \tag{6.3}$$

$$G(y)=1/\sqrt{y}\,. \tag{6.4}$$

The kernel $K(x|y)$, as well as all of its iterates, has an infinite Schmidt norm

$$\|K\|^2=\int_1^\infty dx\int_1^\infty dy\,|K(x|y)|^2=\infty\,. \tag{6.5}$$

The eigensolution of (6.1) is obviously

$$f_\lambda(x)=\text{const}\times H(x)\,, \tag{6.6}$$

with

$$\lambda=1\Big/\int_1^\infty dy\frac{1}{y^2}=1\,. \tag{6.7}$$

These results may be obtained by solving the modified eigenvalue equation

$$f^{(N)}(x)=\lambda(N)\int_1^\infty dy\,K(x|y)\theta(N-y)f^{(N)}(y)\,, \tag{6.8}$$

$$\theta(N-y)=0\,,\quad \text{for } y>N\,,$$
$$\theta(N-y)=1\,,\quad \text{for } y<N\,, \tag{6.9}$$

and extrapolating the results to the limit $N\to\infty$. Obviously,

$$\lambda(N)=\lambda(\infty)N/(N-1)\,. \tag{6.10}$$

Hopefully, we will soon find a mathematically rigorous justification for the numerical procedures used by KT.[4] We are now doing calculations for bound-state, scattering, and form-factor parameters of trinucleon systems for the case of BCM two-body interactions in a finite number of partial waves. Calculations are being done using essentially the methods of KT,[4] and the results are being checked by extrapolation of the results of soft-core calculations to ths singular-core limit.

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