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Elastically Equivalent Potentials*

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We define a nondenumerably infinite class of two-body potentials that give T matrix elements that are identical on the energy shell but are different off shell. The existence of this class is shown to be related to a nonuniqueness in the off-shell continuation of the Schrödinger equation. A relation for the difference between off-shell elements of T matrices that are equal on shell is given in a form that is convenient for the systematic study of off-shell effects in systems of more than two nucleons. In particular, these relations allow one to investigate subclasses of elastically equivalent interaction potentials that satisfy restrictions such as finite range. The possibility that elastically equivalent interactions have different off-shell symmetry properties is discussed, and some observations concerning the use and interpretation of elastically equivalent potentials in many-body calculations are presented.

I. INTRODUCTION

Considerable effort has recently been directed toward an understanding of the off-energy-shell properties of the nucleon-nucleon interaction. This work has been motivated primarily by the realization that our present knowledge of the two-nucleon interaction is not adequate to permit unambiguous calculations of the properties of systems of more than two nucleons. Rather, it seems necessary also to investigate the properties of few- and many-nucleon systems in order to determine the essential character of the two-nucleon interaction. A systematic investigation of this kind presupposes a formalism that makes explicit the arbitrariness that persists in the off-shell behavior of the two-nucleon interaction after all on-shell properties are taken into account. In this paper we present such a formalism. One result is a convenient relation for the difference between off-shell elements of T matrices that are equal on shell.

The present discussion is restricted to a nonrelativistic quantum-mechanical description of the two-body system or, equivalently, to a system described by the one-body Schrödinger equation in the center-of-mass system. For notational convenience, all formulas are given for a single partial wave in an uncoupled partial-wave description of the system. This restriction is easily removed. We use in a fundamental way the assumption that the two-body system is described by a Hermitian Hamiltonian H that can be expressed in the form

$$H = H_0 + V, \tag{1.1}$$

where H_0 describes the unperturbed relative motion of the isolated particles and V describes the interaction between them. It is further assumed that H does not depend explicitly on the parametric center-of-mass energy E. The only other limitations on the interaction are those necessary for the application of conventional scattering theory. In particular, V need not be a local potential. The nucleons are considered as point particles so that the discussion is for values of E such that inelastic processes, such as meson production, can be ignored.

Many of the results in this paper are not new. The on-shell equivalence of systems described by the interactions V and $\hat{V} = UHU^{\dagger} - H_0$, where U is a unitary operator of finite range, has been pointed out previously.[1] Also, the restrictions on the two-body interaction that follow from the completeness and orthogonality of the scattering (plus bound) states have been discussed.[2-4] However, the incorporation of these ideas within the framework of conventional scattering theory and the explicit exposition of the resulting relations between them may be of considerable heuristic, as well as practical, value.

The organization of the paper is as follows. In Sec. II it is shown that the off-shell continuation of the Schrödinger equation is not unique. This, in turn, is related to the existence of a nondenumerably infinite class of potentials that give identical on-shell, but different off-shell, T matrix elements. Limitations on the off-shell continuation of the T matrix are implied by the relation between the asymptotic wave function and the T matrix; these are discussed in Sec. III. The relation between off-shell elements of T matrices that correspond to elastically equivalent potentials is derived in Sec. IV. Restrictions on the off-shell ele-

ments of T are also imposed by the conditions of completeness and orthogonality of the scattering plus bound states of the system. These restrictions are discussed in Sec. V, and a practical procedure is given for the systematic study of off-shell effects in nuclear processes that involve more than two particles. The possibility that elastically equivalent systems have different off-shell symmetry properties is considered briefly in Sec. VI. In Sec. VII we make some observations concerning the use of elastically equivalent potentials in the many-body problem.

II. OFF-SHELL CONTINUATION OF THE SCHRÖDINGER EQUATION

We suppress the index that designates a particular partial wave and write the Schrödinger equation for the relative motion of the two-body system in the form

$$(k^2 - H)|\psi_k\rangle = 0 , \tag{2.1}$$

where $k^2 = E$ is the relative energy. We assume (a) that as the distance of separation r increases, the interaction potential V goes to zero sufficiently fast that Eq. (2.1) has at most a finite number of bound-state solutions, i.e., that V approaches zero at least as fast as $r^{-1-\epsilon}$ with $\epsilon > 0$; and (b) that the continuum solutions satisfy the usual T matrix boundary conditions, i.e., that for $k^2 > 0$ the function $\langle r|\psi_k\rangle$ is asymptotically a plane wave of unit amplitude plus an outgoing spherical wave. For the sake of simplicity the particles are also assumed to be spinless.

The integral equation

$$|\psi^{(+)}_{k,k^2}\rangle = |k\rangle + (k^2 - H_0 + i0)^{-1} V |\psi^{(+)}_{k,k^2}\rangle , \tag{2.2}$$

incorporates this boundary condition for the positive-energy solutions of Eq. (2.1). The ket $|k\rangle$ represents a plane wave and satisfies the free-particle equation

$$(k^2 - H_0)|k\rangle = 0 . \tag{2.3}$$

[In the angular momentum decomposition, the ket $|k\rangle$ really represents not a plane wave but rather $\langle r|k\rangle = (2/\pi)^{1/2} k r j_l(kr)$. However, as a matter of convenience, we continue to refer to this as a plane wave.]

In order to discuss the off-shell continuation of Eq. (2.2), it is convenient to introduce the wave operator $\Omega(z)$, which is the solution of the integral equation

$$\Omega(z) = 1 + (z - H_0)^{-1} V \Omega(z) , \tag{2.4}$$

or, in "solved" form,

$$\Omega(z) = 1 + (z - H)^{-1} V . \tag{2.4'}$$

Equation (2.2) may be rewritten as

$$|\psi^{(+)}_{k,k^2}\rangle = \Omega(k^2 + i0)|k\rangle , \tag{2.2'}$$

and its continuation off shell is

$$|\psi^{(+)}_{k',k^2}\rangle = \Omega(k^2 + i0)|k'\rangle , \qquad k'^2 \neq k^2 . \tag{2.5}$$

The wave amplitude defined by Eq. (2.5) is not an eigenfunction of H; in fact, it satisfies a Bethe-Goldstone equation of the form

$$(k^2 - H)|\psi^{(+)}_{k',k^2}\rangle = (k^2 - k'^2)|k'\rangle . \tag{2.6}$$

Although Eq. (2.5) uniquely defines the off-shell amplitude $|\psi^{(+)}_{k',k^2}\rangle$, the off-shell continuation itself is not unique. To show this let us rewrite Eq. (2.1) in the form

$$[k^2 - U^\dagger H_0 U - (H - U^\dagger H_0 U)]|\psi_k\rangle = 0 , \tag{2.7}$$

where U is a unitary operator that approaches the unit operator sufficiently fast that $(U^\dagger - 1)|k\rangle$ is a vector of finite norm. This implies that

$$r r' \langle r|U^\dagger - 1|r'\rangle \to 0 , \tag{2.8}$$

as r or r' approaches infinity. Equations (2.1) and (2.7) are, of course, identical. However, we make the off-shell continuation of Eq. (2.7) in terms of a complete set of states $|\bar{k}\rangle$, where

$$(k^2 - U^\dagger H_0 U)|\bar{k}\rangle = 0 \tag{2.9}$$

and

$$|\bar{k}\rangle = U^\dagger |k\rangle . \tag{2.10}$$

The wave operator $\bar{\Omega}(z)$ in the $|\bar{k}\rangle$ basis satisfies the operator relation

$$\bar{\Omega}(z) = 1 + (z - U^\dagger H_0 U)^{-1}(H - U^\dagger H_0 U)\bar{\Omega}(z) \tag{2.11}$$

or

$$\bar{\Omega}(z) = 1 + (z - H)^{-1}(H - U^\dagger H_0 U), \tag{2.11'}$$

and the corresponding off-shell continuation of Eq. (2.7) gives the wave amplitude

$$|\bar{\psi}^{(+)}_{k,k^2}\rangle = \bar{\Omega}(k^2 + i0)|\bar{k}'\rangle = \bar{\Omega}(k^2 + i0)U^\dagger |k'\rangle , \qquad k^2 \neq k'^2 . \tag{2.12}$$

From Eqs. (2.5) and (2.12) it follows that

$$|\bar{\psi}^{(+)}_{k',k^2}\rangle - |\psi^{(+)}_{k',k^2}\rangle = [\bar{\Omega}(k^2 + i0)U^\dagger - \Omega(k^2 + i0)]|k'\rangle . \tag{2.13}$$

Substitution of Eqs. (2.11') and (2.4') for the wave operators in Eq. (2.13), together with the conditions (2.8), gives

$$|\bar{\psi}^{(+)}_{k',k^2}\rangle - |\psi^{(+)}_{k',k^2}\rangle = (k^2 - k'^2)(k^2 - H + i0)^{-1}(U^\dagger - 1)|k'\rangle . \tag{2.14}$$

For $k^2 \neq k'^2$, the right-hand side of Eq. (2.14) vanishes identically if and only if $U^\dagger - 1 \equiv 0$. Conse-

quently the off-shell continuation of the Schrödinger equation is not unique, but rather any distinct unitary operator other than the identity can generate a different continuation.

Although the treatment of scattering in terms of the kets $|\bar{k}\rangle$ that satisfy Eq. (2.9) may appear somewhat artificial, this representation is certainly as valid as that given originally in terms of the plane-wave states $|k\rangle$. In fact, if $U^\dagger$ satisfies conditions (2.8), $|\bar{k}\rangle$ is asymptotically a plane wave; i.e., for sufficiently large r

$$\langle r|\bar{k}\rangle \to \langle r|k\rangle = (2/\pi)^{1/2} kr j_l(kr) . \tag{2.15}$$

Furthermore, since all observable properties of the two-body system depend only on the on-shell amplitudes ($k^2 = k'^2$), Eq. (2.14) shows that the two representations lead to identical predictions for the results of any allowed observation.

It is of interest to examine the consequences of the nonuniqueness in the off-shell continuation of the Schrödinger equation in terms of the elements of the associated T matrices. In the usual plane-wave basis, the interaction V is defined to be $H - H_0$, where H_0 is the generator of the basis states $|k\rangle$, and the elements of the T matrix are

$$\langle k''|T(k^2)|k'\rangle = \langle k''|V|\psi^{(+)}_{k',k^2}\rangle = \langle k''|V\Omega(k^2+i0)|k'\rangle . \tag{2.16}$$

In the transformed basis, the interaction $\bar{V}$ is defined to be $\bar{V} = H - U^\dagger H_0 U$, and $\bar{H}_0 = U^\dagger H_0 U$ is the generator of the basis states $|\bar{k}\rangle$. The elements of the corresponding $\bar{T}$ matrix may thus be taken to be

$$\langle \bar{k}''|T(k^2)|\bar{k}'\rangle = \langle \bar{k}''|\bar{V}|\bar{\psi}^{(+)}_{k',k^2}\rangle = \langle \bar{k}''|\bar{V}\bar{\Omega}(k^2+i0)|\bar{k}'\rangle . \tag{2.17}$$

In order to compare T matrices it is convenient to introduce the operator $\hat{T}(k^2)$, where

$$\hat{T}(k^2) = U\bar{T}(k^2)U^\dagger = U\bar{V}\bar{\Omega}(k^2+i0)U^\dagger , \tag{2.18}$$

whose matrix elements are defined with respect to the *same* basis states as are those of $T(k^2)$. We also introduce the operators

$$\hat{V} = U\bar{V}U^\dagger = UHU^\dagger - H_0 \tag{2.19}$$

and

$$\hat{\Omega}(z) = U\bar{\Omega}(z)U^\dagger = 1 + (z - H_0)^{-1}\hat{V}\hat{\Omega}(z) , \tag{2.20}$$

in terms of which Eq. (2.18) becomes

$$\hat{T}(k^2) = \hat{V}\hat{\Omega}(k^2+i0) . \tag{2.18'}$$

All interaction potentials in the class $\hat{V}$ defined by Eq. (2.19) have identical on-shell properties. This may be shown as follows. The Schrödinger Eq. (2.1) may be transformed by a unitary transformation to give

$$(k^2 - UHU^\dagger)|\hat{\psi}_{k,k^2}\rangle = 0 , \tag{2.21}$$

where

$$|\hat{\psi}_{k,k^2}\rangle = U|\psi_{k,k^2}\rangle . \tag{2.22}$$

In order to obtain an integral equation in the $|k\rangle$ basis, Eq. (2.21) may be rewritten in the form

$$(k^2 - H_0 - \hat{V})|\hat{\psi}_{k,k^2}\rangle = 0 , \tag{2.23}$$

where $\hat{V}$ is defined by Eq. (2.19). The off-shell continuation of the corresponding integral equation is

$$|\hat{\psi}^{(+)}_{k',k^2}\rangle = \hat{\Omega}(k^2+i0)|k'\rangle , \tag{2.24}$$

where the wave operator $\hat{\Omega}$ is defined by Eq. (2.20). Since Eqs. (2.1) and (2.23) are equivalent representations of the same two-body system, they must predict the same results for any observation on the system. In particular, they must yield identical discrete spectra and identical elastic scattering phase shifts or, equivalently, identical on-shell T matrix elements. For this reason, all interaction potentials $\hat{V}$ defined by Eq. (2.19) are called *elastically equivalent potentials*.

III. LIMITATIONS ON OFF-SHELL CONTINUATIONS OF THE T MATRIX

In Sec. II we have concerned ourselves with a certain kind of off-shell continuation of the T matrix. One might perhaps think that this sort of continuation is not the only possible continuation. We might, for example, apply the foregoing off-shell continuation to the wave operator. That is to say, we choose, as before,

$$|\bar{\psi}^{(+)}_{k'',k^2}\rangle = |\bar{k}''\rangle + (k^2 - U^\dagger H_0 U + i0)^{-1} \times (H - U^\dagger H_0 U)|\bar{\psi}^{(+)}_{k'',k^2}\rangle , \tag{3.1}$$

and we define the barred wave operator through the relation

$$|\bar{\psi}^{(+)}_{k'',k^2}\rangle = \bar{\Omega}(k^2+i0)|\bar{k}''\rangle . \tag{3.2}$$

We may, if we wish, define the wave operator $\tilde{\Omega}(k^2+i0)$ to be

$$\tilde{\Omega}(k^2+i0) = \bar{\Omega}(k^2+i0)U^\dagger = U^\dagger\Omega(k^2+i0), \tag{3.3}$$

so that

$$|\bar{\psi}^{(+)}_{k'',k^2}\rangle = \tilde{\Omega}(k^2+i0)|k''\rangle . \tag{3.4}$$

If we then define a scattering operator $\tilde{T}(k^2)$ as

$$\tilde{T}(k^2) = V\tilde{\Omega}(k^2+i0) , \tag{3.5}$$

with matrix elements

$$\langle k'|\tilde{T}(k^2)|k''\rangle = \langle k'|V\tilde{\Omega}(k^2+i0)|k''\rangle , \tag{3.6}$$

we find that we have achieved another off-shell extension of the T matrix, and that its matrix elements are identical with those of the T matrix given by $T(k^2) = V\Omega(k^2+i0)$ on the energy shell. In fact, the matrix elements of $T(k^2)$ and $\tilde{T}(k^2)$ are identical on the "half-shell," i.e.,

$$\langle k'|\,T(k^2)\,|k\rangle = \langle k'|\,\tilde{T}(k^2)\,|k\rangle\,, \tag{3.7}$$

for all values of k'. To see this we observe that

$$\begin{aligned}\tilde{\Omega}(k^2+i0) &= U^\dagger + (k^2 - H + i0)^{-1}(H - U^\dagger H_0 U)U^\dagger \\ &= \Omega(k^2+i0) + (k^2 - H + i0)^{-1} \\ &\quad \times [k^2(U^\dagger - 1) - (U^\dagger - 1)H_0],\end{aligned} \tag{3.8}$$

so that

$$\begin{aligned}\tilde{\Omega}(k^2+i0)\,|k''\rangle &= \Omega(k^2+i0)\,|k''\rangle \\ &\quad + (k^2 - k''^2)(k^2 - H + i0)^{-1}(U^\dagger - 1)\,|k''\rangle,\end{aligned} \tag{3.9}$$

and thus

$$\begin{aligned}\langle k'|\,\tilde{T}(k^2)\,|k''\rangle &= \langle k'|\,T(k^2)\,|k''\rangle + (k^2 - k''^2) \\ &\quad \times \langle k'|\,V(k^2 - H + i0)^{-1}(U^\dagger - 1)\,|k''\rangle.\end{aligned} \tag{3.10}$$

This continuation is thus one for which $\langle k'|\,\tilde{T}(k^2)\,|k''\rangle$ and $\langle k'|T(k^2)|k''\rangle$ are identical if $k''^2 = k^2$.

It is not difficult to see, however, that this continuation is unsatisfactory. To show this we examine the asymptotic behavior of the wave function $\langle r\,|\bar{\psi}^{(+)}_{k'',k2}\rangle$ in coordinate space. We have

$$\begin{aligned}\langle r\,|\,\bar{\psi}^{(+)}_{k'',k2}\rangle &= \langle r\,|\tilde{\Omega}(k^2+i0)\,|k''\rangle \\ &\underset{r\to\infty}{\longrightarrow} \left(\frac{2}{\pi}\right)^{1/2}\Big[\sin(k''r - \tfrac{1}{2}l\pi) \\ &\quad - i^{-l}\frac{\pi}{2k}e^{ikr}\langle k|(UH - H_0U)\tilde{\Omega}(k^2+i0)\,|k''\rangle\Big].\end{aligned} \tag{3.11}$$

Now we note that the coefficient of the outgoing spherical wave in Eq. (3.11) is just

$$\begin{aligned}\langle k|(UH - H_0U)\tilde{\Omega}(k^2+i0)\,|k''\rangle &= \langle k|(UHU^\dagger - H_0)\hat{\Omega}(k^2+i0)\,|k''\rangle \\ &= \langle k|\,\hat{T}(k^2)\,|k''\rangle = \langle \bar{k}|\,\bar{T}(k^2)\,|\bar{k}''\rangle\,.\end{aligned} \tag{3.12}$$

If we wish to retain the usual relationship between the asymptotic wave function and the T matrix as part of our definition of the off-shell continuation, then the continuation (3.5) of the T matrix is not acceptable. This rejection is equivalent to saying that an acceptable T matrix must satisfy an equation of the form

$$T = V + VGT\,. \tag{3.13}$$

It is easy to see that the extension (3.5) does not have this property.

IV. T MATRICES FOR ELASTICALLY EQUIVALENT POTENTIALS

Next we consider the relation between the off-shell elements of the T matrices that correspond to the elastically equivalent potentials V and $\hat{V}$. The notation $\hat{V}$ is used here to designate any member of the class of potentials defined by Eq. (2.19). To obtain the desired relation, it is convenient to use the two-potential formula in its operator form. We begin with a brief discussion of this formula.

Let $T(k^2)$ denote the transition operator for the interaction $V_1 + V_2$, i.e., let

$$T(k^2) = (V_1 + V_2)\Omega(k^2+i0)\,, \tag{4.1}$$

where

$$\Omega(z) = 1 + (z - H_0)^{-1}(V_1 + V_2)\Omega(z)\,. \tag{4.2}$$

In terms of the wave operator $\Omega_1(z)$ given by

$$\Omega_1(z) = 1 + (z - H_1)^{-1}V_1\,, \tag{4.3}$$

where

$$H_1 = H_0 + V_1\,, \tag{4.4}$$

Eq. (4.2) becomes

$$\Omega(z) = \Omega_1(z) + (z - H_1)^{-1}V_2\Omega(z)\,. \tag{4.5}$$

By use of Eq. (4.5), it is possible to rewrite Eq. (4.1) as

$$\begin{aligned}T(k^2) &= V_1\Omega_1(k^2+i0) \\ &\quad + [1 + V_1(k^2 - H_1 + i0)^{-1}]\,V_2\Omega(k^2+i0)\end{aligned} \tag{4.6}$$

or

$$T(k^2) = T_1(k^2) + \Omega_1^\dagger(k^2 - i0)V_2\Omega(k^2+i0), \tag{4.7}$$

where $T_1(k^2)$ is the transition operator for the interaction V_1 alone and $\Omega_1^\dagger(z)$ is the Hermitian conjugate of the wave operator defined by Eq. (4.3). Equation (4.7) is easily recognized as an operator form of the familiar two-potential formula.

The relation between transition matrices corresponding to V and $\hat{V}$ is given by Eq. (4.7) if we make the substitutions

$$V_1 \to V, \quad V_2 \to \hat{V} - V,$$

and (4.8)

$$H_1 \to H_0 + V = H\,.$$

This gives

$$\hat{T}(k^2) = T(k^2) + \Omega^\dagger(k^2 - i0)(\hat{H} - H)\hat{\Omega}(k^2+i0), \tag{4.9}$$

where $\hat{\Omega}$ is given by Eq. (2.20). In the "solved" form, Eq. (2.20) is

$$\hat{\Omega}(z) = 1 + (z - \hat{H})^{-1}\hat{V}, \tag{4.10}$$

where

$$\hat{H} = UHU^{\dagger}, \tag{4.11}$$

and $\Omega^{\dagger}(z)$ is the Hermitian conjugate of the wave operator defined by Eq. (2.4′).

From Eqs. (2.12), (2.20), and (2.24), it follows that

$$\hat{\Omega}(k^2 + i0)\,|\,k'\rangle = U\,|\,\overline{\psi}^{(+)}_{k',k^2}\rangle \tag{4.12}$$

for all k and k'. This result, together with Eqs. (2.5) and (2.14), gives the relation

$$\hat{\Omega}(z) = U\Omega(z) + U(z - H)^{-1}(U^{\dagger} - 1)(z - H_0)\ . \tag{4.13}$$

Substitution of this result into Eq. (4.9) gives

$$\hat{T}(k^2) = T(k^2) + \Omega^{\dagger}(k^2 - i0)(U^{\dagger} - 1)(k^2 - H_0) + (k^2 - H_0)(U - 1)\Omega(k^2 + i0) + (k^2 - H_0)(U - 1)(k^2 - H + i0)^{-1}(U^{\dagger} - 1)(k^2 - H_0)\ . \tag{4.14}$$

The condition that $(U^{\dagger} - 1)\,|\,k\rangle$ be of finite norm is sufficient to ensure that

$$\lim_{k \to k'} (k^2 - k'^2)\langle k'\,|(U - 1)\Omega(k^2 + i0)\,|\,k\rangle = 0\ . \tag{4.15}$$

Consequently, it follows from Eq. (4.14) that the on-shell elements of $\hat{T}(k^2)$ and $T(k^2)$ are identical, that the half-shell elements are related as

$$\langle k'\,|\,\hat{T}(k^2)\,|\,k\rangle = \langle k'\,|\,T(k^2)\,|\,k\rangle + (k^2 - k'^2)\langle k'\,|(U - 1)\Omega(k^2 + i0)\,|\,k\rangle \tag{4.16}$$

and

$$\langle k\,|\,\hat{T}(k^2)\,|\,k'\rangle = \langle k\,|\,T(k^2)\,|\,k'\rangle + (k^2 - k'^2)\langle k\,|\,\Omega^{\dagger}(k^2 - i0)(U^{\dagger} - 1)\,|\,k'\rangle\ , \tag{4.17}$$

and that the difference between fully-off-shell elements is

$$\langle k'\,|\,\hat{T}(k^2) - T(k^2)\,|\,k''\rangle = (k^2 - k''^2)\langle k'\,|\,\Omega^{\dagger}(k^2 - i0)(U^{\dagger} - 1)\,|\,k''\rangle + (k^2 - k'^2)\langle k'\,|(U - 1)\Omega(k^2 + i0)\,|\,k''\rangle + (k^2 - k''^2)(k^2 - k'^2)\langle k'\,|(U - 1)(k^2 - H + i0)^{-1}(U^{\dagger} - 1)\,|\,k''\rangle\ . \tag{4.18}$$

Note that $\hat{T}(k^2) \equiv T(k^2)$ only if U is the unit operator in the space of the scattering states $|\psi^{(+)}_{k,k^2}\rangle = \Omega(k^2 + i0)\,|\,k\rangle$. The possible use of these results in an investigation of the off-shell properties of the two-nucleon interaction will be considered in Sec. V.

V. RESTRICTIONS ON THE OFF-SHELL ELEMENTS OF THE T MATRIX

The off-shell elements of T are not completely arbitrary. The completeness condition

$$\int |\psi^{(+)}_{k,k^2}\rangle\, dk\, \langle\psi^{(+)}_{k,k^2}| = 1 - P, \tag{5.1}$$

where $P = \sum_b |\psi_b\rangle\langle\psi_b|$ is the projection onto the bound states of the system, is sufficient to define the fully-off-shell elements in terms of the remaining ones. Multiplication of Eq. (5.1) from the left by V leads to the relation

$$\langle k'\,|\,V\,|\,k''\rangle = \langle k'\,|\,T(k''^2)\,|\,k''\rangle + \int \langle k'\,|\,T(q^2)\,|\,q\rangle \left(\frac{dq}{q^2 - k''^2 - i0}\right)\langle q\,|\,T^{\dagger}(q^2)\,|\,k''\rangle + \sum_b (\omega_b - k'^2)\langle k'\,|\,\psi_b\rangle\langle\psi_b\,|\,k''\rangle\ , \tag{5.2}$$

where

$$H\,|\,\psi_b\rangle = \omega_b\,|\,\psi_b\rangle\ .$$

Equation (5.2) defines the interaction potential in terms of bound-state properties of the system and the half-shell elements of the T matrix. Once V is determined, all elements of T are uniquely defined.

An explicit expression for the fully-off-shell elements may be obtained by using Eq. (5.1) to expand the wave operator in Eq. (2.16). The result is

$$\langle k'|T(k^2)|k''\rangle = \langle k'|V|k''\rangle + \int \langle k'|V|\psi^{(+)}_{q,q^2}\rangle \left(\frac{dq}{k^2-q^2+i0}\right)\langle \psi^{(+)}_{q,q^2}|V|k''\rangle + \sum_b \frac{(\omega_b - k'^2)(\omega_b - k''^2)}{k^2-\omega_b}\langle k'|\psi_b\rangle\langle\psi_b|k''\rangle , \tag{5.3}$$

where of course,

$$\langle k'|V|\psi^{(+)}_{q,q^2}\rangle = \langle k'|T(q^2)|q\rangle . \tag{5.4}$$

Elimination of the interaction potential between Eqs. (5.2) and (5.3) gives the desired relation, namely

$$\langle k'|T(k^2)|k''\rangle = \langle k'|T(k''^2)|k''\rangle + \int \langle k'|T(q^2)|q\rangle\, dq[(k^2-q^2+i0)^{-1} - (k''^2-q^2+i0)^{-1}]\langle q|T^\dagger(q^2)|k''\rangle$$
$$+ (k''^2-k^2)\sum_b \left(\frac{\omega_b - k'^2}{\omega_b - k^2}\right)\langle k'|\psi_b\rangle\langle\psi_b|k''\rangle . \tag{5.5}$$

In a similar way, the orthonormality condition

$$\langle \psi^{(+)}_{k,k^2}|\psi^{(+)}_{k',k'^2}\rangle = \delta(k-k') \tag{5.6}$$

yields another off-shell extension of the optical theorem, namely

$$\langle k|T(k'^2) - T^\dagger(k^2) + T^\dagger(k^2)\left[\frac{\mathcal{P}}{k^2-H_0} - \frac{\mathcal{P}}{k'^2-H_0} + i\pi\delta(k^2-H_0) + i\pi\delta(k'^2-H_0)\right]T(k'^2)|k'\rangle = 0 . \tag{5.7}$$

Recently Baranger *et al.*[2] have shown that even more restrictive conditions on the off-shell elements of T are implied by Eqs. (5.1) and (5.6). They define a half-shell matrix element

$$\phi(k,k') = e^{-i\eta(k)}\langle k'|T(k^2)|k\rangle , \tag{5.8}$$

and a wave function

$$\langle k'|\psi^0_{k,k^2}\rangle = e^{-i\eta(k)}\langle k'|\psi^{(+)}_{k,k^2}\rangle$$
$$= \delta(k-k')\cos\eta(k) + \frac{\mathcal{P}}{k^2-k'^2}\phi(k,k') , \tag{5.9}$$

where $\eta(k)$ is the elastic scattering phase shift for the partial wave of interest. The advantage of this representation is that $\phi(k,k')$ and $\langle k'|\psi^0_{k,k^2}\rangle$ are real. Since the half-shell elements determine the fully-off-shell ones, it is necessary to consider only the former, i.e., the $\phi(k,k')$. The real operator W is defined by its matrix elements

$$\langle k'|W|k\rangle = \langle k'|\psi^0_{k,k^2}\rangle ,$$

so that, in this representation, conditions (5.1) and (5.6) become

$$WW^\dagger = 1 - P \tag{5.10}$$

and

$$W^\dagger W = 1 . \tag{5.11}$$

The matrix ϕ is written as a sum of a symmetric matrix σ and an antisymmetric matrix α, i.e.,

$$\phi(k,k') = \sigma(k,k') + \alpha(k,k') . \tag{5.12}$$

In the case of no bound states ($P=0$), Baranger *et al.* show that conditions (5.10) and (5.11) are sufficient for the determination of $\phi(k,k')$ if $\sigma(k,k')$ is given for all k, k'. Thus the arbitrariness in the continuation of T off the energy shell is isolated to an arbitrariness in the off-diagonal elements of the symmetric part of the T matrix. The generalization of this result to systems with bound states was given in Refs. 3 and 4 and was discussed further by Amado[5] and by Van Dijk and Razavy.[6]

The practical advantage of this result is that it shows one how to vary the off-shell elements of T without changing the on-shell elements and without violating the quantum-mechanical conditions (5.10) and (5.11). This, in turn, makes possible a systematic study of the effects in other nuclear phenomena of a variety of T matrices that are exactly the same on shell. The main disadvantage of the suggested procedure is that it provides no simple way by which reasonable assumptions about the two-body interaction can be incorporated in the choice of the off-diagonal elements of the σ matrix.[6]

An alternative procedure for the systematic study of different on-shell-equivalent T matrices is suggested by Eq. (4.16). In the real representation introduced in this section, Eq. (4.16) becomes

$$\hat{\phi}(k,k') = \phi(k,k') + (k^2-k'^2)\langle k'|(U-1)W|k\rangle , \tag{5.13}$$

where

$$\hat{\phi}(k,k') = e^{-i\eta(k)}\langle k'|\hat{T}(k^2)|k\rangle \tag{5.14}$$

and U is orthogonal, i.e., real. Note that $\hat{\eta}(k) = \eta(k)$, since the on-shell elements of $T(k^2)$ and $\hat{T}(k^2)$ are equal. Thus, simply by varying the choice of the orthogonal transformation U, the off-

shell elements of T (or ϕ) can be varied without changing the on-shell elements.

The initial ϕ matrix in Eq. (5.13) need not be generated from a potential but may be determined from a chosen σ matrix as suggested by Baranger *et al.* On the other hand, if ϕ is obtained from a two-body potential V that satisfies given conditions of finite range, etc., then by a proper choice of transformations U it is possible to investigate on-shell-equivalent T matrices that also satisfy these conditions. More exactly, the potentials implied by these on-shell-equivalent matrices would also satisfy the conditions initially imposed on V. Note, however, that the implied potential need not be calculated in any application of Eq. (5.13).

A convenient general expression for a unitary matrix is

$$\langle r'|(U-1)|r\rangle = \sum_{i,j=1}^{n} g_i(r')(\lambda_{ij}-\delta_{ij})g_j(r), \quad (5.15)$$

where $g_i(r)$ and $g_j(r)$ are members of an orthonormal set of functions, i.e., where

$$\int_0^\infty dr\, g_i(r)g_j(r) = \delta_{ij}, \quad (5.16)$$

and the λ_{ij} are elements of a unitary matrix.

For $N=1$, Eq. (5.15) becomes

$$\langle r'|(U-1)|r\rangle = -2g(r')g(r)\ . \quad (5.17)$$

For $N=2$, it becomes the transformation of rank 2 described by Coester *et al.*,[7] namely

$$\langle r'|U-1|r\rangle = (g_1(r'), g_2(r')) \times \begin{pmatrix} \cos\theta - 1 & \sin\theta \\ -\sin\theta & \cos\theta - 1 \end{pmatrix} \begin{pmatrix} g_1(r) \\ g_2(r) \end{pmatrix}, \quad (5.18)$$

where $-\pi \le \theta \le \pi$. If, in addition, $g_1(r)$ and $g_2(r)$ are chosen such that the transforms

$$\tilde{g}_i(k) = (2/\pi)^{1/2} \int dr\, kr\, j_l(kr) g_i(r)\ , \quad (5.19)$$

are easily obtained analytically, then the evaluation of the matrix elements

$$\langle k'|(U-1)W|k\rangle = \int \langle k'|U-1|k''\rangle\, dk'' \langle k''|W|k\rangle \quad (5.20)$$

is simplified considerably.

VI. SYMMETRY PROPERTIES OFF SHELL

The canonical transformations (2.21) and (2.22) describe completely equivalent two-body systems. In particular, all systems described by these equations must possess identical symmetry properties. On the other hand, the off-shell continuation of the Schrödinger equation (2.23) leads to off-shell T matrix elements that are not equal for different choices of the unitary transformation U that defines the transformed potential $\hat{V}$, where

$$\hat{V} = UHU^\dagger - H_0 = \hat{H} - H_0\ . \quad (6.1)$$

It is of interest, therefore, to inquire about possible off-shell violations of a given symmetry within the class of interactions defined by Eq. (6.1).

As an example, consider a system described by a Hamiltonian H that is invariant under time reversal. In terms of the half-shell elements of the T matrix, this implies that an antiunitary time-reversal operator K exists for which

$$\langle k'|V\Omega(k^2+i0)|k\rangle = \langle Kk|\Omega^\dagger(k^2-i0)V|Kk'\rangle\ . \quad (6.2)$$

For any of the on-shell-equivalent systems with interaction potential given by Eq. (6.1), it follows from Eq. (4.13) that

$$\langle k'|\hat{V}\hat{\Omega}(k^2+i0)|k\rangle = \langle Kk|\hat{\Omega}^\dagger(k^2-i0)\hat{V}|Kk'\rangle + \langle Kk|\hat{\Omega}^\dagger(k^2-i0)\{\hat{V}(UKU^\dagger K^\dagger - 1) + [H_0, UKU^\dagger K^\dagger]\}|Kk'\rangle\ . \quad (6.3)$$

Thus the system described by $\hat{H}$ is time reversal invariant off shell only if U and K commute, i.e., only if

$$[U, K] = 0\ . \quad (6.4)$$

It follows that a unitary transformation of the form

$$\langle r'|U|r\rangle = e^{i\omega(r)}\delta(r'-r)\ , \quad (6.5)$$

where $\omega(r)$ is a real function of r, does not relate two systems that are both (off shell) invariant under time reversal. Clearly no elastic scattering experiment can reveal differences in off-shell symmetry properties, since, on the energy shell, the two systems have identical T matrix elements.

VII. ELASTICALLY EQUIVALENT POTENTIALS AND THE MANY-BODY PROBLEM

In a number of recent calculations of the properties of nuclear matter, elastically equivalent potentials have been used to describe the interaction between two nucleons.[7,8] These calculations have determined that the binding energy and density in nuclear matter are sensitive to the off-shell character of this interaction.

In this section we wish to point out that it is pos-

sible to adopt two different philosophies with respect to the use of elastically equivalent potentials in the many-body problem. The first point of view is quite simple. One may assume that one has no knowledge of the free nucleon-nucleon interaction other than the information concerning phase shifts and the deuteron binding energy. Interactions that reproduce these properties of the two-body system are then used to calculate properties of the many-body system. The Hamiltonian for this problem may be written in the standard form

$$H\{\hat{V}\} = \sum_{\alpha\beta} \langle \alpha \,|\, t \,|\, \beta \rangle a_\alpha^\dagger a_\beta + \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta \,|\, \hat{V} \,|\, \gamma\delta \rangle a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma \,, \tag{7.1}$$

where t is the kinetic energy operator and $\hat{V}$ is a member of the class of elastically equivalent potentials that reproduce the known properties of the two-nucleon systems. The letters $\{\alpha, \beta, \gamma, \delta\}$ denote any complete set of single-particle states.

The determination of the eigenvalues and eigenfunctions of $H\{\hat{V}\}$ may be made via perturbation theory if $\hat{V}$ is sufficiently smooth; otherwise, various reaction-matrix methods may be used. As mentioned previously, the results of such calculations are sensitive to the off-shell character of $\hat{V}$.

We now turn to a discussion of a second approach to the many-body problem. One assumes that there exists a definite many-body Hamiltonian containing known two-body interactions, namely

$$H = \sum_{\alpha\beta} \langle \alpha \,|t|\, \beta \rangle a_\alpha^\dagger a_\beta + \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta \,|\, V \,|\, \gamma\delta \rangle a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma \,. \tag{7.2}$$

In this case the solution of the many-body Schrödinger equation may be facilitated by the introduction of a unitary transformation of the basis states.[9] We may denote this unitary transformation as e^{iS}, and note that

$$H_{\text{eff}} = e^{-iS} H e^{iS} \tag{7.3}$$

has the same spectrum as H. In some cases it may be more convenient to calculate with H_{eff} rather than with H. If one is mainly interested in two-body correlation, it is convenient to choose S to be the two-body operator

$$S = S^{(2)} = \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta \,|\, S \,|\, \gamma\delta \rangle a_\alpha^\dagger a_\beta^\dagger a_\delta \alpha_\gamma \,. \tag{7.4}$$

More generally we may write

$$S = \sum_{n=1} S^{(n)} \,, \tag{7.5}$$

where $S^{(n)}$ denotes an n-body operator.

To perform calculations, one carries out a cluster expansion of H_{eff}, namely

$$H_{\text{eff}} = H^{(1)} + H^{(2)} + H^{(3)} + \cdots \,, \tag{7.6}$$

where the $H^{(n)}$ are n-body operators. The effective many-body forces[10] ($n > 2$) appear here as a result of introducing the unitary transformation (7.3), even though the original H contained only two-body forces, as in Eq. (7.2).

As usual we assume that the operator S has no one-body part, as in Eq. (7.4). The various operators appearing in Eq. (7.6) may be determined successively by taking matrix elements of Eq. (7.6) in the space of states of one particle, two particles, etc. In this manner we find that $H^{(1)}$ is just the many-body kinetic energy operator. The two-body term $H^{(2)}$ is found to be

$$H^{(2)} = \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta \,|\, e^{-iS_{12}} (t_1 + t_2 + V_{12}) e^{iS_{12}} - (t_1 + t_2) \,|\, \gamma\delta \rangle a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma \,, \tag{7.7}$$

where the subscripts 1 and 2 indicate the integration variables for the evaluation of the matrix element.

It is now useful to comment on the structure of the matrix element appearing in Eq. (7.7). We may introduce the two-body states

$$|\overline{\alpha\beta}\rangle = e^{iS_{12}} |\alpha\beta\rangle \,, \tag{7.8}$$

which are analogous to the $|\vec{k}\rangle$ states introduced in Eq. (2.10). As in the case of the operator U introduced in Sec. II the operator $e^{iS_{12}}$ does not affect the center-of-mass motion of the pair and introduces only short-range correlations in the relative motion. Thus $|\overline{\alpha\beta}\rangle$ is identical to $|\alpha\beta\rangle$ for $|\vec{r}_1 - \vec{r}_2|$ sufficiently large. More precisely, in the space of two particles the operators e^{iS} and the operator U may be related by:

$$\langle \vec{r}_1 \vec{r}_2 \,|\, e^{iS} \,|\, \vec{r}_1' \vec{r}_2' \rangle = \langle \vec{r}_1 \vec{r}_2 \,|\, e^{iS^{(2)}} \,|\, \vec{r}_1' \vec{r}_2' \rangle = \delta^{(3)}[\tfrac{1}{2}(\vec{r}_1 + \vec{r}_2) - \tfrac{1}{2}(\vec{r}_1' + \vec{r}_2')] U(\vec{r}_1 - \vec{r}_2, \vec{r}_1' - \vec{r}_2') = \delta^{(3)}(\vec{R} - \vec{R}') U(\vec{r}, \vec{r}') \,, \tag{7.9}$$

where $\vec{R}$ is a center-of-mass coordinate and $\vec{r}$ a relative coordinate. Note that in the space of three particles

$$\langle \vec{r}_1 \vec{r}_2 \vec{r}_3 \,|\, e^{iS^{(2)}} \,|\, \vec{r}_1' \vec{r}_2' \vec{r}_3' \rangle = \langle \vec{r}_1 \vec{r}_2 \vec{r}_3 \,|\, e^{i(S_{12} + S_{23} + S_{13})} \,|\, \vec{r}_1' \vec{r}_2' \vec{r}_3' \rangle \neq \langle \vec{r}_1 \vec{r}_2 \vec{r}_3 \,|\, e^{iS_{12}} e^{iS_{23}} e^{iS_{13}} \,|\, \vec{r}_1' \vec{r}_2' \vec{r}_3' \rangle \,, \tag{7.10}$$

since the terms in the exponent do not commute. This feature makes the accurate evaluation of the higher cluster terms ($n>2$) difficult.

We define the operator $\overline{V}_{12}$ as

$$\overline{V}_{12} = (t_1 + t_2 + V_{12}) - e^{iS_{12}}(t_1 + t_2)e^{-iS_{12}} , \qquad (7.11)$$

in complete analogy to the operator $\overline{V}$ introduced in Eq. (2.17). Then Eq. (7.7) may be written as

$$H^{(2)} = \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \overline{\alpha}\overline{\beta} | \overline{V}_{12} | \overline{\gamma}\overline{\delta} \rangle a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma . \qquad (7.12)$$

Again, we have the option of working in the unbarred representation of Eq. (7.7). Correspondence to the foregoing analysis is achieved by writing

$$\hat{V}_{12} = e^{-S_{12}} \overline{V}_{12} e^{iS_{12}} . \qquad (7.13)$$

In terms of $\hat{V}_{12}$, $H^{(2)}$ has the simple structure

$$H^{(2)} = \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | \hat{V}_{12} | \gamma\delta \rangle a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma . \qquad (7.14)$$

As we have seen in the previous analysis, the potential $\hat{V}_{12}$ is elastically equivalent to the potential V_{12}. We are therefore led to the following observations:

(1) If one terminates the cluster expansion for H_{eff} at the second term, the calculations carried out using either the first philosophy (in which only on-shell information is used to determine H) or the calculation of $H_{\text{eff}} = H^{(1)} + H^{(2)}$ with $H^{(2)}$ given by Eq. (7.7) are identical from the computational point of view.

(2) In the case of the unitary-transformation approach, a complete calculation of the entire cluster series will provide a result independent of the unitary transformation e^{iS}. One may take an alternative point of view of the calculations carried out with elastically equivalent potentials and the first philosophy. These calculations may be thought of as giving the results for the truncated problem $H_{\text{eff}}^T \equiv H^{(1)} + H^{(2)}$. Since the results depend quite significantly on the potential used, one may infer that the higher terms in H_{eff} (those with $n>2$) can be large, of the order of several MeV per particle.

(3) Reasonable agreement with nuclear binding energy and other properties achieved with potentials that are not obtained from any fundamental approach is probably fortuitous in that various elastically equivalent potentials (which are just as acceptable from a theoretical point of view) will give quite different results.

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[10]To avoid confusion, it is important to distinguish between the specific higher cluster terms generated by the unitary transformation and the classification of "cluster" terms in the Goldstone perturbation theory. The latter classification, which is made by considering the number of hole lines appearing in a diagram, may also be used to discuss the evaluation of the perturbation series for H_{eff}.