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Modified Faddeev Equations for Three Bound α Particles in ^{12}C

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The bound state of three α particles is considered. For the α-cluster model, we consider the Coulomb Green's function in the nucleus ^{12}C. The calculated binding energy of the ^{12}C nucleus is consistent with previous calculations.

I. INTRODUCTION

The α-cluster model consists of taking the nucleus to be composed of α clusters as elementary constituents, in view of the strong binding of an α cluster. Therefore, we consider the ^{12}C nucleus, on the basis of the α-cluster model, as three bound α particles. The aim of this work is to study the three-body system when all of these particles are charged. We consider two approximations for the Coulomb Green's functions. The first, suggested by Schulman,[1] yields an approximate form of the Coulomb Green's function in momentum space. The other is the improved version of the Schulman approximation based on the Yamaguchi[2] potential. This will be discussed in Sec. II.

In Sec. III we introduced the Faddeev[3] equations which are modified to include the Coulomb potential between the three α particles. Neglecting the internal construction of the α cluster, we take the two-α-cluster short-range interaction to be of the separable form

$$\langle q'|V|q\rangle = \lambda v(q')v(q),$$

with $v(q) = (\beta^2 + q^2)^{-1}$. Thus, the aim of this paper is to calculate the binding energy of ^{12}C in the three-α-cluster model in the framework of the Faddeev formalism.

The results for the different approximations introduced are compared with previous results in Sec. IV.

II. COULOMB GREEN'S FUNCTION

Taking the momenta of the three α particles in the c.m. system to be $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$, the relative momenta of the pairs (2, 3), (3, 1), and (1, 2) will be given by

$$\vec{q}_1 \equiv \vec{q}_{23} = \tfrac{1}{2}(\vec{p}_3 - \vec{p}_2), \quad \text{etc.} \tag{1}$$

The total Hamiltonian H is given by

$$H = H_0 + U + V, \tag{2}$$

where U is the sum of the two-body Coulomb potential; V is the sum of the short-range nuclear interactions; and the kinetic energy, in the c.m. system, is given by

$$H_0 = p_i^2/2M + q_i^2/2m, \tag{3}$$

where

$$M = 2m_\alpha/3\,, \quad m = m_\alpha/2\,,$$

m_α is the α-particle mass, and $i = 1$, 2, or 3. We denote the three α particles by the symbols i, j, and k, where i, j, and k take the numbers 1, 2, or 3 and $i \neq j \neq k$.

Then we can write for the two-body Coulomb Green's function

$$\langle \vec{p}_i' \vec{q}_i' | G_{jk}^C(z) | \vec{p}_i \vec{q}_i \rangle = \delta(\vec{p}_i' - \vec{p}_i) \langle \vec{q}_i' | G_{jk}^C \left(z - \frac{p_i^2}{2M} \right) | \vec{q}_i \rangle\,, \tag{4}$$

where

$$G_{jk}^C(z) = (z - H_0 - U_{jk})^{-1}\,. \tag{5}$$

The total Coulomb Green's function for three α particles will be given by the equation

$$\begin{pmatrix} G_{01}^C \\ G_{02}^C \\ G_{03}^C \end{pmatrix} = \begin{pmatrix} G_{ij}^C \\ G_{jk}^C \\ G_{ki}^C \end{pmatrix} + \begin{pmatrix} G_{ij}^C(U - U_{ij}) & 0 & 0 \\ 0 & G_{jk}^C(U - U_{jk}) & 0 \\ 0 & 0 & G_{ki}^C(U - U_{ki}) \end{pmatrix} \begin{pmatrix} G_{01}^C \\ G_{02}^C \\ G_{03}^C \end{pmatrix}. \tag{6}$$

Since the Coulomb scattering is peaked in the forward direction, the Coulomb functions in the integrand of Eq. (4) will be peaked where $\vec{q}_i$ coincides in direction and magnitude with the integral variable, where it will have the form

$$\int \psi_k(\vec{p}) \frac{1}{\beta^2 + p^2}\, d^3p \approx \frac{1}{\beta^2 + k^2} \int \psi_k(\vec{p}) d^3p$$

$$= \frac{1}{\beta^2 + k^2} [\Psi_k^C(\vec{r})]_{r=0}\,, \tag{7}$$

where

$$[\Psi_k^C(\vec{r})]_{r=0} = \left[\frac{2\pi\eta}{e^{2\pi\eta} - 1} \right]^{1/2}. \tag{8}$$

$\eta = me^2/k$ is the Coulomb parameter.

Schulman has approximated the Coulomb wave function (8), so that the Green's function (4) will be given by

$$\langle \vec{p}_i' \vec{q}_i' | G_{jk}^C(z) | \vec{p}_i \vec{q}_i \rangle \approx \delta(\vec{p}_i' - \vec{p}_i) \delta(\vec{q}_i' - \vec{q}_i) \frac{|[\Psi_k^C(\vec{r})]_{r=0}|^2}{z - p_i^2/2M - q_i^2/2m}\,. \tag{9}$$

At the same time, evaluating the integral in Eq. (7) exactly, we have for the Green's function

$$\langle \vec{p}_i' \vec{q}_i' | G_{jk}^C(z) | \vec{p}_i \vec{q}_i \rangle \approx \delta(\vec{p}_i' - \vec{p}_i) \delta(\vec{q}_i' - \vec{q}_i) \frac{|[\Psi_k^C(\vec{r})]_{r=0} \exp[2\eta \tan^{-1}(q_i/\beta)]|^2}{z - p_i^2/2M - q_i^2/2m}\,. \tag{10}$$

Equations (9) and (10) represent the two approximations suggested by Schulman for the Coulomb Green's function.

III. MODIFIED FADDEEV EQUATIONS

The Faddeev equations for three uncharged particles are well established. These equations have also been studied for three nucleons with two charged particles.[4-6] Our aim now is to proceed to the three-body problem with all the particles charged.

Defining the following resolvent for the Green's functions as

$$G_0(z) = (z - H_0)^{-1}\,, \quad G(z) = (z - H)^{-1}\,, \tag{11}$$

then

$$G(z) = G_0(z) + G_0(z) V G(z)\,. \tag{12}$$

If the particle i is going straight, while the other two particles scatter, then the amplitude of this process is given by[4]

$$\langle \vec{p}_i' \vec{q}_i' | A_i(z) | \vec{p}_i \vec{q}_i \rangle = \delta(\vec{p}_i' - \vec{p}_i)\, v(\vec{q}_i') v(\vec{q}_i) B\left(z - \frac{p_i^2}{2M}\right), \tag{13}$$

where

$$B(z) = \left[\lambda^{-1} - \frac{1}{(2\pi)^3} \int d^3q\, v^2(\vec{q}) \left(z - \frac{q^2}{2m}\right)^{-1} \right]^{-1}. \tag{14}$$

Now, in our case of ^{12}C composed of three α particles, we define

$$G^C(z) = (z - H)^{-1}, \tag{15}$$

$$G_0^C(z) = (z - H_0 - U)^{-1}, \tag{16}$$

where U is the sum of the two-body Coulomb interactions. $G_0^C(z)$ is given in terms of $G_{ij}^C(z), \ldots$, etc., by the relation (6).

From Eqs. (15) and (16) we can write

$$G^C(z) = G_0^C(z) + G_0^C(z) T(z) G_0^C(z), \tag{17}$$

where

$$T(z) = \sum_{i=1}^{3} T^{(i)}(z), \tag{18}$$

$$T^{(i)}(z) = A_i^C(z) + A_i^C(z) G_0^C(z) T(z), \tag{19}$$

where $A_i^C(z)$ is given by

$$\langle \vec{p}_i' \vec{q}_i' | A_i^C(z) | \vec{p}_i \vec{q}_i \rangle = \delta(\vec{p}_i' - \vec{p}_i) v(\vec{q}_i') v(\vec{q}_i) B_i^C\left(z - \frac{p_i^2}{2M}\right), \tag{20}$$

$$B_i^C(z) = \left[\lambda^{-1} - \frac{1}{(2\pi)^3} \int d^3q_i v^2(\vec{q}_i) \left(z - \frac{q_i^2}{2m}\right)^{-1} \{[\Psi_{q_i}^C(\vec{r})]_{r=0}\}^2 \right]^{-1}. \tag{21}$$

Now, following the same procedure suggested by Adya[4] in the case of the ^3He nucleus, we can write for Eq. (19)

$$\langle \vec{p}_i' \vec{q}_i' | T^{(i)}(z) | \vec{p}_i \vec{q}_i \rangle = \langle \vec{p}_i' \vec{q}_i' | A_i^C(z) | \vec{p}_i \vec{q}_i \rangle + \int \langle \vec{p}_i' \vec{q}_i' | A_i^C(z) | \vec{p}_i'' \vec{q}_i'' \rangle \langle \vec{p}_i'' \vec{q}_i'' | G_0^C(z) | \vec{p}_i''' \vec{q}_i''' \rangle \langle \vec{p}_i''' \vec{q}_i''' | T(z) | \vec{p}_i \vec{q}_i \rangle d\vec{p}_i'' d\vec{q}_i'' d\vec{p}_i''' d\vec{q}_i'''. \tag{22}$$

Neglecting the three-body Coulomb potentials (i.e., considering only two-body Coulomb potentials), with the aid of Eq. (9) for the factor

$$\langle \vec{p}_i'' \vec{q}_i'' | G_0^C(z) | \vec{p}_i''' \vec{q}_i''' \rangle,$$

and using Eq. (20), we get for Eq. (22) the following expression:

$$\langle \vec{p}_i' \vec{q}_i' | T^{(i)}(z) | \vec{p}_i \vec{q}_i \rangle = v(\vec{q}_i') B_i^C\left(z - \frac{p_i'^2}{2M}\right) \left[\delta(\vec{p}_i' - \vec{p}_i) v(\vec{q}_i) + \int d^3p_i''\, d^3q_i'' v(\vec{q}_i'') \frac{\langle \vec{p}_i'' \vec{q}_i'' | T(z) | \vec{p}_i \vec{q}_i \rangle \{[\Psi_{q_i}^C(\vec{r})]_{r=0}\}^2}{z - p_i'^2/2M - q_i''^2/2m} \right]. \tag{23}$$

Since q_i can be expressed in terms of pairs (p_j, q_j) with $j \neq i$, we can integrate over d^3q_i'', and only the integration over d^3p_i'' remains in Eq. (23).

Writing the solution of Eq. (23) in the form

$$\langle \vec{p}_i' \vec{q}_i' | T^{(i)}(z) | \vec{p}_i \vec{q}_i \rangle = v(\vec{q}_i') v(\vec{q}_i) X_i(\vec{p}_i', \vec{p}_i, z), \tag{24}$$

we get the following integral equations for the bound state:

$$X_i(\vec{p}_i', z) - \frac{B_i^C(z - p_i'^2/2M)}{(2\pi)^3} \int \frac{d^3p_j'' X_j(\vec{p}_j'', z)\{[\Psi^C_{|\frac{1}{2}(\vec{p}_j'' - \vec{p}_i')|}(\vec{r})]_{r=0}\}^2}{[\beta^2 + (\vec{p}_j'' + \frac{1}{2}\vec{p}_i')^2][z - p_i'^2/2M - (1/2m)(\vec{p}_j'' + \frac{1}{2}\vec{p}_i')^2][\beta^2 + (\vec{p}_i' + \frac{1}{2}\vec{p}_j'')^2]}$$

$$- \frac{B_i^C(z - p_i'^2/2M)}{(2\pi)^3} \int \frac{d^3p_k'' X_k(\vec{p}_k'', z)\{[\Psi^C_{|\frac{1}{2}(\vec{p}_k'' + 2\vec{p}_i')|}(\vec{r})]_{r=0}\}^2}{[\beta^2 + (\vec{p}_k'' + \frac{1}{2}\vec{p}_i')^2][z - p_i'^2/2M - (1/2m)(\vec{p}_k'' + \frac{1}{2}\vec{p}_i')^2][\beta^2 + (\vec{p}_i' + \frac{1}{2}\vec{p}_k'')^2]} = 0 . \tag{25}$$

With the successive permutations between i, j, and k we get three separate equations for Eq. (25).

The corresponding three one-dimensional integral equations for the $l=0$ partial waves of X_i are

$$X_i^0(p_i', z) + \int_0^\infty dp_j'' \, Y(p_i', p_j'') X_j^0(p_j'', z) + \int_0^\infty dp_k'' \, Y(p_i', p_k'') X_k^0(p_k'', z) = 0 , \tag{26}$$

with

$$Y(p_i, p_j) = \frac{y}{8\pi^2 p_i^3 p_j} B_i^C\left(z - \frac{p_i^2}{2M}\right) \int_{-1}^{+1} \frac{dt}{(\alpha + t)(\gamma + t)(\delta + t)(x - t)[e^{y/(x-t)^{1/2}} - 1]} , \ldots , \text{ etc.}, \tag{27}$$

$$\alpha = \frac{\beta^2 + \frac{1}{4}p_i^2 + p_j^2}{p_i p_j} ,$$

$$\gamma = \frac{\beta^2 + p_i^2 + \frac{1}{4}p_j^2}{p_i p_j} ,$$

$$\delta = \frac{p_i^2 + p_j^2 - z m_\alpha}{p_i p_j} , \tag{28}$$

$$x = \frac{p_i^2 + 4p_j^2}{4p_i p_j} ,$$

and $y = 2\pi\eta k/(p_i p_j)^{1/2}$, or in matrix form, we get for Eq. (26) the representation

$$\begin{bmatrix} 1 & Y_{ij} & Y_{ik} \\ Y_{ji} & 1 & Y_{jk} \\ Y_{ki} & Y_{kj} & 1 \end{bmatrix} \begin{bmatrix} X_i^0 \\ X_j^0 \\ X_k^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} , \tag{29}$$

where for example

$$Y_{ij} = Y(p_i', p_j'') .$$

IV. RESULTS AND CONCLUSIONS

The integrals in Eq. (29) are solved using the Kopal[7] method. These integrals are replaced by a 25-point mesh. The parameter β was chosen to be 1.35 fm^{-1}. Then calculating the binding energy of ^{12}C in the three-α-particle model it is found to be 6.08 MeV.

Repeating all the above calculations, but replacing $[\Psi_q^C(\vec{r})]_{r=0}$ in Eq. (25) by $[\Psi_q^C(\vec{r})]_{r=0} \exp[2\eta \times \tan^{-1}(q/\beta)]$, we obtain the binding energy of ^{12}C to be 7.5 MeV, according to the improved version of the first-mentioned Schulman approximation.

From these calculations, one sees that the calculated binding energy for ^{12}C using three-body calculations is in agreement with the measured values. Also, our calculated value agrees with the Wang[8] result based on using an α-α potential fitted to α-α scattering.

Thus, it is safe to say that the ^{12}C nucleus can be well described in a three-α-particle model in spite of the large error in the value of the calculated binding energy due to the Schulman approximation which is of the order of 16%. It is clear that the error in the calculations of the second approximation, which is the improved version of Schulman, does not exceed 2.7% from the measured values. Thus we can say that our calculation for the ^{12}C binding energy is in good agreement with the measured value.

Now, it is worthwhile to refer to a recent work of Leung and Park,[9] who considered this problem using a wave-function method. They used for the Yamaguchi potential parameters the same values suggested by Harrington,[10] who bases his formulation on the T-matrix method of Faddeev and Lovelace.[11] In their study, they were concerned with the nuclear binding energies; i.e., they neglected the Coulomb energy which is 5.44 MeV for ^{12}C, according to Ref. 10. In the same way, Leung and Park have obtained for the ^{12}C binding energy a value of 7.36 MeV which is similar to Harrington's result, and not far from ours.

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Exchange Processes in $n + \alpha$ Scattering*†

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The features of the kernel function in the nonlocal $n+\alpha$ interaction, derived with the resonating-group method, which uses a totally antisymmetric wave function and a nucleon-nucleon potential, is studied. This kernel function is made up of three terms, corresponding to knockout, heavy-particle-pickup, and nucleon-rearrangement processes. In the medium- and high-energy regions, the knockout process contributes mainly in the forward directions, while the heavy-particle-pickup and nucleon-rearrangement processes contribute mainly in the backward directions. In particular, it is found that these latter two processes are almost entirely responsible for the occurrence of large backward-angle cross sections in the $n+\alpha$ problem. An equivalent local potential between the neutron and the α particle is also constructed, which, in the Born approximation, yields the same results as does the resonating-group calculation. This equivalent local potential has an explicit energy dependence and a significant amount of Majorana space-exchange component. Finally, approximation methods are proposed which contain the essential features of the antisymmetrization procedure and yet could be used to consider such more complicated problems as the scattering of nucleons by medium- and heavy-weight nuclei.

I. INTRODUCTION

In recent years, a number of calculations[1,2] have been performed to examine the properties of light nuclear systems with $A \leq 8$ using the method of the resonating-group structure.[3,4] The results of these calculations have been very encouraging, since for all these systems, the agreement between the calculated and experimental results was found to be quite satisfactory over a wide range of energies.

A logical step is to extend these calculations to heavier systems, such as the scattering of protons by medium- and heavy-weight nuclei where extensive phenomenological analyses have been performed using the optical model with local potentials.[5] Here, however, one encounters practical difficulties, since in a resonating-group calculation one employs a completely antisymmetric wave function which, from a computational point of view, is feasible only for systems with a relatively small number of nucleons. On the other hand, the results of the resonating-group calculations clearly indicated that the effects introduced by the antisymmetrization procedure are important and cannot be omitted if satisfactory agreement with the experimental data is to be obtained. Thus, in a heavier system where there is no *a priori* reason to expect that the antisymmetrization effects are unimportant, one must seek an approximation method which can simplify the computation to a significant extent and yet preserve the main features of antisymmetrization. In this investigation, we make an initial attempt in this direction by performing a detailed study of the antisymmetrization effects in the case of the scattering of neutrons by α particles. This particular case is chosen, since here these effects are manifested in a particularly transparent manner and hence are amenable to clear and simple interpretation.

In the one-channel resonating-group formalism, this complicated five-nucleon problem is reduced to a two-body problem in which the wave function $F(\vec{R})$ describing the relative motion of the neutron and the α particle is given by

$$\left[\frac{\hbar^2}{2\mu}\nabla_R^2 + E - V_D(\vec{R})\right]F(\vec{R}) = \int K(\vec{R},\vec{R}')F(\vec{R}')d\vec{R}' , \quad (1)$$