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Two-Nucleon T Matrix Half Off the Energy Shell: A Direct Approach*

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The T matrix half off the energy shell can be expressed in terms of the on-shell T matrix and the difference between the full scattering wave function and the phase-shifted free wave function. This representation allows one to investigate variations in the half-shell T matrix by means of a parametrization of the scattering wave function in the interaction region. Most of our knowledge of the physical constraints on that wave function may then be included. In particular we have studied the arbitrariness remaining in the half-shell T matrix given the following constraints: (1) The on-shell T matrix is given; (2) a specific local potential acts beyond a certain radius; and (3) the wave function is suppressed at short distances. We construct models of the $^1S_0$ two-nucleon half-shell T matrix. In a simple one-parameter model satisfying these constraints, the range of variation of the near off-shell behavior is displayed.

I. INTRODUCTION

Calculations of the properties of multi-nucleon systems generally require as input a description of the two-nucleon interaction off the energy shell. This interaction is usually represented in the context of the nonrelativistic Schrödinger equation by some phenomenological potential whose parameters have been adjusted to give a fit to phase shifts extracted from nucleon-nucleon elastic scattering experiments. It is known that the constraint provided by the phase shifts can be satisfied by an infinity of potentials; many such potentials have been constructed.[1] Since a potential is related only indirectly to observable quantities, any evidence for choosing one in preference to others must be adduced indirectly and at the expense of considerable computation.[2]

In this paper, we study an alternative description of the two-nucleon interaction advocated by Baranger, Giraud, Mukhopadhyay, and Sauer[3] (BGMS). These authors suggest that, instead of making models of an unobservable potential, one should directly parametrize the two-nucleon partial-wave T matrix half off the energy shell (hereafter called the half-shell T). The quantity is itself measurable, in principle,[4] and it carries as much information as the potential.[5] For most purposes, the half-shell T is actually a more convenient representation of the interaction than the potential.[3]

BGMS have discussed the restrictions imposed on the half-shell T by the requirement that the two-nucleon scattering states form a complete orthonormal set. There are, as they note, other physical constraints, such as the range of the interaction. It is the effect of these physical requirements, together with generally accepted qualitative features of the two-nucleon wave function in coordinate space, that we investigate here.

We present a simple scheme for devising models of the half-shell T subject to the following constraints: (1) The on-shell T is prescribed by the scattering data; (2) beyond a certain radius the interaction is described by a specific local potential; and (3) the two-nucleon wave function is suppressed at short distances. For convenience, we discuss only the $^1S_0$ partial wave. However, all formulas are written in a form that permits immediate transcription to corresponding results for higher uncoupled partial waves in which there are no bound states. These results are given explicitly in Appendix A. Partial waves that are

coupled or that contain bound states are not treated in this paper.

In Sec. II, we derive an equation which expresses the half-shell T in terms of the on-shell T and the spherical Bessel transform of a scattering difference function. This function is the difference between the full scattering wave function and the phase-shifted free wave function. Since the on-shell T is to be taken from experiment, we are left with the problem of finding reasonable models of the difference function. Procedures for doing this are developed in Secs. III and IV.

We illustrate our method by using it to generate a simple one-parameter model of the difference function and the corresponding half-shell T's in Sec. V. Invoking some plausible assumptions, we study the range of variation of the half-shell T in this model. Section VI summarizes the paper.

II. HALF-SHELL T AND THE SCATTERING DIFFERENCE FUNCTION

We begin with the customary optimistic assumption that a potential may be found which, when inserted in the nonrelativistic Schrödinger equation, gives an accurate wave function for two interacting nucleons when their relative momentum (in units such that $\hbar = 1$) is much less than λ_N^{-1}, where λ_N is the nucleon Compton wave length. The s-wave Schrödinger equation at center-of-mass energy k^2 is[6]

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+k^2\right]\psi_0^{(+)}(k,r) = \int_0^\infty U(r,r')\psi_0^{(+)}(k,r')r'^2dr' . \tag{1}$$

$U(r,r')$ is the s-wave projection of the potential in question, which in general will depend on the relative orbital angular momentum and probably must contain additional nonlocality,[7] and $\psi_0^{(+)}$ is the outgoing-wave solution normalized to

$$\psi_0^{(+)}(k,r) \underset{r\to\infty}{\sim} \frac{1}{kr}e^{i\delta_0(k)}\sin[kr+\delta_0(k)], \tag{1'}$$

with $\delta_0(k)$ the s-wave phase shift.

Having assumed the existence of $U(r,r')$, we wish to avoid the difficult (if not impossible) task of completely determining it. It has been pointed out that equivalent information is carried by the half-shell T in which the initial momentum is on shell and the final momentum is off shell.[8] This half-shell T is the object of our study; for the s wave it is given by the expression[9]

$$t_0(p,k;k^2) = \int_0^\infty dr\, r^2 j_0(pr)\int_0^\infty dr'\, r'^2 U(r,r')\psi_0^{(+)}(k,r') . \tag{2}$$

$j_0(pr) = \sin pr/pr$ is the zeroth spherical Bessel function. The on-shell limit of Eq. (2)

$$t_0(k) \equiv t_0(k,k;k^2) = \int_0^\infty dr\, r^2 j_0(kr)\int_0^\infty dr'\, r'^2 U(r,r')\psi_0^{(+)}(k,r') \tag{2'}$$

is empirically determined; it may be expressed in terms of the phase shift,

$$t_0(k) = -\frac{1}{k}e^{i\delta_0(k)}\sin\delta_0(k) . \tag{2''}$$

Our first objective is to rewrite Eq. (2) in such a fashion that: (a) The underlying potential $U(r,r')$ does not explicitly appear; and (b) the constraints on $t_0(p,k;k^2)$ imposed by fixing $t_0(k)$ from experiment are made as explicit as possible.

The potential may be eliminated from Eq. (2) by means of Eq. (1), which gives

$$t_0(p,k;k^2) = \int_0^\infty dr\, r^2 j_0(pr)\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+k^2\right]\times\psi_0^{(+)}(k,r) . \tag{3}$$

Next, we introduce the phase-shifted free wave function[10]

$$v_0(kr) = e^{i\delta_0(k)}[\cos\delta_0(k)j_0(kr)+\sin\delta_0(k)n_0(kr)] . \tag{4}$$

Since $v_0(kr)$ is annihilated by the operator in square brackets in the integrand of Eq. (3), we may write

$$t_0(p,k;k^2) = \int_0^\infty dr\, r^2 j_0(pr)\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+k^2\right]\times[\psi_0^{(+)}(k,r)-v_0(kr)] . \tag{5}$$

In so doing, we have replaced $\psi_0^{(+)}(k,r)$, which oscillates undamped for large r, by a function which vanishes for large r. This allows us to get rid of the differential operator in the integrand by integrating by parts twice. It is convenient to introduce the real "difference function"

$$\Delta_0(k,r) = kr\,e^{-i\delta_0(k)}[\psi_0^{(+)}(k,r)-v_0(kr)] \tag{6}$$

and to write $u_0(pr) = prj_0(pr) = \sin pr$, in terms of which Eq. (5) becomes

$$t_0(p,k;k^2) = \frac{e^{i\delta_0(k)}}{pk}\int_0^\infty dr\, u_0(pr)\left[\frac{d^2}{dr^2}+k^2\right]\Delta_0(k,r) . \tag{7}$$

Integrating by parts twice, we find

$$t_0(p,k;k^2) = t_0(k)+(k^2-p^2)\frac{e^{i\delta_0(k)}}{pk}\int_0^\infty dr\, u_0(pr)\Delta_0(k,r) . \tag{8}$$

This representation is the desired result, for it does not require us to exhibit a potential, and it brings to the foreground the role of the on-shell T in determining the half-shell T. Although Eq. (8) has been found previously,[11] it apparently has not been used to obtain direct parametrizations of half-shell T's corresponding to a prescribed on-shell T. The problem of constructing such parametrizations is now seen to reduce to the task of making suitable models of the difference function.

Two noteworthy properties of $\Delta_0(k,r)$ may be read directly from its definition. They limit the class of functions which might be used as model difference functions. The first is that $\Delta_0(k,r)$ is vanishingly small for r greater than the range of interaction. This follows from Eq. (6) and the observation that, outside the interaction region, $\psi_0^{(+)}(k,r)$ is practically identical to $v_0(kr)$. Secondly, since[12] $kr\psi_0^{(+)}(k,r) \underset{r\to 0}{\longrightarrow} 0$, it is seen that

$$\Delta_0(k,r) \underset{r\to 0}{\longrightarrow} -\sin\delta_0(k)\cos kr \underset{r\to 0}{\longrightarrow} -\sin\delta_0(k)\,. \tag{9}$$

The foregoing remarks indicate the importance of the short range of the interaction in specifying off-shell behavior. In the limit that the range becomes very small, they lead to the self-evident conclusion that $\Delta_0(k,r)$ and $t_0(p,k;k^2)$ are completely determined by the phase shift. More generally, we may expect that if $\Delta_0(k,r)$ is a reasonably smooth function of r, the two boundary conditions just cited should impose a significant correlation between the on-shell T and the half-shell T for any short-range force, local or nonlocal.

In the evaluation of model difference functions, it is convenient to divide the interaction region into an exterior region, $r>R$, in which the interaction is accurately described by a local potential, and an interior region, $0\le r\le R$. For reasons given in the next section, we take $R=\lambda\!\!\!^{-}_{\pi}=1.43$ F, where $\lambda\!\!\!^{-}_{\pi}$ is the pion Compton wavelength.

III. EFFECT OF A LOCAL EXTERIOR POTENTIAL

In the $^1S_0$ wave, there seems to be a reasonably unique phenomenological local potential description of the interaction in the exterior region,[13] and it is to be expected on fairly general theoretical grounds that $U(r,r')$ should be local when $r\ge\lambda\!\!\!^{-}_{\pi}$.[14] Although we have eliminated the potential from the expression for $t_0(p,k;k^2)$ because it is generally not well determined, there is no reason to ignore the information carried by the fixed local exterior potential. This is easily incorporated in our models of $\Delta_0(k,r)$ by means of an integral equation.

Our starting point is the integral equation for $\psi_0^{(+)}(k,r)$ equivalent to Eqs. (1) and (1′),

$$\psi_0^{(+)}(k,r)=j_0(kr)-k\int_0^\infty dr'\,r'^2 j_0(kr_<)h_0^{(+)}(kr_>)\int_0^\infty U(r',r'')\psi_0^{(+)}(k,r'')r''^2dr''\,, \tag{10}$$

where

$$h_0^{(+)}(kr)=n_0(kr)+ij_0(kr)=\frac{e^{ikr}}{kr}$$

and $(r_<,r_>)$ denote the lesser and greater of (r,r'), respectively. Note that Eq. (4) may also be written as

$$v_0(kr)=j_0(kr)-kt_0(k)h_0^{(+)}(kr) \tag{11}$$

and that $t_0(k)$ is given by (2′), so that

$$v_0(kr)=j_0(kr)-kh_0^{(+)}(kr)\int_0^\infty dr'\,r'^2 j_0(kr')\int_0^\infty U(r',r'')\psi_0^{(+)}(k,r'')r''^2dr''\,. \tag{12}$$

Subtracting Eq. (12) from Eq. (10), we obtain

$$\psi_0^{(+)}(k,r)-v_0(kr)=k\Big[h_0^{(+)}(kr)\int_r^\infty dr'\,r'^2 j_0(kr')\int_0^\infty U(r',r'')\psi_0^{(+)}(k,r'')r''^2dr''$$
$$-j_0(kr)\int_r^\infty dr'\,r'^2 h_0^{(+)}(kr')\int_0^\infty U(r',r'')\psi_0^{(+)}(k,r'')r''^2dr''\Big]. \tag{13}$$

If we define

$$\begin{aligned} y_0(kr)&=krn_0(kr)\,,\\ w_0(k,r)&=e^{-i\delta_0(k)}kr\psi_0^{(+)}(k,r)\end{aligned} \tag{14}$$

and refer to Eq. (6), we find that

$$\Delta_0(k,r)=\frac{1}{k}\left[y_0(kr)\int_r^\infty dr' u_0(kr')r'\int_0^\infty U(r',r'')r''w_0(k,r'')dr''-u_0(kr)\int_0^\infty dr' y_0(kr')r'\int_0^\infty U(r',r'')r''w_0(k,r'')dr''\right]. \tag{15}$$

Equation (15) is cast as an integral equation for $\Delta_0(k,r)$ upon inserting

$$w_0(k,r)=\Delta_0(k,r)+\cos\delta_0(k)u_0(kr)+\sin\delta_0(k)y_0(kr), \tag{16}$$

which gives

$$\Delta_0(k,r)=\Delta_0^{(0)}(k,r)+\frac{1}{k}\int_r^\infty dr' g_0(k;r,r')r'\int_0^\infty U(r',r'')r''\Delta_0(k,r'')dr'', \tag{17}$$

where

$$\Delta_0^{(0)}(k,r)=\frac{1}{k}\int_r^\infty dr' g_0(k;r,r')r'\int_0^\infty U(r',r'')r''[\cos\delta_0(k)u_0(kr'')+\sin\delta_0(k)y_0(kr'')]dr'' \tag{17'}$$

and

$$g_0(k;r,r')=y_0(kr)u_0(kr')-u_0(kr)y_0(kr')=-\sin k(r-r'). \tag{17''}$$

As it stands, Eq. (17) requires a complete knowledge of $U(r,r')$. However, as remarked previously, it is believed that the nonlocality in the potential (in a given partial wave) is confined to distances shorter than $\lambda\!\!\!^{-}_\pi$. This means that we can write

$$U(r,r')=N(r,r')+\delta(r-r')L(r)/rr' \tag{18}$$

such that

$$|L(r)|\gg\int_0^\infty dr' r'^2|N(r,r')|, \tag{18'}$$

when $r>R$. For $r>R$ Eq. (17) then simplifies to

$$\Delta_0(k,r)=\Delta_0^{(0)}(k,r)+\frac{1}{k}\int_r^\infty dr' g_0(k;r,r')L(r')\Delta_0(k,r'), \quad r>R \tag{19}$$

with

$$\Delta_0^{(0)}(k,r)=\frac{1}{k}\int_r^\infty dr' g_0(k;r,r')L(r')[\cos\delta_0(k)u_0(kr')+\sin\delta_0(k)y_0(kr')], \quad r>R.$$

Equation (19) allows us to isolate the influence of a local potential on $\Delta_0(k,r)$ for $r>R$. In fact, it gives us an exact expression for $\Delta_0(k,r)$, $r>R$, given just the phase shift $\delta_0(k)$ and the local potential $L(r)$ in this exterior region. Note that Eq. (19) is a Volterra equation. It may therefore be solved by iteration with the assurance that the series thus obtained converges uniformly in r, as long as $\int_R^\infty dr\, r|L(r)|<\infty$.[15] In the calculations described in Sec. V, we found $\Delta_0^{(0)}(k,r)$ to be an excellent approximation to $\Delta_0(k,r)$. The rapid convergence of the iteration series is demonstrated explicitly for a soluble but realistic exterior potential in Appendix B.

IV. POLYNOMIAL PARAMETRIZATION OF $\Delta_0(k,r)$ IN THE INTERIOR REGION

As a working hypothesis, we suppose that $\Delta_0(k,r)$ varies smoothly enough over the interior region $[0,R]$ that we may represent it there by a polynomial in r, $P(k,r)$, of reasonably low degree. This is in fact the case for the difference function generated by the $^1S_0$ Reid soft-core potential[16] (RSC), as will be seen in Sec. V.

One can certainly imagine functions on $[0,R]$ for which no polynomial of tractable degree would give an adequate representation. However, $\Delta_0(k,r)$ is supposed to be generated from a reasonable potential[17] by means of the nonrelativistic Schrödinger equation. This equation is not exact; it is most plausibly interpreted as the low-energy limit of a relativistic two-nucleon equation (for example, the Bethe-Salpeter equation) in which the nucleons propagate according to the Dirac equation. When a nonrelativistic reduction is carried out, it is found that the observable position of a Dirac nucleon corresponds to a nonlocal "mean-position operator" which smears functions of r over an interval of size $\lambda\!\!\!^{-}_N$. Accordingly, it makes no sense to speak of a nucleon as being more localized in r than this; functions of r are interpreted consistent-

ly only when averaged over an interval of order λ_N.[18] Since $R \approx 7\lambda_N$, this heuristic argument suggests that the observable features of a two-nucleon wave function in $[0,R]$ could be summarized by its values at eight representative points. Thus, for example, a seventh-degree polynomial could be employed.

We arrived at the following method for specifying the polynomial representation of $\Delta_0(k,r)$,

$$\Delta_0(k,r) = P(k,r), \quad 0 \le r \le R, \quad k \ll \lambda_N^{-1}, \tag{20}$$

using the available information and a minimal number of free parameters. Once Eq. (19) has been solved, $\Delta_0(k,r)$ is known for $r > R$. Under reasonable assumptions of smoothness, this solution also gives us $\Delta_0(k,r)$ in a neighborhood of $r = R$, so we take $\Delta_0(k,R)$, $\Delta_0'(k,R)$, and $\Delta_0''(k,R)$ (the primes denoting derivatives with respect to r) as known quantities to be obtained from Eq. (19). In addition, Eq. (9) gives $\Delta_0(k,0) = -\sin\delta_0(k)$. For the time being, we regard $\Delta_0'(k,0)$ and $\Delta_0''(k,0)$ as free parameters. Once we have chosen them, we have specified the value, slope, and curvature of $\Delta_0(k,r)$ at the two endpoints of $[0,R]$. Since this is a small interval on the scale of variation we are considering, we expect these endpoint conditions to be significant constraints on $\Delta_0(k,r)$. In the polynomial representation this means we set

$$\left.\begin{aligned} P(k,R) &= \Delta_0(k,R) \\ P'(k,R) &= \Delta_0'(k,R) \\ P''(k,R) &= \Delta_0''(k,R) \end{aligned}\right\} \text{ obtained from the solution of Eq. (19)}, \tag{21a}$$

and

$$\left.\begin{aligned} P(k,0) &= \Delta_0(k,0) = -\sin\delta_0(k) \\ P'(k,0) &= \Delta_0'(k,0) \\ P''(k,0) &= \Delta_0''(k,0) \end{aligned}\right\} \text{ free parameters}. \tag{21b}$$

Clearly, there are an infinite number of polynomials which satsify the six conditions (21a) and (21b), but the one of lowest degree – the smoothest one – is a unique fifth-degree polynomial, $P_5(k,r)$. The determination of P_5 may be regarded as a six-point interpolation problem in the limit that three of the points coincide at $r = 0$ and three at $r = R$. This suggests that we write

$$\begin{aligned} P_5(k,r) = {} & p_0(k) + p_1(k)(r-R) + p_2(k)(r-R)^2 \\ & + p_3(k)r(r-R)^2 + p_4(k)r^2(r-R)^2 \\ & + p_5(k)r^2(r-R)^3 . \end{aligned} \tag{22}$$

Equations (21a) and (21b) then give

$$p_0(k) = \Delta_0(k,R), \tag{23a}$$

$$p_1(k) = \Delta_0'(k,R), \tag{23b}$$

$$p_2(k) = \frac{\Delta_0(k,0) - \Delta_0(k,R) + R\Delta_0'(k,R)}{R^2}, \tag{23c}$$

$$p_3(k) = \frac{\Delta_0'(k,0) - \Delta_0'(k,R) + 2Rp_2(k)}{R^2}, \tag{23d}$$

$$p_4(k) = \frac{\Delta_0''(k,R) - 2p_2(k) - 2Rp_3(k)}{2R^2}, \tag{23e}$$

$$p_5(k) = -\frac{[\Delta_0''(k,0) - 2p_2(k) + 4Rp_3(k) - 2R^2p_4(k)]}{2R^3}. \tag{23f}$$

Although we expect $\Delta_0(k,r)$ to be reasonably smooth, it may well include variations which cannot be represented by P_5, since this is the smoothest polynomial that can satisfy Eqs. (21a) and (21b) simultaneously. In particular, as shown in Sec. V, the RSC difference function displays such behavior. We therefore introduce an extra degree of freedom by adding to P_5 a term $\eta r^3(r-R)^3$, with η a free parameter. Since this term and its first two derivatives vanish at $r = 0$ and $r = R$, the resulting polynomial,

$$P(k,r) = P_5(k,r) + \eta r^3(r-R)^3, \tag{24}$$

satisfies Eqs. (21a) and (21b) if P_5 does. The polynomial defined by Eq. (24) and Eqs. (22) and (23) is our parametrization of $\Delta_0(k,r)$ over the interior region. It is evaluated in a simple model in the next section. As an aside, we note that *if the interaction is assumed to be purely local*, one has

$$\begin{aligned} \Delta_0'(k,0) &= w_0'(k,0) - k\cos\delta_0(k) \\ &= k/|f_0(k)| - k\cos\delta_0(k), \end{aligned} \tag{25}$$

where

$$f_0(k) = \exp\left(-\frac{2}{\pi}\int_0^\infty dq\, \frac{\delta_0(q)}{q-k+i\epsilon}\right) \tag{25'}$$

is the s-wave Jost function.[19] If it is further assumed that this local potential is finite at the origin, or if the wave function goes to zero at the origin more rapidly than the potential diverges there, it is apparent from the Schrödinger equation that $w_0''(k,0) = 0$, so that

$$\Delta_0''(k,0) = k^2 \sin\delta_0(k). \tag{26}$$

To some extent, then, by varying the first of these parameters one may study the influence of the high-energy phase shifts on $\Delta_0(k,r)$ (if a local potential is assumed), according to Eq. (25). The second is a measure of the effect of short-range nonlocality, since for a nonlocal interaction, Eq. (26) is replaced by

$$\Delta_0''(k,0) = k^2 \sin\delta_0(k) + \lim_{r\to 0} r\int_0^\infty dr'\, U(r,r')r'w_0(k,r'). \tag{27}$$

V. ONE-PARAMETER MODEL OF THE $^1S_0$ HALF-SHELL T

A one-parameter family of model difference functions may be obtained by invoking the common and plausible assumption that the two-nucleon wave function is very small at short distances. This suppression is naturally expressed in the parametrization of the preceding section by setting not only $w_0(k,0)$, but also $w_0'(k,0)$ and $w_0''(k,0)$ to zero, independent of k. We then have

$$\Delta_0'(k,0) = -k\cos\delta_0(k) \tag{28a}$$

and

$$\Delta_0''(k,0) = k^2\sin\delta_0(k)\,, \tag{28b}$$

leaving η as a free parameter.

We have calculated difference functions and half-shell T's in this one-parameter model. For illustrative purposes, they were evaluated at three values of the (on-shell) relative momentum: $k = 0.75$, 1.05, and 1.50 F^{-1}, corresponding to laboratory energies of about 47, 92, and 187 MeV, respectively.

Since there are no experimental data with which to confront our results, we compare them with the predictions of the RSC potential, which fits the $^1S_0$ phase shifts very well and generates half-shell T's which are typical of most potential models.[20] Wave functions and half-shell T's for the RSC potential were kindly made available to us by Lerner and Haftel.[21]

Although Eqs. (8), (19), and (24) enable us to use a set of empirical phase shifts directly, we use the RSC phase shifts as input instead. This is done so that differences between our half-shell T's and those of the RSC may be attributed to purely off-shell effects, the on-shell T's being identical by construction. Because the RSC phase shifts represent the empirical ones quite faithfully, we do not expect the substitution of one set for the other to affect our results significantly.

A. Models of $\Delta_0(k,r)$ and $w_0(k,r)$ in the Interaction Region

Because the results of the calculations of Lerner and Haftel were readily available, we chose to represent the exterior potential by the RSC for $r > \lambda\!\!\!^{-}_{\pi}$, so that we could study the accuracy of our approximations. It should be noted, however, that this reversion to a specific model of the exterior potential does not lead to any significant restriction on the validity of our final results and conclusions. For one thing, when $r > \lambda\!\!\!^{-}_{\pi}$, all of the phenomenological potentials which approach the one-pion-exchange potential smoothly at large distances become virtually identical.[13] In addition, it turns out that $\Delta_0(k,r)$ and $t_0(p,k;k^2)$ are not sensitive to the exterior potential once $\delta_0(k)$ has been fixed. As will be seen in part B of this section, the exterior potential may be omitted entirely without changing $t_0(p,k;k^2)$ very much.

The RSC potential in units of F^{-2} is

$$L(r) = G_1\frac{e^{-\mu r}}{\mu r} + G_4\frac{e^{-4\mu r}}{\mu r} + G_7\frac{e^{-7\mu r}}{\mu r}\,, \tag{29a}$$

where

$$\mu = 0.7\ F^{-1} \tag{29b}$$

and

$$\begin{aligned} G_1 &= -10.463/41.47\ F^{-2}\,,\\ G_4 &= -1650.6/41.47\ F^{-2}\,,\\ G_7 &= \ \ 6484.2/41.47\ F^{-2}\,. \end{aligned} \tag{29c}$$

Inserting Eqs. (29a)–(29c) in Eq. (19), we find that

$$\Delta_0^{(0)}(k,r) = [\cos kr \sum_{\beta=1,4,7} G_\beta I_\beta(k,r) - \sin kr \sum_{\beta=1,4,7} G_\beta \tilde{I}_\beta(k,r)]/k\,, \tag{30a}$$

where

$$I_\beta(k,r) = \{\cos\delta_0(k)[E_1(\beta\mu r) - \mathrm{Re}E_1((\beta\mu - 2ik)r)] + \sin\delta_0(k)\mathrm{Im}E_1((\beta\mu - 2ik)r)\}/2\mu \tag{30b}$$

and

$$\tilde{I}_\beta(k,r) = \{\sin\delta_0(k)[E_1(\beta\mu r) + \mathrm{Re}E_1((\beta\mu - 2ik)r)] + \cos\delta_0(k)\mathrm{Im}E_1((\beta\mu - 2ik)r)\}/2\mu\,. \tag{30c}$$

Here Re and Im denote the real and imaginary parts, respectively, of the functions they prefix, and

$$E_1(z) = \int_1^\infty dt\frac{e^{-zt}}{t} \tag{31}$$

is the exponential integral.[22] We computed the above expressions on the University of Maryland CEIR Multi-Access Computer Service, using formulas (5.1.53) and (5.1.56) in Ref. 23 together with formulas (6) and (7) of Ref. 22 to generate $E_1(z)$. The results were checked against hand calculations using the tabulation of $E_1(z)$ given in Ref. 22.

Upon adding $\sin[kr + \delta_0(k)]$ to Eqs. (30a)–(30c) and comparing the resulting approximation to $w_0(k,r)$ with the numerical wave functions of Lerner, we found agreement to three significant figures. [A comparison of our $w_0(k,r)$ and Lerner's is presented below in Fig. 2.] The higher-order iterates of Eq. (19) cannot be given in closed form, and the upper bounds that can be established for

them are much too crude to demonstrate that they are as negligible as our results indicate. However, the exponential exterior potential of the Kallio-Kolltveit force[24] approximates the right-hand side of Eq. (29a) quite closely when $r > \lambda_\pi$. When $L(r)$ is an exponential, the iteration series is known in closed form. For the Kallio-Kolltveit exterior potential, we show in Appendix B that the terms beyond $\Delta_0^{(0)}$ are indeed negligible. It seems safe to assume that equally rapid convergence of the iteration series obtains if the RSC form of Eqs. (29a)–(29c) is used. Together with the previously cited *a posteriori* observation that changes in the exterior potential affect $\Delta_0(k,r)$ and $t_0(p,k;k^2)$ only slightly, the above comments lead us to conclude that the approximation of retaining only the first term in the iteration series for $\Delta_0(k,r)$ is quite adequate for our purpose.

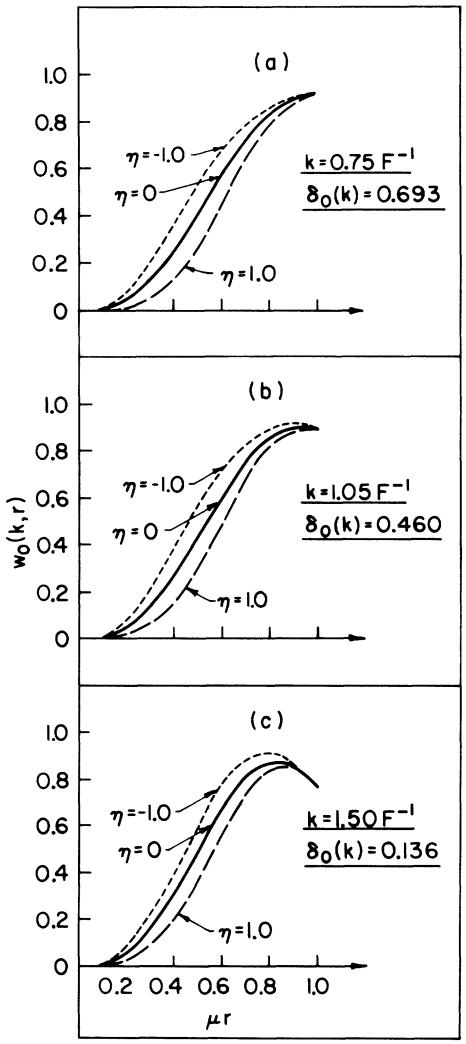

FIG. 1. (a)–(c) The model interior wave function $w_0(k,r)$ for $\eta=-1$ (---), $\eta=0$ (——), and $\eta=1$(———), $\mu=0.7$ F^{-1}.

In order to compute $P(k,r)$, we need expressions for $\Delta_0'(k,r)$ and $\Delta_0''(k,r)$ in the exterior region. Differentiating Eq. (30a), one finds

$$\Delta_0'(k,r) = \sum_{\beta=1,4,7} G_\beta[\sin kr I_\beta(k,r) + \cos kr \tilde{I}_\beta(k,r)] \tag{32}$$

and

$$\Delta_0''(k,r) = L(r)\sin[kr+\delta_0(k)] - k^2\Delta_0(k,r)\,. \tag{33}$$

Using the values of $\Delta_0(k,R)$, $\Delta_0'(k,R)$, and $\Delta_0''(k,R)$ [given by Eqs. (30a)–(30c), (32), and (33)] together with the values of $\Delta_0(k,0)$, $\Delta_0'(k,0)$, and $\Delta_0''(k,0)$ [given in Eqs. (21b), (28a), and (28b)], the $P_i(k)$ of Eq. (22) are calculated by means of Eqs. (23a)–(23f). This gives us the fifth-degree polynomial $P_5(k,r)$ of Eq. (22), to which we add $\eta r^3(r-R)^3$ to get the sixth-degree polynomials $P(k,r)$ of Eq. (24). These are our parametric models of $\Delta_0(k,r)$. The $w_0(k,r)$ corresponding to these models are obtained by adding $\sin[kr+\delta_0(k)]$ to $P(k,r)$.

Results for $\eta=-1$, 0, 1 at the three chosen values of k are plotted in Fig. 1. We show $w_0(k,r)$ rather than $\Delta_0(k,r)$ because, by force of habit, we find it easier to visualize the qualitative features one expects to find in wave functions than those of difference functions.

Without exhaustive searching on the parameter η, we found that our model resembles the RSC wave function very closely when $\eta=-0.55$ (independent of k), as shown in Fig. 2. This indicates that our parametrization is at least flexible enough to encompass the behavior of potential models.

Finally, since the difference functions are the quantities of ultimate interest for the calculation of the half-shell T, it is helpful to have an idea of their behavior. In Fig. 3 we show $\Delta_0(k,r)$ for $\eta=-0.55$.

B. Models of $t_0(p,k;k^2)$

All that remains to be done in the calculation of the half-shell T is the evaluation of the integral in Eq. (8) which we denote by $I(p,k)$. We write it as

$$I(p,k) = \int_0^\infty dr \sin pr \Delta_0(k,r)$$
$$= \int_0^R dr \sin pr P(k,r) + \int_R^\infty dr \sin pr \Delta_0^{\text{ext}}(k,r)\,, \tag{34}$$

since in our models we have

$$\Delta_0(k,r) = \begin{cases} P(k,r) & 0 \le r \le R \\ \Delta_0^{\text{ext}}(k,r) & r > R\,. \end{cases} \tag{34'}$$

The superscript ext is simply a reminder that $\Delta_0^{\text{ext}}(k,r)$ is the function obtained from the exterior potential by means of Eqs. (30a)–(30c).

Straightforward evaluation of

$$I_{\text{int}}(p,k) \equiv \int_0^R dr \sin pr P(k,r) \tag{35}$$

gives

$$I_{\text{int}}(p,k) = \sum_{n=0}^{6} c_n(k) I_n(p), \tag{36a}$$

where

$$I_n(p) = -p^{-n-1} \sum_{j=0}^{n} j! \binom{n}{j} \left[(pr)^{n-j} \cos\left(pr + \tfrac{1}{2} j\pi\right) \right]_{r=0}^{r=R}, \tag{36b}$$

and

$$\begin{aligned}
c_0(k) &= p_0(k) - R p_1(k) + R^2 p_2(k), \\
c_1(k) &= p_1(k) - 2R p_2(k) + R^2 p_3(k), \\
c_2(k) &= p_2(k) - 2R p_3(k) + R^2 p_4(k) - R^3 p_5(k), \\
c_3(k) &= p_3(k) - 2R p_4(k) + 3R^2 p_5(k) - \eta R^3, \\
c_4(k) &= p_4(k) - 3R p_5(k) + 3\eta R^2, \\
c_5(k) &= p_5(k) - 3\eta R, \\
c_6(k) &= \eta .
\end{aligned} \tag{36c}$$

The integration of

$$I_{\text{ext}}(p,k) = \int_R^\infty dr \sin pr \Delta_0^{\text{ext}}(k,r) \tag{37}$$

simplifies if one notices that

$$(k^2 - p^2) I_{\text{ext}}(p,k) = \int_R^\infty dr \sin pr \left[\frac{d^2}{dr^2} + k^2 \right] \Delta_0^{\text{ext}}(k,r) - \int_R^\infty dr \frac{d}{dr} \left[\sin pr \frac{d}{dr} \Delta_0^{\text{ext}}(k,r) - \Delta_0^{\text{ext}}(k,r) \frac{d}{dr} \sin pr \right] \tag{38a}$$

and, according to Eq. (33),

$$\left(\frac{d^2}{dr^2} + k^2 \right) \Delta_0^{\text{ext}}(k,r) = L(r) \sin[kr + \delta_0(k)] . \tag{38b}$$

One then obtains

$$(k^2 - p^2) I_{\text{ext}}(p,k) = \int_R^\infty dr \sin pr L(r) \sin[kr + \delta_0(k)] + \sin pR \Delta_0'(k,R) - p \cos pR \Delta_0(k,R), \tag{39}$$

where $\Delta_0(k,R)$ and $\Delta_0'(k,R)$ are given by Eqs. (30a)–(30c) and (32). With $L(r)$ given by Eqs. (29a)–(29c), the integral on the right-hand side of Eq. (39) reduces to

$$\int_R^\infty dr \sin pr L(r) \sin[kr + \delta_0(k)] = \sum_{\beta = 1, 4, 7} G_\beta I_\beta(p,k;R), \tag{40a}$$

where

$$\begin{aligned}
I_\beta(p,k;R) = (2\mu)^{-1} \big(&\cos\delta_0(k) \operatorname{Re}\{E_1([\beta\mu - i(p-k)]R) - E_1([\beta\mu - i(p+k)]R)\} \\
&+ \sin\delta_0(k) \operatorname{Im}\{E_1([\beta\mu - i(p-k)]R) + E_1([\beta\mu - i(p+k)]R)\} \big) .
\end{aligned} \tag{40b}$$

Combining the above results and inserting them in Eq. (8), we obtain the half-shell T. It is convenient, in comparing the variations in p for different values of k, to plot instead of $t_0(p,k;k^2)$ the Noyes-Kowalski off-shell factor $f_0(p,k) = t_0(p,k;k^2)/t_0(k)$.[8] Figures 4(a)–4(c) show our models of $f_0(p,k)$ for $\eta = -1, 0, 1$. In Fig. 5, we plot the contributions of the parametric term $\eta r^3 (r-R)^3$ and of the exterior difference function $\Delta_0^{\text{ext}}(k,r)$ [as determined from Eqs. (39), (40a), and (40b)] to $f_0(p,k)$. For ease in drawing the curves, we plotted only the parametric contribution $f_0^\eta(p,k)$ for $\eta = -1$. Since $f_0^\eta(p,k)$ depends linearly on η, our curves may be easily scaled to give $f_0^\eta(p,k)$ for other values of this parameter. The term $f_0^{\text{ext}}(p,k)$ coming from the exterior difference function is seen to be quite small everywhere except when $k = 1.50\ \text{F}^{-1}$, when it contributes appreciably at low p.

In Figs. 6(a)–6(c) we compare our models for $\eta = 0$ and $\eta = -0.55$ with the RSC half-shell T calculated by Haftel. Note that the curve for $\eta = 0$ actually lies closer to the RSC curve than does the one for $\eta = -0.55$, even though $\eta = -0.55$ gives a better fit to the RSC wave function. Referring to Eq. (34), one sees that this is simply a graphic illustration of the obvious remark that a good pointwise approximation to a function does not imply an equally good approximation to its Fourier sine transform. Since this is the case, one might ask why we did not expand the interior difference function in trigonometric polynomials instead of using an inter-

polating polynomial parametrization. The answer is that, in general, it is not possible to construct a trigonometric polynomial which simultaneously satisfies the conditions (21a) and (21b), which we felt to be important constraints. One would have to give up the requirement that the interior difference function join smoothly onto the known exterior difference function at $r=R$.

Since the contribution of $f_0^{\text{ext}}(p,k)$ is generally quite small, one expects that an idealized model in which the interaction is simply cut off at $r=\lambda\!\!\!^{-}_{\pi}$ should give rise to a half-shell T which contains the essential features of the more realistic models which include an exterior potential. Setting $\Delta_0(k,r)=0$ when $r>\lambda\!\!\!^{-}_{\pi}$, and taking $\eta=0$ for simplicity, we find the predictions of such a "Manx" model shown in Figs. 7(a)–7(c) along with those of a previous calculation including an exterior potential. The two curves lie fairly close to one another, showing that once the phase shift is fixed, the exterior region is unimportant in determining the behavior of the half-shell T.

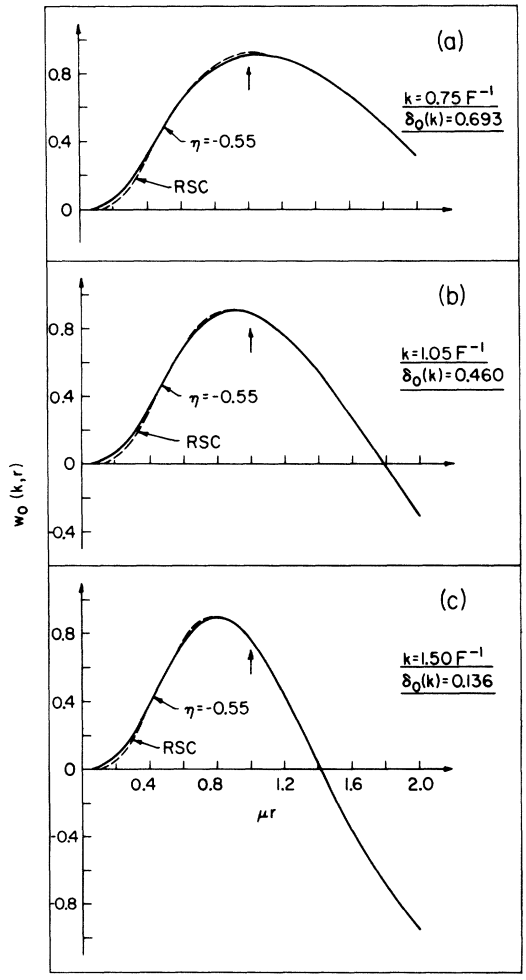

FIG. 2. (a)–(c) Comparison of model wave function for $\eta=-0.55$ (——) with RSC wave function (– – –). The arrow indicates the radius $R=1.43$ F, which separates the exterior and interior regions.

C. Range of Variation of $t_0(p,k;k^2)$

It is clear from Fig. 5 that wide variations in the behavior of $t_0(p,k;k^2)$, even when p is near k, are possible if η is left completely arbitrary. However, we find that when $\eta>1$, the model wave function develops two extra nodes in the interaction region. If the interaction were strictly local, such nodes would imply the existence of two bound states at negative energy in the $^1S_0$ partial wave, in contradiction to observation. Since the interaction is nonlocal, there is another possibility which is not ruled out so simply. There may be two "bound states in the continuum." As Bolsterli[25] has pointed out, the accidental degeneracy which gives rise to bound states in the continuum is unphysical. The slightest perturbation, such as the coupling of the nucleons to the electromagnetic field, will remove the degeneracy and change the bound states in the continuum into narrow resonances. In the absence of evidence for the occurrence of either bound states in the continuum or narrow resonances in the two-nucleon $^1S_0$ state, we exclude these possibilities and restrict η to values less than +1.

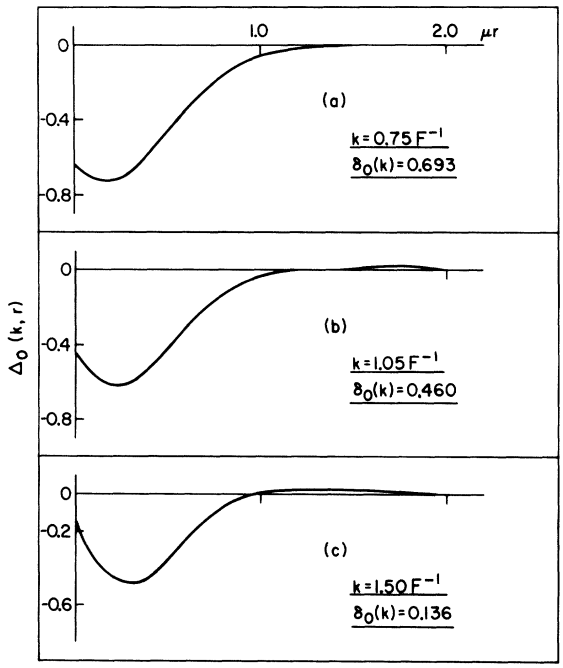

FIG. 3. (a)–(c) Model difference functions, $\Delta_0(k,r)$, for $\eta=-0.55$.

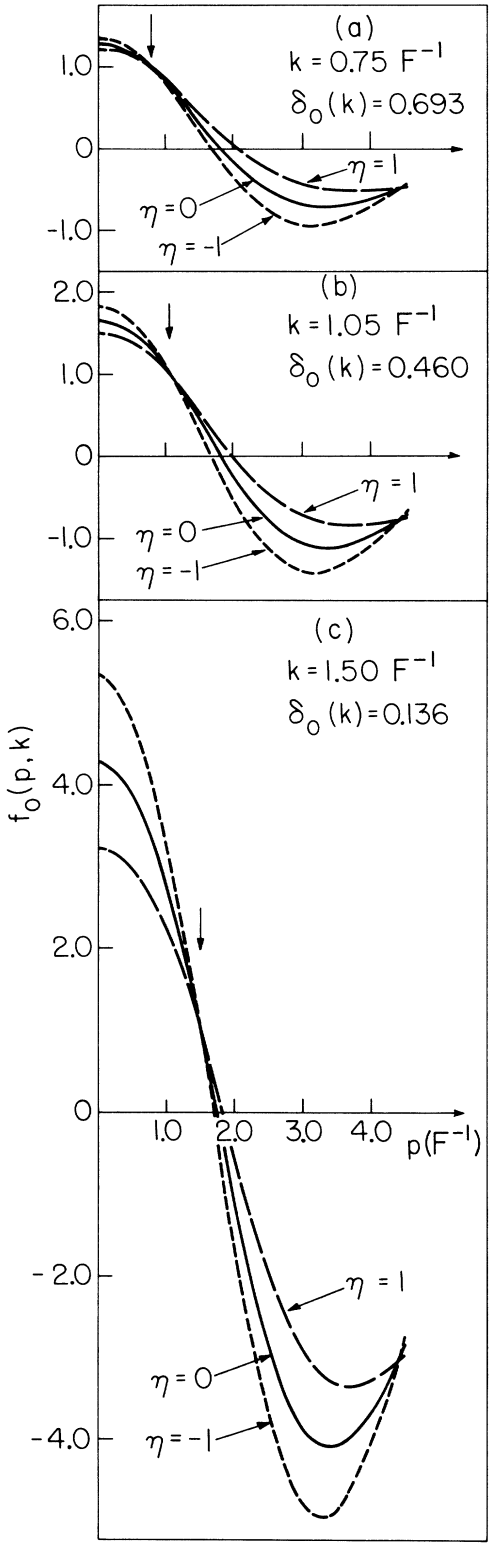

FIG. 4. (a)–(c) Half-shell factors, $f_0(p,k)$, for $\eta=-1$ (---), $\eta=0$ (——), and $\eta=1$ (– – –). The on-shell point, $p=k$, is indicated by a vertical arrow. k is on the on-shell momentum. Note scale change in (c).

When η is less than –1 we find that the maximum amplitude of the wave function inside the interaction region becomes greater than the amplitude of the phase-shifted free wave function. The only mechanism known to us which can cause such an internal enhancement is some sort of resonance "trapping" of the wave function.[26] Again, since we know of no evidence for resonance effects of this sort, we exclude them by restricting η to values greater than –1.

On the basis of these heuristic arguments, we believe that η should be limited to values in the range $|\eta|<1$. Then the curves of Figs. 4(a)–4(c) display the range of variation of $f_0(p,k)$ that is to be expected given (1) the empirical phase shifts, (2) the range of the two-nucleon interaction, (3) the assumptions about the smoothness of the wave function made in Sec. IV, (4) the suppression of the wave function at short distances, and (5) that no sharp resonances in the $^1S_0$ partial wave are observed.

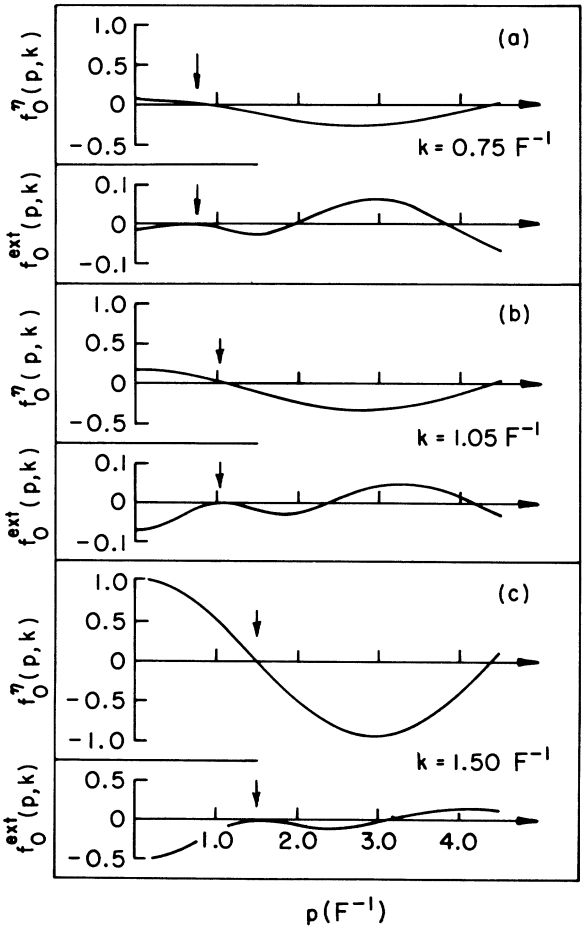

FIG. 5. (a)–(c) The contributions of the parametric term $f_0^{\eta}(p,k)$ (for $\eta=-1$), and of the exterior $f_0^{ext}(p,k)$, to the half-shell factor. Note differences in scale.

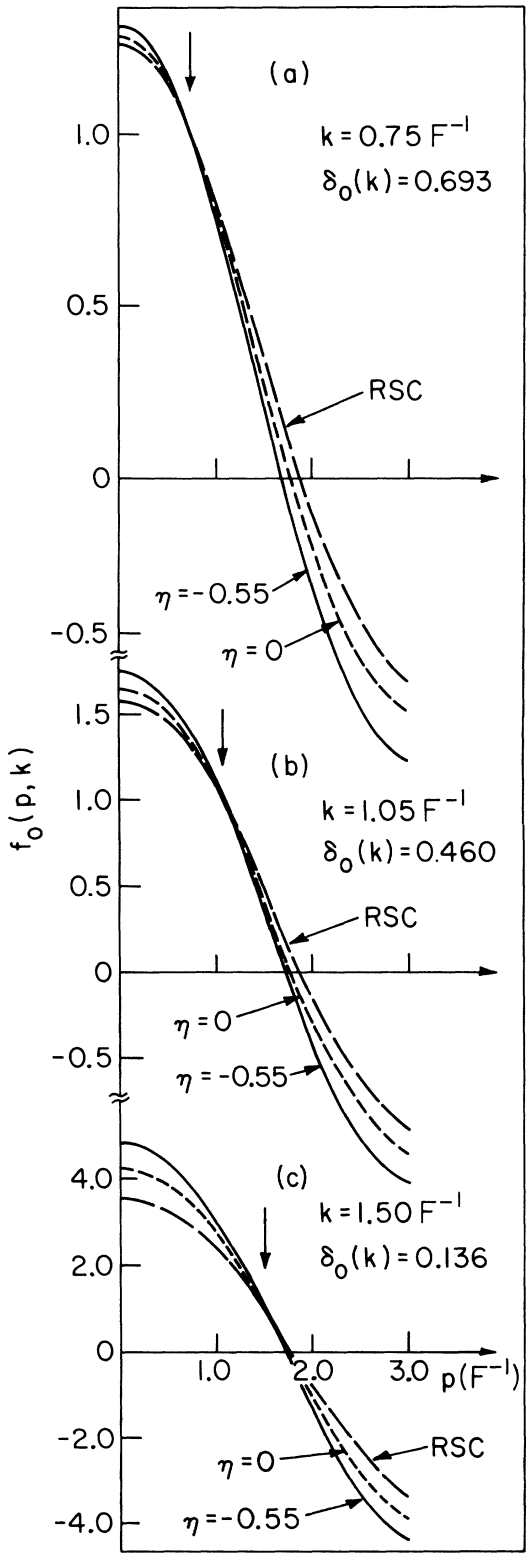

FIG. 6. (a)–(c) Half-shell factors for $\eta=0$ (---), and $\eta=-0.55$ (——) compared with that of the RSC (– – –). Note differences in scale.

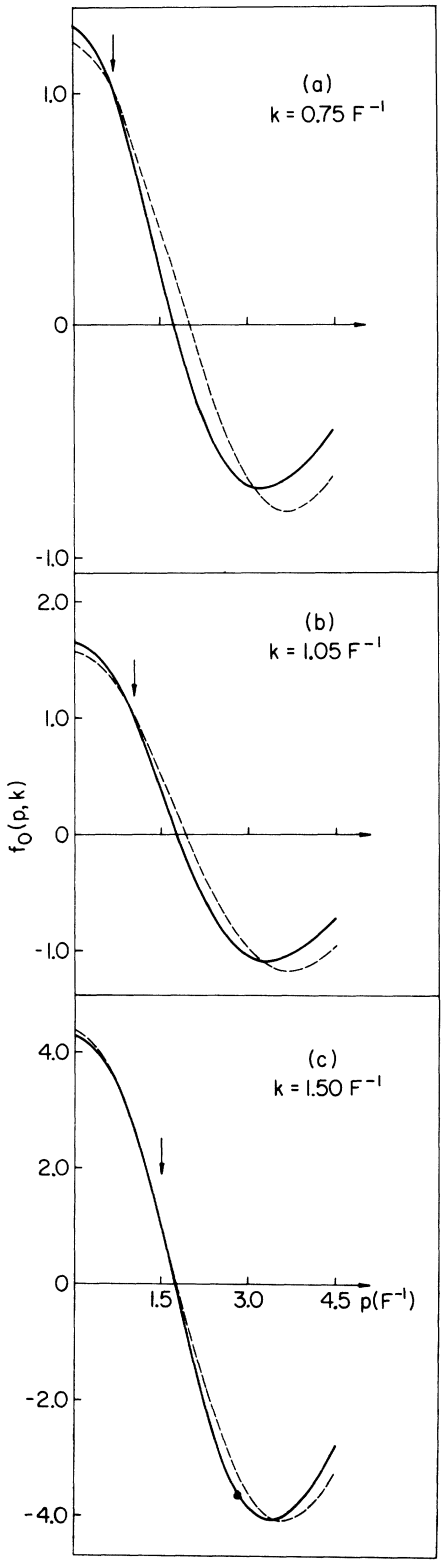

FIG. 7. (a)–(c) Half-shell factors of the Manx model (---) compared with those of the $\eta=0$ model including exterior contributions (——). Note differences in scale.

VI. SUMMARY AND OUTLOOK

For the $^1S_0$ partial wave, we have given a method for extending an empirical on-shell T matrix half off the energy shell which is both more direct and computationally much simpler than the usual procedure based on potential models. It enabled us to make optimal use of the available information about the two-nucleon wave function in coordinate space. Supplementing this with a few plausible physical assumptions, we constructed models of the half-shell T and from them estimated the range of variation of the half-shell T. Although our assumptions seem quite reasonable, they are by no means necessary, and the effects of lifting one or more of them may be easily investigated using our approach.

At the energies considered, we found that once the phase-shift is fixed, the half-shell T matrix is rather insensitive to the local exterior interaction. In fact, the main characteristics of the half-shell T generated from a typical potential model are reproduced by an extremely simple "Manx" model. This model idealizes the two-nucleon interaction by assuming that: (1) It is zero for $r > \lambda\!\!\!^{-}_{\pi}$; (2) it suppresses the wave function at short distances; and (3) it gives rise to a wave function which is as smooth as possible, subject to the constraints imposed by assumptions (1) and (2).

The extension of our method to higher uncoupled partial waves with no bound states is straightforward, as indicated in Appendix A. Coupled partial waves introduce no complications of principle. The generalization needed to account for the bound state in the $^3S_1 + {}^3D_1$ wave can probably be achieved by assuming a wave function for the bound state (which should be fitted to the deuteron data) and then requiring that the model scattering wave functions be orthogonal to it.

We have not required that our model wave functions form a complete orthonormal set. Given that our underlying assumptions are valid only at nonrelativistic energies, completeness seems a moot point; the two-particle scattering wave functions of the real two-nucleon system do not form a complete set when relativity and additional degrees of freedom are taken into account. Even within the framework of nonrelativistic potential theory, there are potentials, notably hard-core potentials, whose wave functions are obviously incomplete. Yet their predictions for the half-shell T are strikingly close to those of potentials for which completeness holds.[21] Our model wave functions are already quite circumscribed and we do not expect any qualitative changes in our results if they are orthogonalized. If one wishes to investigate the additional constraints imposed by completeness and orthogonality, it is a straightforward matter to do so by using our results to make models of the symmetric part of the half-shell T, which is the basic input of the BGMS procedure, and then calculating the antisymmetric part according to the BGMS prescription. Since new constraints are introduced, the resulting half-shell T's should show an even smaller range of variation than those we obtained.

There remains the problem of going from our results to models of fully off-shell T's. Because the Low equation relies on the completeness and orthogonality of the scattering wave functions, the usual method[3] for going from the half-shell T to the fully off-shell T does not apply here. This difficulty is also encountered with hard-core potentials. However, there is a representation which gives the fully off-shell T in terms of the half-shell T and the spherical Bessel transform of an off-shell difference function.[20] It is this relation that is used for hard-core potentials.[27] Using the ideas of this paper, it should be possible to construct models of the off-shell difference function.

Perhaps the most important advantage of our method is that, in contrast to the time-honored potential-model approach, the criteria for its validity are expressed in clear physical terms.

ACKNOWLEDGMENTS

We should like to thank G. Lerner and M. Haftel for making available to us the results of their calculations with the RSC potential. One of us (H.S.P.) thanks M. Baranger for a stimulating discussion, and also acknowledges a great debt to Karen Picker for making his part of this work possible.

APPENDIX A. FORMULAS FOR $l > 0$

The equations given below generalize to uncoupled partial waves of orbital angular momentum $l > 0$ the correspondingly numbered equations for $l = 0$ appearing in the body of the paper, i.e., Eq. (1A) is the transcription of Eq. (1), etc. As before, it is assumed that there are no bound states:

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + k^2 - \frac{l(l+1)}{r^2}\right]\psi_l^{(+)}(k,r) = \int_0^\infty U_l(r,r')\psi_l^{(+)}(k,r')r'^2dr', \tag{1A}$$

$$\psi_l^{(+)}(k,r) \underset{r\to\infty}{\sim} (kr)^{-1} e^{i\delta_l(k)} \sin[kr + \delta_l(k) - \tfrac{1}{2}l\pi], \tag{1'A}$$

$$t_l(p,k;k^2) = \int_0^\infty dr\, r^2 j_l(pr) \int_0^\infty U_l(r,r')\psi_l^{(+)}(k,r') r'^2 dr', \tag{2A}$$

$$t_l(k) = -k^{-1} e^{i\delta_l(k)} \sin\delta_l(k), \tag{2''A}$$

$$t_l(p,k;k^2) = \int_0^\infty dr\, r^2 j_l(pr) \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + k^2 - \frac{l(l+1)}{r^2}\right]\psi_l^{(+)}(k,r), \tag{3A}$$

$$v_l(kr) = e^{i\delta_l(k)}[\cos\delta_l(k) j_l(kr) + \sin\delta_l(k) n_l(kr)], \tag{4A}$$

$$\Delta_l(k,r) = kr e^{-i\delta_l(k)}[\psi_l^{(+)}(k,r) - v_l(kr)], \tag{6A}$$

$$u_l(pr) = pr\, j_l(pr),$$

$$t_l(p,k;k^2) = t_l(k)\left(\frac{p}{k}\right)^l + (pk)^{-1}(k^2 - p^2) e^{i\delta_l(k)} \int_0^\infty dr\, u_l(pr)\Delta_l(k,r), \tag{8A}$$

$$\Delta_l(k,r) \xrightarrow[r\to 0]{} -\sin\delta_l(k)\, kr n_l(kr) \xrightarrow[r\to 0]{} -(2l-1)!!(kr)^{-l}\sin\delta_l(k). \tag{9A}$$

Note that, since $u_l(pr) \underset{r\to 0}{\sim} (pr)^{l+1}$, the integrand on the right-hand side of Eq. (8A) is regular despite the singularity of $\Delta_l(k,r)$ at $r=0$ displayed in (9A).

$$\left.\begin{aligned} y_l(kr) &= kr n_l(kr) \\ w_l(k,r) &= e^{-i\delta_l(k)} kr \psi_l^{(+)}(k,r) \end{aligned}\right\}, \tag{14A}$$

$$w_l(k,r) = \Delta_l(k,r) + \cos\delta_l(k) u_l(kr) + \sin\delta_l(k) y_l(kr), \tag{16A}$$

$$g_l(k;r,r') = y_l(kr)u_l(kr') - u_l(kr)y_l(kr'), \tag{17''A}$$

$$U_l(r,r') = N_l(r,r') + \delta(r-r')L_l(r)/rr', \tag{18A}$$

$$|L_l(r)| \gg \int_0^\infty dr' r'^2 |N_l(r,r')|, \quad r > R_l, \tag{18'A}$$

$$\left.\begin{aligned} \Delta_l(k,r) &= \Delta_l^{(0)}(k,r) + \frac{1}{k}\int_0^\infty dr' g_l(k;r,r')L_l(r')\Delta_l(k,r') \\ \Delta_l^{(0)}(k,r) &= \frac{1}{k}\int_r^\infty dr' g_l(k;r,r')L_l(r')[\cos\delta_l(k)u_l(kr') + \sin\delta_l(k)y_l(kr')] \end{aligned}\right\} \quad r > R_l. \tag{19A}$$

Because $\Delta_l(k,r)$ is singular at the origin, it cannot be represented by a polynomial in the interior region. Instead, one can find a polynomial representation of the wave function $w_l(k,r)$ and obtain $\Delta_l(k,r)$ from it by means of Eq. (16A). One writes

$$w_l(k,r) = P_l(k,r), \quad 0 \le r \le R_l,$$

and finds

$$P_l(k,r) = P_{5l}(k,r) + \eta r^3 (r-R)^3$$

from the conditions

$$\left.\begin{aligned} P_l(k,R_l) &= w_l^{\text{ext}}(k,R_l) \\ P_l'(k,R_l) &= w_l^{\text{ext}\,\prime}(k,R_l) \\ P_l''(k,R_l) &= w_l^{\text{ext}\,\prime\prime}(k,R_l) \end{aligned}\right\} \text{ obtained from the solution of Eq. (19A)} \tag{21aA}$$

and

$$\left.\begin{aligned} P_l(k,0) &= 0 \\ P_l'(k,0) &= w_l'(k,0) \\ P_l''(k,0) &= w_l''(k,0) \end{aligned}\right\} \text{ free parameters}. \tag{21bA}$$

$w_l^{ext}(k,r)$ is obtained by adding $\cos\delta_l(k)u_l(kr)+\sin\delta_l(k)y_l(kr)$ to the solution of Eq. (19A). The rest of the analysis proceeds as before.

APPENDIX B. CONVERGENCE OF THE ITERATION OF EQ. (19) FOR THE KALLIO-KOLLTVEIT POTENTIAL

The $^1S_0$ potential of Kallio and Kolltveit[24] in MeV is

$$V^{KK}(r)=\begin{cases}-330.8e^{-\gamma(r-c)} & r\geq c,\\ +\infty & r<c,\end{cases} \tag{1B}$$

where

$$c=0.4\ \text{F},\quad \gamma=2.4021\ \text{F}^{-1}. \tag{1'B}$$

The exterior potential corresponding to Eq. (1B) is

$$L^{KK}(r)=-ge^{-\gamma r},\quad r>\lambda\!\!\!^{-}_{\pi}, \tag{2B}$$

with

$$g=330.8e^{\gamma c}\ \text{MeV}. \tag{2'B}$$

As shown in Fig. 8, in the region $\lambda\!\!\!^{-}_{\pi}<r<2\lambda\!\!\!^{-}_{\pi}$, the exterior potential given by Eq. (2B) resembles quite closely the RSC exterior potential used in our calculations. For $r>2\lambda\!\!\!^{-}_{\pi}$, both potentials are small and, as may be seen from Fig. 3, the difference function is negligible. It therefore seems reasonable to expect that a study of the convergence of the iterative solution of Eq. (19) using (2B) as the exterior potential should indicate the rate of convergence which obtains when the RSC exterior potential is used.

Let $f(k,r)$ denote the solution of the Schrödinger equation which goes as

$$f(k,r)\underset{r\to\infty}{\sim}e^{-ikr}. \tag{3B}$$

For the potential (2B) it is easily shown[28] that

$$f(k,r)=e^{-ikr}\left[1-\frac{ge^{-\gamma r}}{\gamma(\gamma+2ik)}+\frac{g^2e^{-2\gamma r}}{\gamma(2\gamma)(\gamma+2ik)(2\gamma+2ik)}-\cdots\right] \tag{4aB}$$

$$=\left(\frac{\gamma^2}{g}\right)^{ik/\gamma}\Gamma\left(1+\frac{2ik}{\gamma}\right)J_{2ik/\gamma}\left(\frac{2\sqrt{g}}{\gamma}e^{-\frac{1}{2}\gamma r}\right). \tag{4bB}$$

Here $\Gamma(z)$ is the Γ function and J_ν is a Bessel function of order ν. The regular solution $w_0(k,r)$ which approaches $\sin[kr+\delta_0(k)]$ for large r may be written in terms of $f(k,r)$ as

$$w_0(k,r)=(2i)^{-1}[e^{i\delta_0(k)}f(-k,r)-e^{-i\delta_0(k)}f(k,r)] \tag{5aB}$$

or

$$w_0(k,r)=\text{Im}[e^{i\delta_0(k)}f(-k,r)]. \tag{5bB}$$

Then

$$\Delta_0(k,r)=w_0(k,r)-\sin[kr+\delta_0(k)]$$

is

$$\Delta_0(k,r)=\sum_{n=0}^{\infty}\Delta_0^{(n)}(k,r) \tag{6aB}$$

with

$$\Delta_0^{(0)}(k,r)=\text{Im}\left[e^{i\delta_0(k)}e^{ikr}\frac{-ge^{-\gamma r}}{\gamma(\gamma-2ik)}\right], \tag{6bB}$$

$$\Delta_0^{(1)}(k,r)=\mathrm{Im}\left[e^{i\delta_0(k)}e^{ikr}\frac{g^2e^{-2\gamma r}}{\gamma(2\gamma)(\gamma-2ik)(2\gamma-2ik)}\right], \tag{6cB}$$

$$\vdots$$

$$\Delta_0^{(n)}(k,r)=\mathrm{Im}\left[e^{i\delta_0(k)}e^{ikr}\frac{(-g)^{n+1}e^{-(n+1)\gamma r}}{\gamma(2\gamma)\cdots[(n+1)\gamma](\gamma-2ik)(2\gamma-2ik)\cdots[(n+1)\gamma-2ik]}\right]. \tag{6dB}$$

More explicitly, we have

$$\Delta_0^{(0)}(k,r)=-\frac{ge^{-\gamma r}}{\gamma(\gamma^2+4k^2)}\{2k\cos[kr+\delta_0(k)]+\gamma\sin[kr+\delta_0(k)]\} \tag{7aB}$$

and

$$\Delta_0^{(1)}(k,r)=\frac{g^2e^{-2\gamma r}}{4\gamma^2(\gamma^2+4k^2)(\gamma^2+k^2)}\{3k\gamma\cos[kr+\delta_0(k)]+(\gamma^2-2k^2)\sin[kr+\delta_0(k)]\}. \tag{7bB}$$

Setting $r=R=1.43$ F, we find the following values:

k (F^{-1})	$\Delta_0^{(0)}(k,R)$	$\Delta_0^{(1)}(k,R)$
0.75	−0.079	+0.001
1.05	−0.045	+0.0002
1.50	−0.001	+0.000 01

It is clear from Eq. (6dB) that $\Delta_0^{(2)}(k,R)$ and higher terms will be extremely small and that considerable cancellation among these contributions will occur. Since the above values show that $\Delta_0^{(1)}(k,R)$ already is negligible, it is apparent that $\Delta_0(k,R)\approx\Delta_0^{(0)}(k,R)$ is an excellent approximation for the Kallio-Kolltveit exterior potential.

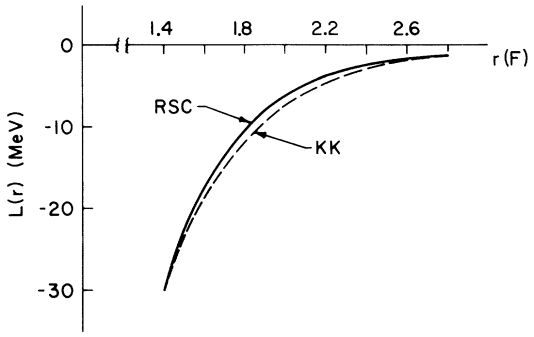

FIG. 8. Comparison of the Kallio-Kolltveit (KK) exterior potential (---) with the RSC exterior potential (——).

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Modified Faddeev Equations for Three Bound α Particles in ^{12}C

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The bound state of three α particles is considered. For the α-cluster model, we consider the Coulomb Green's function in the nucleus ^{12}C. The calculated binding energy of the ^{12}C nucleus is consistent with previous calculations.

I. INTRODUCTION

The α-cluster model consists of taking the nucleus to be composed of α clusters as elementary constituents, in view of the strong binding of an α cluster. Therefore, we consider the ^{12}C nucleus, on the basis of the α-cluster model, as three bound α particles. The aim of this work is to study the three-body system when all of these particles are charged. We consider two approximations for the Coulomb Green's functions. The first, suggested by Schulman,[1] yields an approximate form of the Coulomb Green's function in momentum space. The other is the improved version of the Schulman approximation based on the Yamaguchi[2] potential. This will be discussed in Sec. II.

In Sec. III we introduced the Faddeev[3] equations which are modified to include the Coulomb potential between the three α particles. Neglecting the internal construction of the α cluster, we take the two-α-cluster short-range interaction to be of the separable form

$$\langle q'|V|q\rangle = \lambda v(q')v(q),$$

with $v(q) = (\beta^2 + q^2)^{-1}$. Thus, the aim of this paper is to calculate the binding energy of ^{12}C in the three-α-cluster model in the framework of the Faddeev formalism.

The results for the different approximations introduced are compared with previous results in Sec. IV.

II. COULOMB GREEN'S FUNCTION

Taking the momenta of the three α particles in the c.m. system to be $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$, the relative momenta of the pairs (2, 3), (3, 1), and (1, 2) will be given by

$$\vec{q}_1 \equiv \vec{q}_{23} = \tfrac{1}{2}(\vec{p}_3 - \vec{p}_2), \quad \text{etc.} \tag{1}$$

The total Hamiltonian H is given by

$$H = H_0 + U + V, \tag{2}$$

where U is the sum of the two-body Coulomb potential; V is the sum of the short-range nuclear interactions; and the kinetic energy, in the c.m. system, is given by

$$H_0 = p_i^2/2M + q_i^2/2m, \tag{3}$$