Boson Expansions for Fermion Pair Operators: The Single j Level*

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(Received 2 June 1971)

A new version of a method due originally to Marumori for the representation of fermion pair operators as expansions in boson variables is developed and applied to nucleons confined to a single j shell. In the present formulation, a seniority basis of good angular momentum states is imaged in a subspace of the boson space – the physical boson space – characterized by the same quantum numbers. This transformation has the following properties, only the first of which is shared with previous work: (i) Pauli-principle restrictions are satisfied. (ii) The fermion pair operators approach the leading terms of their expansions as $j\rightarrow\infty$. (iii) For fixed j one can identify maximal subspaces of the physical boson space in which the boson expansions converge and which define the maximum extent to which the model physical system can exhibit vibrational behavior. An extension appropriate to systems with an odd number of nucleons is also described. By consideration of the direct product, the method described here should be useful for shell-model calculations of a restricted set of states for large particle number.

I. INTRODUCTION AND SUMMARY

The boson expansion method was introduced into nuclear physics by Beliaev and Zelevinsky (BZ)[1] as a means of studying the anharmonic corrections to the vibrational spectra of spherical even nuclei. These authors obtained "convergent" expansions in the sense that each term in the series is $0(\Omega^{-1})$ times the preceding one, where Ω is a "large" number associated with the dimension of the fermion space [$\Omega=\frac{1}{2}(2j+1)$ for a single j shell]. In the procedure described, the expansions are required to satisfy only the Lie algebra generated by fermion or quasifermion pair and multipole operators. However, the satisfaction of the Pauli principle is not guaranteed by the algebra alone, and it turns out that the original BZ expansion did violate the Pauli conditions. To solve this difficulty, Marumori, Yamamura, and Tokunaga[2] then proposed a method which utilized a transformation between the fermion space and a subspace of the boson space (called the physical boson space). Yet Marumori's expansion is not "convergent" in the sense described above.

Although the original boson-expansion method was intended for the study of vibrational nuclei, it becomes clear from Marumori's point of view that the boson expansion is in fact a transformation between the fermion space and the physical boson space. If there exists a "convergent" expansion for the shell-model Hamiltonian, then it can be truncated and used as an approximation for the exact shell-model Hamiltonian. The boson-expansion method may thus provide a way to simplify and make possible approximate shell-model calculations for nuclei relatively far removed from closed shells.

Our purpose is to investigate the possibility of constructing a convergent boson expansion using a modification of Marumori's method. For an account of the relation between the theory of vibrational nuclei and the boson-expansion method, one can consult Ref. 2. Here we shall only give a brief outline of the BZ and Marumori methods, and then suggest a modification of Marumori's method which gives convergent boson expansions in certain subspaces of the physical boson space, but at the same time shows that because of the effect of the Pauli principle it is impossible in general to have convergent boson expansions within the entire physical boson space.

We shall first develop our method for an even system in a single j shell. Next this method is extended to include odd systems in a single shell. There we follow a procedure similar to that used by Yamamura[3] and Simard[4-6] in extending the original Marumori expansion to odd nuclei.

Our method can also be extended to the problem of several j shells by considering the multishell space as a direct product of single shells. We have chosen to omit any presentation of this extension not only to keep this paper within bounds, but also because it fails to do as much for the multishell problem as we feel the developments of this paper do to clarify the significance of boson expansions for the single-j case. For this case, given the value of j and relative to a given Hamiltonian, we can specify the maximum extent to which the system can exhibit vibrational degrees of freedom. The same specification suggests that for an appropriate subset of levels, we can free ourselves from the usual practical restrictions of the shell model which confine us to problems with small particle number. As anticipated, the results can be interesting and useful if j is large enough.

Now our experience with real physical systems suggests that a system with several closely spaced levels, with moderate average value of j, should be capable of "vibrational" behavior with an effective j determined by the total number of available orbits. The direct-product method does not put this possibility clearly into evidence from the outset, though we believe it would do so in practice.

These remarks suggest the need for further theoretical development along the lines anticipated in this work. One obvious suggestion which will be exploited is to utilize from the start two-particle eigenoperators of a suitable model Hamiltonian (such as the surface δ interaction). We also plan to present in a separate brief account some numerical illustrations of the general points developed in this paper.

II. GENERAL CONSIDERATIONS CONCERNING THE BOSON-EXPANSION METHOD

We consider specifically the case of an even number of nucleons of one charge outside a closed core. The single-particle states $|\alpha\rangle$ are characterized by the quantum numbers $(nljm)$ and can be written in the second-quantized form as

$$|\alpha\rangle \equiv a_\alpha^\dagger|0\rangle\,, \tag{2.1}$$

where $|0\rangle$ is the inert-core state. We also use a Latin subscript a to denote all the quantum numbers in α except m. Next, we define the pair and multipole operators as follows:

$$A_M^{(J)\dagger}(ab) \equiv \frac{1}{\sqrt{2}} \sum_{m_a(m_b)} \begin{bmatrix} j_a & j_b & J \\ m_a & m_b & M \end{bmatrix} a_\alpha^\dagger a_\beta^\dagger\,, \tag{2.2a}$$

$$A_M^{(J)}(ab) \equiv [A_M^{(J)\dagger}(ab)]^\dagger\,, \tag{2.2b}$$

and

$$B_M^{(J)} \equiv \frac{1}{(\hat{J})^{1/2}} \sum_{m_a(m_b)} \begin{bmatrix} j_a & j_b & J \\ m_a & -m_b & M \end{bmatrix} s_\beta a_\alpha^\dagger a_\beta \tag{2.2c}$$

with

$$s_\beta \equiv (-)^{j_b - m_b} \tag{2.3a}$$

and

$$s_{-\beta} \equiv (-)^{j_b + m_b}\,, \tag{2.3b}$$

where $\hat{J} \equiv 2J+1$, [ ] is a Clebsch-Gordan coefficient, and the quantum numbers within the parentheses are not summed over. From the above definitions we have

$$A_M^{(J)\dagger}(ab) = -\theta(abJ) A_M^{(J)\dagger}(ba)\,, \tag{2.4}$$

and

$$B_M^{(J)\dagger}(ab) = -\theta(abJ) s_{JM} B_{-M}^{(J)}(ba)\,, \tag{2.5}$$

where

$$\theta(abJ) \equiv (-)^{j_a + j_b - J}\,. \tag{2.6}$$

These operators form an algebra under the commutation relations

$$[A_1^\dagger, A_2^\dagger] = 0\,, \tag{2.7a}$$

$$[A_1, A_2^\dagger] = \delta_{12}^{(+)} - 2\sum_3 \left\{ \frac{1+\hat{p}_1}{2} \frac{1+\hat{p}_2}{2} Y(123) \right\} (\hat{J}_3)^{1/2} B_3^\dagger\,, \tag{2.7b}$$

$$[B_1^\dagger, A_2^\dagger] = \frac{2}{(\hat{J}_1)^{1/2}} \sum_3 \left\{ \frac{1+\hat{p}_2}{2} \frac{1+\hat{p}_3}{2} Y(231) \right\} A_3^\dagger\,, \tag{2.7c}$$

and

$$[B_1^\dagger, B_2] = \frac{1}{(\hat{J}_1 \hat{J}_2)^{1/2}} \sum_3 \{(1 - \hat{p}_1\hat{p}_2\hat{p}_3) Y(132)\} (\hat{J}_3)^{1/2} B_3^\dagger\,. \tag{2.7d}$$

In the above equations we introduced the unified subscripts $1 \equiv (a_1 b_1, J_1 M_1)$, etc., and $\hat{p}_1$ is a permutation operator acting on this subscript. For an arbitrary function $f(1)$ we have

$$\hat{p}_1 f(1) \equiv \hat{p}_1 f(a_1 b_1, J_1 M_1) = -\theta(a_1 b_1 J_1) f(b_1 a_1, J_1 M_1)\,. \tag{2.8}$$

The functions $\delta_{12}^{(+)}$ and $Y(123)$ are then defined as

$$\delta_{12}^{(+)} \equiv \frac{1+\hat{p}_1}{2} \delta_{12} = \tfrac{1}{2} \delta_{J_1 J_2} \delta_{M_1 M_2} \{ \delta_{a_1 a_2} \delta_{b_1 b_2} - \theta(a_1 b_1 J_1) \delta_{a_1 b_2} \delta_{a_2 b_1} \}\,, \tag{2.9}$$

and

$$Y(123) \equiv \delta_{b_1 b_2} \delta_{a_1 a_3} \delta_{a_2 b_3} (-)^{j_{b_1} + j_{a_3} + J_2 + J_3} (\hat{J}_2 \hat{J}_3)^{1/2} \times \begin{bmatrix} J_2 & J_3 & J_1 \\ M_2 & M_3 & M_1 \end{bmatrix} \begin{Bmatrix} J_2 & J_3 & J_1 \\ j_{b_1} & j_{a_3} & j_{a_2} \end{Bmatrix}\,, \tag{2.10}$$

where { } is a 6-j symbol.

We now define 2Ω as the average shell size of the shell-model space, i.e.,

$$2\Omega \equiv \frac{1}{n} \sum_{i=1}^{n} (2j_i + 1)\,, \tag{2.11}$$

where n is the total number of shells considered in the problem. As $\Omega\to\infty$, the asymptotic value of the function $Y(123)$ is given by[1]

$$Y(123)\sim\delta_{b_1b_2}\delta_{a_1a_3}\delta_{a_2b_3}(\hat{j})^{-1/2}(\hat{J}_2\hat{J}_3/\hat{J}_1)^{1/2}\times\begin{bmatrix}J_2 & J_3 & J_1\\ M_2 & M_3 & M_1\end{bmatrix}\begin{bmatrix}J_2 & J_3 & J_1\\ j_{a_2}-j_{b_1} & j_{a_3}-j_{a_2} & j_{a_3}-j_{b_1}\end{bmatrix}\tag{2.12}$$

with

$$\hat{j}\equiv[\hat{j}_{a_3}\hat{j}_{a_2}\hat{j}_{b_1}]^{1/3}\sim(2\Omega).\tag{2.13}$$

Also the asymptotic value of the vacuum expectation value of $B_M^{(J)}(ab)$ becomes, with N_a the occupation of level a and N the total number of particles,

$$\begin{aligned}\langle B_M^{(J)}(ab)\rangle&=\frac{1}{(\hat{J})^{1/2}}\sum_{m_a(m_b)}\begin{bmatrix}j_a & j_b & J\\ m_a & -m_b & M\end{bmatrix}s_\beta\langle a_\alpha^\dagger a_\beta\rangle\\&\cong\frac{1}{(\hat{J})^{1/2}}\sum_{m_a(m_b)}\begin{bmatrix}j_a & j_b & J\\ m_a & -m_b & M\end{bmatrix}s_\beta\delta_{\alpha\beta}N_a\\&\cong\frac{1}{(\hat{J})^{1/2}}\sum_{m_a(m_b)}\begin{bmatrix}j_a & j_b & J\\ m_a & -m_b & M\end{bmatrix}s_\beta\delta_{\alpha\beta}\left(\frac{N}{2\Omega}\right)\\&\sim\left(\frac{N}{2\Omega}\right)\delta_{ab}\delta_{J0}\delta_{M0}.\end{aligned}\tag{2.14}$$

From Eqs. (2.12) and (2.14), we shall expect that for the low-lying states $Y(123)\to 0$ and $B^{(J)}\to 0$, as $\Omega\to\infty$ and $N\ll 2\Omega$. Consequently, the commutation relation (2.7b) becomes

$$[A_1,A_2^\dagger]\cong\delta_{12}^{(+)},\tag{2.15}$$

which means that the pair operator $A^\dagger$ can be approximated by a boson operator $\mathfrak{a}^\dagger$ defined as

$$[\mathfrak{a}_1^\dagger,\mathfrak{a}_2^\dagger]=0,\tag{2.16a}$$

and

$$[\mathfrak{a}_1,\mathfrak{a}_2^\dagger]=\delta_{12}^{(+)}.\tag{2.16b}$$

This argument, at best heuristic, contains no hint of how any suspected vibrational structure dissipates with excitation of the system.

A. Boson-Expansion Method of Beliaev and Zelevinsky

In this method one considers the correction to Eq. (2.15) and assumes the following expansions for the fermion operators $A_1^\dagger$ and B_1:

$$A_1^\dagger\equiv\mathfrak{a}_1^\dagger+\sum_{234}g_{234}^{(1)}\mathfrak{a}_2^\dagger\mathfrak{a}_3^\dagger\mathfrak{a}_4+0(5),\tag{2.17}$$

and

$$B_1=\sum_{23}d_{23}^{(1)}\mathfrak{a}_2^\dagger\mathfrak{a}_3+0(4),\tag{2.18}$$

where the expansions are normal-ordered. For example, the fifth-order term in Eq. (2.17) will be

$$f_{23456}^{(1)}\mathfrak{a}_2^\dagger\mathfrak{a}_3^\dagger\mathfrak{a}_4^\dagger\mathfrak{a}_5\mathfrak{a}_6.$$

These expansions are then required to satisfy the algebra (2.17). First, from Eq. (2.7c) we have

$$d_{23}^{(1)}=\frac{2}{(\hat{J}_1)^{1/2}}Y(231).\tag{2.19}$$

Next, with the above expression and Eqs. (2.7a) and (2.7b) we obtain a set of equations for the coefficients $g_{234}^{(1)}$:

$$g_{234}^{(1)}=g_{231}^{(4)},\tag{2.20a}$$

and

$$2(1+\hat{p}_2)g_{234}^{(1)}+2(1+\hat{p}_1)g_{143}^{(2)}+\sum_{56}g_{564}^{(1)}(1+\hat{p}_5)(1+\hat{p}_6)g_{563}^{(2)}=-2\sum_5(1+p_1)(1+p_2)Y(125)Y(345),\tag{2.20b}$$

where we have symmetrized $g_{234}^{(1)}$ so that

$$g_{234}^{(1)}=g_{324}^{(1)}.\tag{2.20c}$$

Assuming that $g_{234}^{(1)}$ is of the order $0(Y^2)\sim 0(\Omega^{-1})$, one can discard the quadratic term in Eq. (2.20b) and get a solution[7]

$$g_{234}^{(1)}=-\frac{1+p_1}{2}\sum_5 Y(125)Y(345).\tag{2.21}$$

Using the above formula and Eq. (2.7c) again, we can obtain the fourth-order coefficients of B_1, which in turn give the fifth-order ones in $A_1^\dagger$. In this way one can calculate the coefficients one by one to all orders.

The BZ method described above is unsatisfactory because the Lie algebra of pair and multipole operators does not include every possible constraint between these operators due to the Pauli principle. Therefore it is not clear that any expansion satisfying the algebra alone will not violate the Pauli principle in some respect. Recently, Marshalek[8] has shown for a problem other than the one considered in this paper, how to obtain expansions which satisfy all Pauli restrictions within a physical boson space which is defined as an image of the fermion space, provided that one makes some further assumptions about the boson vacuum state. (The present authors[9] have applied the method of this paper to the same problem.) It would appear, therefore, that correct completions of the BZ method can be carried out.

In order to do this, one needs, however, to consider more equations than just the Lie algebra. One must either impose conditions on the boson space, as shown by Marshalek,[8] or include every possible Pauli constraint between the pair and multipole operators, or equivalently satisfy all the

Casimir operator relations. [Note that in obtaining Eq. (2.21) we arbitrarily discarded the quadratic term in Eq. (2.20b). If one were to include the required additional equations for these coefficients, then this term might not be small and could not be neglected.] There would thus appear to be several possible ways to generate correct BZ-type boson expansions. We, however, have chosen to look into the Marumori-type[2] expansion because of the straightforward way in which the Pauli principle is satisfied in the construction of the boson expansions.

B. Marumori's Method of Boson Expansion and Its Properties

In the space of an even number N of nucleons, a complete orthonormal basis can be designated as $\{|Np\rangle_F\}$, where p labels the fermion states, $p=1,2,\ldots,p_{\max}$, with $p_{\max}$ being a finite integer. On the other hand, in the boson space for n bosons a complete orthonormal basis will be $\{|nk)_B\}$ where k labels the boson states.

In the method of Marumori, a subset $\{|nk)_B\}$ of the entire n boson space $\{|nk)_B\}$ is chosen as the image of the fermion basis $\{|2n,p\rangle_F\}$. Once this choice has been made, a transformation V can be introduced:

$$V\equiv\sum_{Np}|\tfrac{1}{2}N,p)_B\,{}_F\langle Np|\,. \tag{2.22}$$

We note that V is not unitary. In fact, we have

$$V^\dagger V=\sum_{Np}|Np\rangle_F\,{}_F\langle Np|=1\,, \tag{2.23}$$

but

$$VV^\dagger=\sum_{Np}|\tfrac{1}{2}N,p)_B\,{}_B(\tfrac{1}{2}N,p|\,, \tag{2.24}$$

where the right-hand side is not a sum over a complete set of states and consequently is not the identity operator. The operator V transforms the states in the following way:

$$V|Np\rangle_F=|\tfrac{1}{2}N,p)_B\,, \tag{2.25a}$$

and

$$V^\dagger|\tfrac{1}{2}N,p)_B=|Np\rangle_F\,, \tag{2.25b}$$

but

$$V^\dagger|\tfrac{1}{2}N,q)_B=0 \tag{2.25c}$$

for any state $|\tfrac{1}{2}N,q)_B$ not in the physical boson space. It is therefore clear that V^{-1} does not exist.

With the transformation V, one can obtain the image of any fermion operator T in the boson space

$$\begin{aligned}T_B&\equiv VTV^\dagger\\&=\sum_{NN'pp'}|\tfrac{1}{2}N,p)_B\,{}_F\langle Np|T|N'p'\rangle_F\,{}_B(\tfrac{1}{2}N',p'|\\&\equiv\sum_{NN'pp'}T_{Np,N'p'}|\tfrac{1}{2}N,p)_{BB}(\tfrac{1}{2}N',p'|\,.\end{aligned} \tag{2.26}$$

If we write the boson state as

$$|\tfrac{1}{2}N,k)_B\equiv\tilde{O}^\dagger_{\frac{1}{2}N,k}|0)_B\,, \tag{2.27}$$

where $|0)_B$ is the boson vacuum state, and $\tilde{O}^\dagger_{\frac{1}{2}N,k}$ is a polynomial of boson operators, then Eq. (2.26) becomes

$$T_B=\sum_{NN'pp'}T_{Np,N'p'}\tilde{O}^\dagger_{\frac{1}{2}N,p}|0)_B\,{}_B(0|\tilde{O}_{\frac{1}{2}N',p'}\,. \tag{2.28}$$

By the iteration method, the completeness relation

$$\begin{aligned}1&=\sum_{Nk}|\tfrac{1}{2}N,k)_{BB}(\tfrac{1}{2}N,k|\\&=\sum_{Nk}\tilde{O}^\dagger_{\frac{1}{2}N,k}|0)_B\,{}_B(0|\tilde{O}_{\frac{1}{2}N,k}\end{aligned} \tag{2.29}$$

can be solved for the projection operator $|0)_B\,{}_B(0|$ in terms of boson operators. Substituting back into Eq. (2.28), one obtains the boson expansion for any fermion operator T. Thus, specifying the set $|np)_B$ determines the boson expansions uniquely in a manner which guarantees satisfaction of the Pauli principle. Also the expansion coefficients are matrix elements of fermion operators. For illustration, we shall consider only the lowest anharmonic terms; consequently only matrix elements between states containing up to four fermions will have to be evaluated.

The transformed operator has the property that its matrix elements in the unphysical boson space always vanish. This is easily shown by using Eqs. (2.25),

$${}_B(\tfrac{1}{2}N,p|T_B|\tfrac{1}{2}N',q')_B={}_B(\tfrac{1}{2}N,p|VTV^\dagger|\tfrac{1}{2}N',q')_B=0\,. \tag{2.30}$$

As a consequence, if one restricts himself to the physical boson space, he can add to or subtract from the expansion terms with vanishing matrix elements within the physical boson space. Also, the operator V can be regarded as a unitary transformation, even though strictly speaking it is not unitary and has no inverse. It is to be emphasized here that these aforementioned properties of the Marumori expansion do not depend on the choice of the physical boson space. This is the freedom which we will utilize later to obtain a boson expansion different from Marumori's original one.

In their original paper,[2] Marumori, Yamamura, and Tokunaga chose the fermion states

$$|2n,p\rangle_F=a^\dagger_{\alpha_1}a^\dagger_{\beta_1}a^\dagger_{\alpha_2}a^\dagger_{\beta_2}\cdots a^\dagger_{\alpha_n}a^\dagger_{\beta_n}|0\rangle_F\,, \tag{2.31}$$

with $|0\rangle_F$ the fermion vacuum state, and the corresponding physical boson states as

$$|np)_B = \frac{1}{(2n-1)!!} \sum_p{}' (-)^P P(\mathcal{C}^\dagger_{\alpha_1\beta_1}\mathcal{C}^\dagger_{\alpha_2\beta_2}\cdots\mathcal{C}^\dagger_{\alpha_n\beta_n})|0)_B , \tag{2.32}$$

where $\mathcal{C}^\dagger_{\alpha\beta}$ is defined as the operator

$$\mathcal{C}^\dagger_{\alpha\beta} \equiv \sqrt{2} \sum_{m_a(m_b)} \begin{bmatrix} j_a & j_b & J \\ m_a & m_b & M \end{bmatrix} \mathcal{A}^{(J)\dagger}_M(ab) , \tag{2.33}$$

with the properties

$$[\mathcal{C}_{\alpha_1\beta_1}, \mathcal{C}^\dagger_{\alpha_2\beta_2}] = \delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2} - \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1} \tag{2.34a}$$

and

$$\mathcal{C}^\dagger_{\alpha\beta} = -\mathcal{C}^\dagger_{\beta\alpha} , \tag{2.34b}$$

while $\sum_p'$ means a restricted summation of permutations which connnects independent boson states of the form

$$\mathcal{C}^\dagger_{\alpha_1\beta_1}\mathcal{C}^\dagger_{\alpha_2\beta_2}\cdots\mathcal{C}^\dagger_{\alpha_n\beta_n}|0)_B .$$

Since the total number of such states is $(2n)!/(2^n n!) \equiv (2n-1)!!$, this gives the normalization factor in Eq. (2.32).

With the definition of the physical boson states and the identity

$$|0)_{B\,B}(0| = 1 + \sum_{l=1}^{\infty} \frac{(-)^l}{2^l l!} \sum_{(\alpha\beta)} \mathcal{C}^\dagger_{\alpha_1\beta_1}\mathcal{C}^\dagger_{\alpha_2\beta_2}\cdots\mathcal{C}^\dagger_{\alpha_l\beta_l} \times \mathcal{C}_{\alpha_l\beta_l}\cdots\mathcal{C}_{\alpha_2\beta_2}\mathcal{C}_{\alpha_1\beta_1} , \tag{2.35}$$

obtained from Eq. (2.29), the Marumori expansion of the operator $A_1^\dagger$ turns out to be[2]

$$VA_1^\dagger V^\dagger = \mathcal{A}_1^\dagger - \left(1 - \frac{1}{\sqrt{3}}\right)\mathcal{A}_1^\dagger \sum_2 \mathcal{A}_2^\dagger \mathcal{A}_2^\dagger - \frac{2}{\sqrt{3}} \times \sum_{2-5} \left\{ \frac{1+\hat{p}_1}{2} Y(125)Y(345) \right\} \mathcal{A}_2^\dagger \mathcal{A}_3^\dagger \mathcal{A}_4 + 0(5) . \tag{2.36}$$

Equation (2.36) now clearly shows the difficulties of the Marumori expansion. The first one is that the coefficient of the third-order term is of the same order as the leading term. In fact, all the higher-order terms have the same order of magnitude as the leading term. The reason for this can be traced back to Eq. (2.35), where we can see that the expansion of $|0)_{B\,B}(0|$ is not an expansion in $1/\Omega$. The second difficulty is when $\Omega\to\infty$, the expansion does not approach the boson operator limit $\mathrm{A}_1^\dagger$. However, in the next section we will show that this can be overcome with a suitable choice of the physical boson space.[10]

C. A Sufficient Condition for a Fermion Pair Operator to Have the Correct Boson Limit

The fermion states can be expressed as

$$|Np\rangle_F \equiv S_p f_p(A^\dagger)|0\rangle_F , \tag{2.37}$$

where $f_p(A^\dagger)$ is a polynomial of the fermion pair operators, S_p is the normalization constant, and $|0\rangle_F$ is the fermion vacuum state. On the other hand, the boson states have the form

$$|nk)_B = \tilde{S}_k \tilde{f}_k(\mathcal{A}^\dagger)|0)_B , \tag{2.38}$$

where $\tilde{f}_k(\mathcal{A}^\dagger)$ is now a polynomial of the boson creation operators, $\tilde{S}_k$ the normalization constant, and $|0)_B$ the boson vacuum state.

From the way the transformation V is constructed, we know that the boson expansions are uniquely determined by choosing a definite set $\{|np)_B\}$. However, any orthonormal subset $\{|np)_B\}$ of the entire boson space is allowed (usually it is chosen so that particle number and angular momentum are conserved), and different choices give different boson expansions related by a unitary transformation. A sufficient condition for selecting a basis $\{|np)_B\}$ so that the boson expansion of $A_1^\dagger$ approaches the ideal boson limit, $A_1^\dagger \to \mathcal{A}_1^\dagger$, as $\Omega\to\infty$ is to choose

$$\tilde{f}_p(\mathcal{A}^\dagger) = f_p(\mathcal{A}^\dagger) . \tag{2.39}$$

That this most natural choice is also a desirable one follows from the fact that the normalization constant S_p of the fermion states is evaluated by means of the commutation relations Eqs. (2.7). As $\Omega\to\infty$ for fixed J_1, J_2, and J_3, $Y(123)\to 0^{(1)}$ and thus $S_p \to \tilde{S}_p$ because $[A_1, A_2^\dagger] \to \delta_{12}^{(+)}$. Combined with Eq. (2.39), this shows that as $\Omega\to\infty$ V becomes the identity transformation, and all we are doing is renaming $A^\dagger$ as $\mathcal{A}^\dagger$ and S_p as $\tilde{S}_p$. Consequently, as $\Omega\to\infty$, we have

$$VA^\dagger V^\dagger \to \mathcal{A}^\dagger . \tag{2.40}$$

We note that the condition (2.39) is expressed in terms of the angular momentum coupled operators $A^\dagger$. This is essential because in this proof we used the property $[A_1, A_2^\dagger] \to \delta_{12}^{(\dagger)}$ which is only true for the coupled operators. If we try to write Eq. (2.31) in the form

$$|2n, p\rangle_F = \frac{1}{(2n-1)!!} \times \sum_P{}' (-)^P P(C^\dagger_{\alpha_1\beta_1}C^\dagger_{\alpha_2\beta_2}\cdots C^\dagger_{\alpha_n\beta_n})|0\rangle_F ,$$

with

$$C_{\alpha_1\beta_1} \equiv a^\dagger_{\alpha_1}a^\dagger_{\beta_1} , \text{ etc.},$$

then we do get

$$\tilde{f}_p(\mathcal{C}^\dagger) = f_p(\mathcal{C}^\dagger) .$$

However, the commutator $[C_1, C_2]$ does not approach $\delta_{12}^{(\dagger)}$, because it does not contain a small parameter corresponding to $Y(123)$. Consequently, this shows that the original Marumori representation does not satisfy the above criterion.

In the next section we will demonstrate the result of such a choice by considering boson expansions in a single j shell. The "convergence" problem will also be considered there. Right now we shall digress for a moment and discuss in the following section some alternatives to Marumori's method.

D. Other Possible Approaches to Include the Effect of the Pauli Principle in the Boson Expansions

Marumori's method gives the boson expansions without directly resorting to the commutation relations in Eqs. (2.7). Yet by applying the transformation V to these relations and using Eqs. (2.23) and (2.24), one can easily show that these relations are satisfied within the physical boson space by the transformed operators.

One alternative which was suggested by Marumori *et al.* in their original paper is to obtain the boson expansions by considering these transformed commutation relations and their effects on the unphysical boson states as well as the physical ones. Nevertheless, this method requires the specification of the physical boson states and is therefore not very much different from the original method.

Another alternative to the procedure described in this work is the one invented by Marshalek as mentioned in the Introduction. This method was in fact used before in a simplified model.[11] It can also be used to solve the problems raised in the pairing model work of Sørenson.[12] Marshalek's work appears to be a complete alternative to the approach of this paper. However, besides the fact that this method has not been generalized to the algebra (2.7), as so far formulated, it still leaves unclear certain basic physical questions to which we shall address attention.

In the next section we will use the modified Marumori method to study in detail the behavior of boson expansions in a single j shell.

III. BOSON EXPANSIONS FOR AN EVEN SYSTEM IN A SINGLE j SHELL

We consider an even number of nucleons in a single j shell. The creation and annihilation operators for a single nucleon are $a_m^\dagger$ and a_m, respectively, with $-j \leq m \leq j$. The pairing and multipole operators defined in Sec. II now become

$$A_M^{(J)\dagger} = \frac{1}{\sqrt{2}}[a^\dagger \times a^\dagger]^{(J)} = \frac{1}{\sqrt{2}}\sum_m \begin{bmatrix} j & j & J \\ m & M-m & M \end{bmatrix} a_m^\dagger a_{M-m}^\dagger \tag{3.1}$$

and

$$B_M^{(J)} = \frac{1}{(\hat{J})^{1/2}}[a^\dagger \times b]^{(J)} = \frac{1}{(\hat{J})^{1/2}}\sum_m \begin{bmatrix} j & j & M \\ M+m & -m & M \end{bmatrix} (-)^{j-m} a_{M+m}^\dagger a_m, \tag{3.2}$$

with

$$b_m \equiv s_{-m} a_{-m} \tag{3.3}$$

and

$$s_{-m} \equiv (-)^{j+m}. \tag{3.4}$$

These operators then have the properties

$$A_M^{(J)\dagger} = (-)^J A_{-M}^{(J)\dagger}, \tag{3.5}$$

and

$$B_M^{(J)\dagger} = (-)^M B_{-M}^{(J)}. \tag{3.6}$$

The commutation relations in Eqs. (2.7) become

$$[A_1^\dagger, A_2^\dagger] = 0, \tag{3.7a}$$

$$[A_1, A_2^\dagger] = \delta_{12} - 2\sum_3 Y(123)(\hat{J}_3)^{1/2} B_3^\dagger, \tag{3.7b}$$

$$[B_1^\dagger, A_2^\dagger] = \frac{2}{(\hat{J}_1)^{1/2}} \sum_3 Y(231) A_3^\dagger, \tag{3.7c}$$

and

$$[B_1^\dagger, B_2] = \frac{1}{(\hat{J}_2\hat{J}_3)^{1/2}} \sum_3 [1 - (-)^{J_1+J_2+J_3}] Y(132)(\hat{J}_3)^{1/2} B_3^\dagger, \tag{3.7d}$$

with $A_1^\dagger \equiv A_{M_1}^{(J_1)\dagger}$, etc. The function $Y(123)$ now reduces to

$$Y(123) = (-)^{2j+J_2+J_3}(\hat{J}_2\hat{J}_3)^{1/2} \begin{bmatrix} J_2 & J_3 & J_1 \\ M_2 & M_3 & M_1 \end{bmatrix} \begin{Bmatrix} J_2 & J_3 & J_1 \\ j & j & j \end{Bmatrix}, \tag{3.8}$$

with the following symmetry properties which can be derived easily from the above definition:

$$\begin{aligned} Y(123) &= (-)^{J_1+J_2+J_3} Y(132) \\ &= (-)^{M_3} Y(21\bar{3}) \\ &= (-)^{J_1+J_2+J_3} Y(\bar{1}\bar{2}\bar{3}), \end{aligned} \tag{3.9}$$

where $A_{\bar{3}}^\dagger \equiv A_{-M_3}^{(J_3)\dagger}$, etc.

Our modified Marumori method is best illustrated in the current model, although the results we shall get are quite general. In this section we start by forming the fermion states in the seniority scheme with the $SU(2j+1) \supset Sp(2j+1) \supset R(3)$ classification so that in contrast to Marumori's basis, ours are eigenstates of the total angular momentum. Since only the lowest anharmonic terms in the boson expansions are usually needed, explicit construction of states with no more than four particles will be enough. Next the physical boson states are chosen according to the criterion in Sec. II C. With these states one can construct the transformation V and obtain the boson expansions of fermion operators $A_1^\dagger$ and B_1 which are different from Marumori's original expansions. It will be verified that as $\Omega \to \infty$, the expansion of $A_1^\dagger$ now has the correct boson operator limit. However, one also discovers that for any finite value of j, it is the Pauli principle which always impairs total "convergence" of the expansions. Consequently, one can only compromise by showing that it is nevertheless possible to find a certain subspace of the boson space in which the expansions can be made convergent. As a corollary, it is also shown that the boson expansions will be totally convergent only when all the boson operators involved are spinless. This explains why every previous successful attempt to obtain convergent boson expansions[11-15] has been confined to special nuclear models, all of which require only expansions in terms of spinless bosons.

A. Construction of the Fermion Basis and the Selection of Physical Boson States in the Seniority Scheme

In the seniority scheme, the fermion states are labeled as $|NvJM\omega\rangle$, where N is the particle number, v is the seniority quantum number, J and M are the total angular momentum and its projection, respectively, and ω denotes additional quantum numbers needed to completely classify the states. The operator $A^{(0)\dagger}$ when operated on a state of seniority v does not change its seniority, while $A_M^{(J)\dagger}$ with $J \neq 0$ changes it to a linear combination of states with seniority $v+2$, v, and $v-2$. The two-particle states are therefore,

$$|2000\rangle = A^{(0)\dagger}|0\rangle \tag{3.10a}$$

and

$$|22JM\rangle = A_M^{(J)\dagger}|0\rangle\,, \tag{3.10b}$$

where $|0\rangle$ is the fermion vacuum state.

The lowest-order anharmonic terms are the third-order terms in the expansion of $A_1^\dagger$ and fourth-order terms in that of B_1. To get them we have to construct the four-particle states. First, the four-particle seniority-0 state is

$$|4000\rangle = S_0^{(00)} A^{(0)\dagger} A^{(0)\dagger}|0\rangle\,, \tag{3.11a}$$

with the normalization constants $S_0^{(00)}$ given in Eq. (B3) of Appendix B:

$$S_0^{(00)} = (2 - 2/\Omega)^{-1/2}\,. \tag{3.11b}$$

Next, the seniority-2 states are

$$|42JM\rangle = S_J^{(0J)} A^{(0)\dagger} A_M^{(J)\dagger}|0\rangle\,, \tag{3.12a}$$

where again the constant $S_J^{(0J)}$ is found from Eq. (B3),

$$S_J^{(0J)} = (1 - 2/\Omega)^{-1/2}\,. \tag{3.12b}$$

Then to get the seniority-4 states, we start from a state

$$|IM(J_1J_2)\rangle = S_I^{(J_1J_2)}[A^{(J_1)\dagger} \times A^{(J_2)\dagger}]_M^{(I)}|0\rangle\,, \tag{3.13}$$

with J_1, $J_2 \neq 0$, and I being one of the allowed angular momenta given by an angular momentum analysis of the four-particle seniority-4 representation.[16,17] The normalization constant in the above equation is just the one given in (B3). Since the above state is a mixture of seniority 0, 2, and 4 parts, we have to project out the seniority-4 part and obtain

$$\begin{aligned}|44JM(J_1J_2)\rangle &= N_I^{(J_1J_2)}[|IM(J_1J_2)\rangle - |42JM\rangle\langle 42JM|IM(J_1J_2)\rangle - |4000\rangle\langle 4000|IM(J_1J_2)\rangle]\\ &= N_I^{(J_1J_2)}[|IM(J_1J_2)\rangle - \rho_I^{(J_1J_2)}|42JM\rangle - \tau_I^{(J_1J_2)}|4000\rangle]\,,\end{aligned} \tag{3.14a}$$

where the coefficients, evaluated with the help of (B2), are

$$\begin{aligned}\rho_I^{(J_1J_2)} &\equiv \langle 42JM|IM(J_1J_2)\rangle\\ &= S_I^{(0I)} S_I^{(J_1J_2)} \langle 0|A^{(0)} A_M^{(I)} [A^{(J_1)\dagger} A^{(J_2)\dagger}]_M^{(I)}|0\rangle\\ &\equiv S_I^{(0I)} S_I^{(J_1J_2)} \tilde{G}_I(0I;J_1J_2)\\ &= \left(\frac{8\hat{J}_1\hat{J}_2}{\Omega}\right)^{1/2} \begin{Bmatrix} J_1 & J_2 & I\\ j & j & j\end{Bmatrix} \left(1 - \frac{2}{\Omega}\right)^{-1/2} \left[1 + \delta_{J_1J_2} - 4\hat{J}_1\hat{J}_2 \begin{Bmatrix} j & j & J_1\\ j & j & J_2\\ J_1 & J_2 & I\end{Bmatrix}\right]^{-1/2} \quad (I \neq 0)\,,\end{aligned} \tag{3.14b}$$

$$\tau_I^{(J_1J_2)} \equiv \langle 4000 | IM(J_1J_2)\rangle$$
$$= S_0^{(00)} S_I^{(J_1J_2)} \langle 0 | A^{(0)} A^{(0)} [A^{(J_1)\dagger} \times A^{(J_2)\dagger}]_M^{(I)} | 0\rangle$$
$$= S_0^{(00)} S_I^{(J_1J_2)} \tilde{G}_I(00; J_1J_2)$$
$$= -\frac{(2\hat{J}_1)^{1/2}}{\Omega}\left(1-\frac{1}{\Omega}\right)^{-1/2} \delta_{I0}\delta_{J_1J_2}, \qquad (3.14c)$$

and

$$N_I^{(J_1J_2)} = \{1 - [\rho_I^{(J_1J_2)}]^2 - [\tau_I^{(J_1J_2)}]^2\}^{-1/2}. \qquad (3.14d)$$

Finally, in case of degeneracy, i.e., when there exist several seniority-4 states with the same angular momentum I, we will choose a set of states $|44IM(J_1J_2)\rangle$ with different J_1 and J_2 and orthogonalize them with the Schmidt process such that:

$$|44IM[J_1J_2]\rangle = R^{(J_1J_2)}[|44IM(J_1J_2)\rangle - \sum_{J_1'J_2'} |44IM[J_1'J_2']\rangle \times \langle 44IM[J_1'J_2'] | 44IM(J_1J_2)\rangle], \qquad (3.15a)$$

in which the subset $|44IM[J_1'J_2']\rangle$ consists of all previously orthogonalized states, the renormalization constant is

$$R_I^{(J\ J\ )} = [1 - \sum_{J_1'J_2'} |\langle 44IM[J_1'J_2'] | 44IM(J_1J_2)\rangle|^2]^{-1/2}, \qquad (3.15b)$$

while the coefficients $\langle 44IM[J_1'J_2'] | 44IM(J_1J_2)\rangle$ can always be evaluated by using Eqs. (B2), (B3), and (B4). We note that as $\Omega \to \infty$, these coefficients tend to vanish and therefore $R_I^{(J_1J_2)} \to 1$. Also from Eqs. (3.14), we have $\rho_I^{(J_1J_2)}$ and $\tau_I^{(J_1J_2)} \to 0$, and $N_I^{(J_1J_2)} \to 1$. Consequently, in the limit $\Omega \to \infty$, $|44IM[J_1J_2]\rangle \to |IM(J_1J_2)\rangle$. This is the result which will be used later to study the asymptotic behavior of boson expansions.

As for the boson space, we now choose the physical boson states such that:

$$|0)_B \leftrightarrow |0\rangle_F, \qquad (3.16a)$$

$$\mathcal{a}_M^{(J)\dagger} |0)_B \leftrightarrow A_M^{(J)\dagger} |0\rangle_F, \qquad (3.16b)$$

$$\frac{1}{\sqrt{2}} \mathcal{a}^{(0)\dagger}\mathcal{a}^{(0)\dagger} |0)_B \leftrightarrow |4000\rangle, \qquad (3.16c)$$

$$\mathcal{a}^{(0)\dagger}\mathcal{a}_M^{(J)\dagger} |0)_B \leftrightarrow |42JM\rangle \quad (J \neq 0), \qquad (3.16d)$$

and

$$\frac{1}{(1+\delta_{J_1J_2})^{1/2}} [\mathcal{a}^{(J_1)\dagger} \times \mathcal{a}^{(J_2)\dagger}]_M^{(I)} |0)_B \leftrightarrow |44IM[J_1J_2]\rangle. \qquad (3.16e)$$

In choosing the physical boson state corresponding to the fermion state $|44IM[J_1J_2]\rangle$, we did not follow the prescription (2.39) exactly. However, since $|44IM[J_1J_2]\rangle \to |IM(J_1J_2)\rangle$ as $\Omega \to \infty$, we see that the condition (2.39) is satisfied in the limit, i.e., $\tilde{f}_p(A^\dagger) \sim f_p(A^\dagger)$ as $\Omega \to \infty$. It will be demonstrated later that with this choice we do get an expansion of $A_1^\dagger$ with the appropriate boson-operator limit.

With these states given, one can now proceed to construct the boson expansions with Marumori's method.

B. Boson Expansions of $A^{(J)\dagger}$ and $B^{(J)}$

We shall need the following expansion of the projection operator $|0)_{B\,B}(0|$:

$$|0)_{BB}(0| = 1 - \sum_{JM} \mathcal{a}_M^{(J)\dagger}\mathcal{a}_M^{(J)} + 0(4), \qquad (3.17)$$

which can be obtained either from Eq. (2.35) or directly from the completeness relation (2.29). Only terms up to the second order are given because those are all that we need to get the lowest anharmonic terms in the expansions of $A_1^\dagger$ and B_1. These expansions are now given in the following subsections.

1. *Expansion of* $A^{(0)\dagger}$

Since $A^{(0)\dagger}$ does not change the seniority, we have

$$VA^{(0)\dagger}V^\dagger = \sum_{\substack{N\nu IM\omega\\ \omega'}} \tilde{O}_{N\nu IM\omega}^\dagger |0)_B \langle N\nu IM\omega | A^{(0)\dagger} | N\nu IM\omega'\rangle \times {}_B(0|\tilde{O}_{N\nu IM\omega'}. \qquad (3.18a)$$

Keeping only the first- and third-order terms in the boson operator, the above expansion becomes

$$VA^{(0)\dagger}V^\dagger \equiv \mathcal{a}^{(0)\dagger}(1 - \sum_{JM}\mathcal{a}_M^{(J)\dagger}\mathcal{a}_M^{(J)}) + C_0'\mathcal{a}^{(0)\dagger}\mathcal{a}^{(0)\dagger}\mathcal{a}^{(0)} + \sum_{J\neq 0,M} C_J'\mathcal{a}^{(0)\dagger}\mathcal{a}_M^{(J)\dagger}\mathcal{a}_M^{(J)} + 0(5), \qquad (3.18b)$$

where we have used Eq. (3.17). The coefficients are

$$C_0' \equiv \frac{1}{\sqrt{2}}\langle 4000 | A^{(0)\dagger} | 2000\rangle$$
$$= \frac{1}{\sqrt{2}} S_0^{(00)} \tilde{G}_0(00; 00)$$
$$= \left(1 - \frac{1}{\Omega}\right)^{1/2}, \qquad (3.18c)$$

and

$$C_J' \equiv \langle 42JM | A^{(0)\dagger} | 22JM\rangle$$
$$= S_J^{(0J)} \tilde{G}_J(0J; 0J)$$
$$= (1 - 2/\Omega)^{1/2}. \qquad (3.18d)$$

Substituting them back into Eq. (3.18b), we get

$$VA^{(0)\dagger}V^\dagger = \mathcal{a}^{(0)\dagger} + \left[-1 + \left(1 - \frac{1}{\Omega}\right)^{1/2}\right]\mathcal{a}^{(0)\dagger}\mathcal{a}^{(0)\dagger}\mathcal{a}^{(0)} + \left[-1 + \left(1 - \frac{2}{\Omega}\right)^{1/2}\right]\mathcal{a}^{(0)\dagger} \sum_{J\neq 0,M} \mathcal{a}_M^{(J)\dagger}\mathcal{a}_M^{(J)} + 0(5). \qquad (3.19)$$

2. *Expansion of* $A^{(J)\dagger}$ *(J ≠ 0)*

The pair operator $A^{(J_1)\dagger}_{M_1}$ with $J_1\neq 0$ changes the seniority quantum number by at most two units. We find after some calculation,[18] utilizing also Eq. (3.17),

$$\begin{aligned} VA^{(J_1)\dagger}_{M_1}V^\dagger &= \sum_{\substack{N\nu IM\omega\\ \nu'I'M'\omega'}} \tilde{O}^\dagger_{N\nu IM\omega}|0)_B\langle N\nu IM\omega|A^{(J_1)\dagger}_{M_1}|N-2,\nu'I'M'\omega'\rangle_B(0|\tilde{O}_{N-2,\nu'I'M'\omega'}\\ &=\mathfrak{a}^{(J_1)\dagger}_{M_1}+C^{(J_1)}_{J_100}\mathfrak{a}^{(J_1)\dagger}_{M_1}\mathfrak{a}^{(0)\dagger}\mathfrak{a}^{(0)}+C^{(J_1)}_{00J_1}(-)^{M_1}\mathfrak{a}^{(0)\dagger}\mathfrak{a}^{(0)\dagger}\mathfrak{a}^{(J_1)}_{-M_1}\\ &\quad+\sum_{\substack{J_2\neq 0\\ J_3\neq 0}} C^{(J_1)}_{0J_2J_3}\mathfrak{a}^{(0)\dagger}[\mathfrak{a}^{(J_2)\dagger}\mathfrak{a}^{(J_3)}]^{(J_1)}_{M_1}+R_1+R_2+0(5)\,, \end{aligned} \tag{3.20a}$$

with

$$[\mathfrak{a}_2^\dagger\times\mathfrak{a}_3]^{(J_1)}_{M_1}\equiv\sum_{M_2(M_3)}\begin{bmatrix}J_2 & J_3 & J_1\\ M_2 & M_3 & M_1\end{bmatrix}(-)^{M_3}\mathfrak{a}^{(J_2)\dagger}_{M_2}\mathfrak{a}^{(J_3)}_{-M_3}\,. \tag{3.20b}$$

Here, the various coefficients, as evaluated with the help of Eqs. (3.15), (3.14), and (B2) are

$$\begin{aligned} C^{(J_1)}_{J_100}&\equiv-1+\langle 42\,J_1M_1|A^{(J_1)\dagger}_{M_1}|2000\rangle\\ &=-1+(1-2/\Omega)^{1/2}\,, \end{aligned} \tag{3.21a}$$

$$\begin{aligned} C^{(J_1)}_{00J_1}&\equiv(-)^{M_1}\frac{1}{\sqrt{2}}\langle 4000|A^{(J_1)\dagger}_{M_1}|22\,J_1-M_1\rangle\\ &=-\frac{1}{\Omega}\left(1-\frac{1}{\Omega}\right)^{-1/2}, \end{aligned} \tag{3.21b}$$

$$\begin{aligned} C^{(J_1)}_{0J_2J_3}&\equiv(-)^{M_3}\begin{bmatrix}J_2 & J_3 & J_1\\ M_2 & -M_3 & M_1\end{bmatrix}^{-1}\langle 42\,JM|A^{(J_1)\dagger}_{M_1}|22\,J_3M_3\rangle\\ &=\left(\frac{8\hat{J}_2\hat{J}_3}{\Omega}\right)^{1/2}\left(1-\frac{2}{\Omega}\right)^{-1/2}\begin{Bmatrix}J_1 & J_2 & J_3\\ j & j & j\end{Bmatrix}, \end{aligned} \tag{3.21c}$$

$$\begin{aligned} R_1&=-\mathfrak{a}^{(J_1)\dagger}_{M_1}\sum_{J_4\neq 0,M_4}\mathfrak{a}^{(J_4)\dagger}_{M_4}\mathfrak{a}^{(J_4)}_{M_4}\\ &=-\sum_{J_4\neq 0,M_4}\sum_I\begin{bmatrix}J_1 & J_4 & I\\ M_1 & M_4 & M\end{bmatrix}[\mathfrak{a}^{(J_1)\dagger}\times\mathfrak{a}^{(J_4)\dagger}]^{(I)}_M\mathfrak{a}^{(J_4)}_{M_4}\,, \end{aligned} \tag{3.22a}$$

and

$$R_2\equiv\sum_{J_4\neq 0,M_4}\sum_{I[J_2J_3]}\begin{bmatrix}J_1 & J_4 & I\\ M_1 & M_4 & M\end{bmatrix}C^{(J_1)}_{[J_2J_3]IJ}[\mathfrak{a}^{(J_2)\dagger}\times\mathfrak{a}^{(J_3)\dagger}]^{(I)}_M\mathfrak{a}^{(J_4)}_{M_4}\,, \tag{3.22b}$$

with

$$\begin{bmatrix}J_1 & J_4 & I\\ M_1 & M_4 & M\end{bmatrix}C^{(J_1)}_{[J_2J_3]IJ_4}\equiv(1+\delta_{J_2J_3})^{-1/2}\langle 44\,IM[J_2J_3]|A^{(J_1)\dagger}_{M_1}|22\,J_4M_4\rangle\,. \tag{3.23}$$

3. *Expansion of* $B^{(J)}$

The multipole operator $B^{(J_1)}_{M_1}$ conserves the particle number but changes the seniority. We find, with the help of Eq. (3.17), that the leading anharmonic terms are of fourth order:

$$\begin{aligned} VB^{(J_1)}_{M_1}V^\dagger &= \sum_{\substack{N\nu IM\omega\\ \nu'I'M'\omega''}} \tilde{O}^\dagger_{N\nu IM\omega}|0)_B\langle N\nu IM\omega|B^{(J_1)}_{M_1}|N\nu'I'M'\omega'\rangle_B(0|\tilde{O}_{N\nu'I'M'\omega'}\\ &=\sum_{J_2J_3} d^{(J_1)}_{J_2J_3}[\mathfrak{a}^{(J_2)\dagger}\times\mathfrak{a}^{(J_3)}]^{(J_1)}_{M_1}+f^{(J_1)}_{0000}\mathfrak{a}^{(0)\dagger}\mathfrak{a}^{(0)\dagger}\mathfrak{a}^{(0)}\mathfrak{a}^{(0)}\\ &\quad+f^{(J_1)}_{J_1000}\{\mathfrak{a}^{(J_1)\dagger}_{M_1}\mathfrak{a}^{(0)\dagger}\mathfrak{a}^{(0)}\mathfrak{a}^{(0)}+(-)^{M_1}\mathfrak{a}^{(0)\dagger}\mathfrak{a}^{(0)\dagger}\mathfrak{a}^{(0)}\mathfrak{a}^{(J_1)}_{M_1}\}\\ &\quad+\sum_{\substack{J_2\neq 0\\ J_3\neq 0}} f^{(J_1)}_{0J_2J_3}\,\mathfrak{a}^{(0)\dagger}[\mathfrak{a}^{(J_2)\dagger}\times\mathfrak{a}^{(J_3)}]^{(J_1)}_{M_1}\mathfrak{a}^{(0)}+T_1+T_2+T_4+0(6)\,, \end{aligned} \tag{3.24}$$

where

$$d^{(J_1)}_{J_2J_3} \equiv (-)^{M_3}\begin{bmatrix} J_2 & J_3 & J_1 \\ M_2 & -M_3 & M_1 \end{bmatrix}^{-1} \langle 22\,J_2M_2 | 22\,J_3M_3\rangle$$

$$= -2(-)^{J_1}\left(\frac{\hat{J}_2\hat{J}_3}{\hat{J}_1}\right)^{1/2}\begin{Bmatrix} J_1 & J_2 & J_3 \\ j & j & j \end{Bmatrix}, \tag{3.25}$$

$$f^{(J_1)}_{0000} = d^{(J_1)}_{00} + \tfrac{1}{2}\langle 4000 | B^{(J_1)}_{M_1} | 4000\rangle$$

$$= 0, \tag{3.26a}$$

$$f^{(J_1)}_{J_1000} = d^{(J_1)}_{J_10} + \frac{1}{\sqrt{2}}\langle 42\,J_1M_1 | B^{(J_1)}_{M_1} | 4000\rangle$$

$$= -\frac{1}{(\hat{J}_1)^{1/2}}\left(\frac{2}{\Omega}\right)^{1/2}\left\{1 - \left(1 - \frac{1}{\Omega}\right)^{-1/2}\left(1 - \frac{2}{\Omega}\right)^{1/2}\right\} \quad (J_1 \neq 0,\ \text{even}), \tag{3.26b}$$

$$f^{(J_1)}_{0J_2J_30} = -\delta_{J_10}\, d^{(J_1)}_{00}(\hat{J}_2)^{1/2} d^{(J_1)}_{J_2J_3} + (-)^{M_3}\begin{bmatrix} J_2 & J_3 & J_1 \\ M_2 & -M_3 & M_1 \end{bmatrix}^{-1}\langle 42\,J_3M_2 | B^{(J_1)}_{M_1} | 42\,J_3M_3\rangle$$

$$= \begin{cases} 0 & (J_1 = 0 \text{ or } J_1 \text{ odd}) \\ \dfrac{4}{\Omega}\left(\dfrac{\hat{J}_2\hat{J}_3}{\hat{J}_1}\right)^{1/2}\left(1 - \dfrac{2}{\Omega}\right)^{-1}\begin{Bmatrix} J_1 & J_2 & J_3 \\ j & j & j \end{Bmatrix} & (J_1 \neq 0,\ \text{even}), \end{cases} \tag{3.26c}$$

and

$$T_1 = -\sum_{J_4\neq 0, M_4}\sum_{I} d^{(J_1)}_{J_10}\begin{bmatrix} J_1 & J_4 & I \\ M_1 & M_4 & M \end{bmatrix}\{[\mathcal{A}^{(J_1)\dagger}\times \mathcal{A}^{(J_4)\dagger}]^{(I)}_M \mathcal{A}^{(J_4)}_{M_4}\mathcal{A}^{(0)}$$
$$+ (-)^{M_1}\mathcal{A}^{(0)\dagger}\mathcal{A}^{(J_4)\dagger}_{-M_4}[\mathcal{A}^{(J_4)}\times \mathcal{A}^{(J_1)}]^{(I)}_{-M}\} \quad (J_1\neq 0,\ \text{even}), \tag{3.27a}$$

$$T_2 = \sum_{J_4\neq 0, M_4}\sum_{I[J_2J_3]} f^{(J_1)}_{[J_2J_3]IJ_4}\begin{bmatrix} J_1 & J_4 & I \\ M_1 & M_4 & M \end{bmatrix}[\mathcal{A}^{(J_2)\dagger}\times \mathcal{A}^{(J_3)\dagger}]^{(I)}_M \mathcal{A}^{(J_4)}_{M_4}\mathcal{A}^{(0)}$$
$$+ (-)^{M_1}\mathcal{A}^{(0)\dagger}\mathcal{A}^{(J_4)\dagger}_{-M_4}[\mathcal{A}^{(J_3)}\times \mathcal{A}^{(J_2)}]^{(I)}_{-M} \quad (J_1\neq 0,\ \text{even}), \tag{3.27b}$$

with

$$f^{(J_1)}_{[J_2J_3]IJ_4}\begin{bmatrix} J_1 & J_4 & I \\ M_1 & M_4 & M \end{bmatrix} = \frac{1}{(1+\delta_{J_2J_3})^{1/2}}\langle 44\,IM[J_2J_3] | B^{(J_1)}_{M_1} | 42\,J_4M_4\rangle, \tag{3.27c}$$

while

$$T_3 = -\sum_{J_2J_4J_5\neq 0}\sum_{II'M'}(\hat{I}\hat{I}')^{1/2}\begin{Bmatrix} I & I' & J \\ J_5 & J_2 & J_4 \end{Bmatrix} d^{(J_1)}_{J_2J_5}(-)^{I'-M'}\begin{bmatrix} I & I' & J_1 \\ M & -M' & M_1 \end{bmatrix}[\mathcal{A}^{(J_2)\dagger}\times \mathcal{A}^{(J_4)\dagger}]^{(I)}_M[\mathcal{A}^{(J_4)}\times \mathcal{A}^{(J_5)}]^{(I')}_{M'}, \tag{3.28a}$$

$$T_4 = \sum_{\substack{I[J_2J_3] \\ I'[J_4J_5]}}\sum_{M'} f^{(J_1)}_{[J_2J_3]I[J_4J_5]I'}(-)^{I'-M'}\begin{bmatrix} I & I' & M_1 \\ M & -M' & M_1 \end{bmatrix}[\mathcal{A}^{(J_2)\dagger}\times \mathcal{A}^{(J_3)\dagger}]^{(I)}_M[\mathcal{A}^{(J_4)}\times \mathcal{A}^{(J_5)}]^{(I')}_{M'}, \tag{3.28b}$$

with

$$f^{(J_1)}_{[J_2J_3]I[J_4J_5]I'}(-)^{I'-M'}\begin{bmatrix} I & I' & J_1 \\ M & -M' & M_1 \end{bmatrix} = [(1+\delta_{J_2J_3})(1+\delta_{J_4J_5})]^{-1/2}\langle 44\,IM[J_2J_3] | B^{(J_1)}_{M_1} | 44\,I'M'[J_4J_5]\rangle. \tag{3.28c}$$

We have thus obtained the boson expansions of $A_1^\dagger$ and B_1 to the lowest anharmonic terms which are the third-order ones for $A_1^\dagger$ and the fourth-order ones for B_1. The coefficients are either evaluated explicitly or given as matrix elements which can always be expressed in terms of the invariant functions defined in Appendix B. Besides, there is no difficulty in writing a computer program to calculate these coefficients.

In the last section we mentioned that the original Marumori expansion contained two undesirable features. So far our modified method has avoided one of them, that the fermion pair operator does not have the appropriate boson operator limit as $\Omega\to\infty$. This problem has been solved by a suitable choice of the physical boson space. On the other hand, an examination of Eq. (3.22a), for example, shows that our method still possesses the other difficulty, i.e., the expansions are not "convergent." However, we shall see in the following section that it is the Pauli principle itself which prevents us from having a totally convergent expansion for any finite value of j. Therefore any boson expansion which takes into account the effect of the Pauli principle correctly will not in general be "convergent." Fortunately, it can nevertheless be shown that there exists a certain sequence of subspaces of the boson space such that we can always find among them one element, whose size depends upon the value of j, in which the expansions can be made convergent.

In the next section we shall study the asymptotic behavior of our expansions as $\Omega\to\infty$. One of the advantages of our choice of the physical boson space is that it clearly shows how the Pauli principle impairs total convergence, while this in turn suggests by itself a natural way of constructing the subspaces mentioned before in which the boson expansions can be made convergent.

C. Asymptotic Behavior of Boson Expansions as $j\to\infty$

We shall begin with the expansion of $A^{(J_1)\dagger}_{M_1}$ with $J_1\neq 0$. First of all, it is found that while the coefficients in Eqs. (3.21) all behave like $0(\Omega^{-1})$ as $\Omega\to\infty$, those in R_1 and R_2 are in general $0(1)$, and therefore the convergence of the expansion must depend on a relative cancellation of contributions from the corresponding terms in them.

As $\Omega\to\infty$, we obtain for the coefficients $C^{(J_1)}_{[J_1J_2]IJ_4}$ of Eq. (3.23), using Eqs. (B5), (B3), and (B2),

$$C^{(J_1)}_{[J_2J_3]IJ_4}\sim(1+\delta_{J_2J_3})^{-1}\{\delta_{J_2J_1}\delta_{J_3J_4}+(-)^I\delta_{J_3J_1}\delta_{J_2J_4}\} +0(\Omega^{-1}). \qquad (3.29)$$

Substituting this back into Eq. (3.22b), we get

$$\begin{aligned}R_2&\sim\sum_{J_4\neq 0,M_4}\sum_{I[J_2J_3]}\begin{bmatrix}J_1 & J_4 & I\\ M_1 & M_4 & M\end{bmatrix}(1+\delta_{J_2J_3})^{-1}[\delta_{J_2J_1}\delta_{J_3J_4}+(-)^I\delta_{J_3J_1}\delta_{J_2J_4}][\mathcal{a}^{(J_2)\dagger}\times\mathcal{a}^{(J_3)\dagger}]^{(I)}_M\mathcal{a}^{(J_4)}_{M_4}\\ &=\sum_{J_4\neq 0,M_4}\sum_{I[J_2J_3]}\begin{bmatrix}J_1 & J_4 & I\\ M_1 & M_4 & M\end{bmatrix}\delta_{(J_2J_3),(J_1J_4)}[\mathcal{a}_2^\dagger\times\mathcal{a}_3^\dagger]^{(I)}_M\mathcal{a}_4,\end{aligned} \qquad (3.30)$$

where

$$\delta_{(J_2J_3),(J_1J_4)}\equiv\begin{cases}1, & J_2=J_1,\ \text{and}\ J_3=J_4\\ 1, & J_2=J_4,\ \text{and}\ J_3=J_1\\ 0, & \text{otherwise}.\end{cases} \qquad (3.31)$$

Here J_1 is fixed and J_4 can assume all allowed values $[J_4=2,4,\ldots,(2j-1)]$. From Eq. (3.30) we find that there are two kinds of terms in R_2. That which is not matched by any term in R_1 ($\delta_{(J_2J_3),(J_1J_4)}=0$) always has a coefficient $0(\Omega^{-1})$. That which is matched by a certain term in R_1 ($\delta_{(J_2J_3),(J_1J_4)}=1$) carries a coefficient $1+0(\Omega^{-1})$. The latter, however, when combined with the corresponding term with the coefficient -1 in R_1, also gives a resultant coefficient of the order Ω^{-1}. Thus every term found in R_2 will eventually contribute a term with a coefficient $0(\Omega^{-1})$.

We must now remember that the sum over J_2 and J_3 in Eq. (3.22b) or (3.30) is restricted to those values which specify the basis of fermion states. Therefore there are, in general, terms in R_1 which are not matched in R_2 and always have the coefficients ~ 1. With the existence of these terms which are of the same order of magnitude as the first-order term, it is now clear that our expansion is still not convergent; but it is also clear that the reason for this can be attributed to the Pauli principle which, instead of allowing all possible fermion pair operators $A^{(J_2)\dagger}$ and $A^{(J_3)\dagger}$ to couple together and form four-particle seniority-4 states, restricts J_2 and J_3 to those values which are initially selected to specify the fermion states.

As $\Omega\to\infty$, we note from Eq. (B4) that

$$\begin{aligned}\langle 44IM[J_1J_2]\,|\,44IM[J_3J_4]\rangle&\sim\langle IM(J_1J_2)\,|\,IM(J_3J_4)\rangle\\ &\sim(1+\delta_{J_1J_2})^{-1/2}(1+\delta_{J_3J_4})^{-1/2}[\delta_{J_1J_3}\delta_{J_2J_4}+(-)^I\delta_{J_1J_4}\delta_{J_2J_3}]\\ &=\delta_{(J_1J_2),(J_3J_4)},\end{aligned} \qquad (3.32)$$

which means that in the limit $\Omega\to\infty$ the overlap integral between any two states among the set of four-particle states $|IM(J_1J_2)\rangle$ tends to vanish and therefore all possible values of J_1 and J_2 would be needed to specify the four-particle states. Consequently, every term in R_1 would be canceled by a corresponding term in R_2, and we have verified the statement (2.40) that the expansion of $A^{(J_1)\dagger}$ has the correct boson-operator limit. Nevertheless, we also discover that this only means that the relative number of terms of 0(1) tends to zero as $\Omega\to\infty$, and for any finite value of j the expansion cannot in general be totally convergent.

We have discussed in detail the asymptotic behavior of the third-order terms in the expansion of $A^{(J_1)\dagger}$ with $J_1\neq 0$, but the results we found are in fact entirely general and can be applied to terms of all orders in the expansion. It is also obvious that the same result can be obtained for the expansion of $B^{(J_1)}$. The third-order terms in the expansion of $A^{(0)\dagger}$ all behave as $0(\Omega^{-1})$, but that is because the matrix elements of seniority-4 states do not contribute to these terms. We shall expect that in the fifth order it will begin to show the same sympton as other expansions.

The previous studies also reveal another important fact, that those terms in R_1 not matched in R_2 always have vanishing matrix elements in the subspace of physical boson states with no more than two bosons. However, in actual calculations boson states with more than two bosons are sometimes included, and therefore the above statement is not a very satisfactory answer to the problem of finding a truly convergent boson degree of freedom. Nevertheless, it has provided a clue to the solution, and in the next section we shall carefully investigate the possibility of having a convergent boson expansion within certain subspaces of the physical boson space.

D. Construction of Subspaces in Which the Boson Expansions Are Convergent

We have seen in the last section that it is in general impossible to have a totally convergent boson expansion in the entire physical boson space. This is due to the fact that there are always terms which have the same order of magnitude as those of a lower order. Nevertheless, in certain subspace of the physical boson space, the matrix elements of these terms will all vanish and one obtains a convergent boson expansion. If the physical states in which we are interested can be transformed largely into this subspace, then the boson expansion will provide a useful representation of the original fermion operator within this convergent subspace. To repeat, the best we can hope for is that the boson expansions will be useful for a subset of all the physical states. The choice of the proper fermion basis involves dynamical considerations, i.e., it depends on the structure of the Hamiltonian. Yet just from the kinematical conditions we have considered so far, it is possible to discuss the construction of these convergent subspaces and their maximum possible size. We shall therefore proceed to find these spaces.

We construct a sequence of boson subspaces $D_J^{(n)}$ such that each $D_J^{(n)}$ contains states constructed from boson operators $\mathcal{A}^{(0)\dagger}$, $\mathcal{A}^{(2)\dagger},\ldots,\mathcal{A}^{(J)\dagger}$ up to n boson states. For example, we can have

$$D_0^{(n)}\equiv\left\{|0)_B,\ \mathcal{A}^{(0)\dagger}|0)_B,\ \frac{1}{\sqrt{2}}\mathcal{A}^{(0)\dagger}\mathcal{A}^{(0)\dagger}|0)_B,\ldots,\ \frac{1}{\sqrt{n!}}\mathcal{A}^{(0)\dagger}\mathcal{A}^{(0)\dagger},\ldots,\ \mathcal{A}^{(0)\dagger}|0)_B\right\},\tag{3.33}$$

or

$$D_2^{(2)}\equiv\left\{|0)_B,\ \mathcal{A}^{(0)\dagger}|0)_B,\ \mathcal{A}^{(2)\dagger}|0)_B,\ \frac{1}{\sqrt{2}}\mathcal{A}^{(0)\dagger}\mathcal{A}^{(0)\dagger}|0)_B,\ \mathcal{A}^{(0)\dagger}\mathcal{A}^{(2)\dagger}|0)_B,\ \frac{1}{\sqrt{2}}[\mathcal{A}^{(2)\dagger}\mathcal{A}^{(2)\dagger}]^{(0,2,4)}|0)_B\right\}.\tag{3.34}$$

From the definition it is clear that $D_J^{(n)}\supset D_{J'}^{(n')}$ if, and only if, $J\geq J'$ and $n\geq n'$.

Now if the physical boson space D_P is chosen so as to conserve both the particle number and the total angular momentum, then we have

$$D_{2j-1}^{(\Omega)}\supset D_P\supset D_0^{(\Omega)}.\tag{3.35}$$

For a given j, if we can find n_0 and J_0 such that the condition

$$D_P\supset D_{J_0}^{(n_0)}\tag{3.36}$$

is satisfied, then from what we have discussed in the last section, it is not difficult to see that the matrix elements

$${}_B(m|VTV^\dagger|n)_B,$$

where $|m)_B$, $|n)_B\in D_{J_0}^{(n_0)}$, and T is either $A^{(J_1)\dagger}$ or $B^{(J_1)}$, will be convergent expansions in the sense that terms of a certain order will be Ω^{-1} times smaller than those of the previous order provided $J_1\leq J_0$. Consequently, the matrix elements of the expansion of a chosen Hamiltonian ${}_B(m|H_B|n)_B$ will also be convergent in the subspace. Finally, if there are states of the physical system which are imaged largely into this subspace, then and only then can we expect to have vibrational spectra.

As an example, take $j=\frac{9}{2}$. Here a Slater count establishes that

$$D_P\supset D_2^{(2)}.$$

For larger j values, we will be able to find larger n_0 and J_0 to satisfy Eq. (3.36), and indeed we should have to expect to go to such larger values of j to establish a self-respecting vibrational behavior.

It is interesting to note that

$$D_{J_0}^{(\Omega)} \not\subset D_P, \tag{3.37}$$

if $J_0 \geq 2$. This is because D_P contains only one Ω-boson state, namely the closed-shell state; but $D_{J_0}^{(\Omega)}$ with $J_0 \geq 2$ will always contain more than one Ω-boson state. Therefore, only in the spin-zero case, when it is possible to have $D_P \supset D_0^{(\Omega)}$, can we have a totally convergent boson expansion. This explains why everyone who succeeded in obtaining convergent Marumori expansion considered models involving spinless bosons only.

We should remark here that the subspaces $D_J^{(n)}$ are by no means the only possible convergent subspaces in the physical boson space. In cases when only the quadrupole vibrations are of interest, for example, one can consider the subspace $d_{J=2}^{(n)}$ which contains states constructed from $\mathfrak{a}^{(2)\dagger}$ up to n bosons. In this way $D_J^{(n)}$ can be expressed as

$$D_J^{(n)} \equiv \sum_{n_0+n_2+\cdots+n_J=n} d_0^{(n_0)} \otimes d_2^{(n_2)} \otimes \cdots \otimes d_J^{(n_J)}, \tag{3.38}$$

where the summation extends over all possible values of $n_0, n_2, \ldots, n_J$ which form a partition of n.

From the above discussion it is obvious that one can also consider any direct-product space with the form

$$\overline{D}_{J_1 J_2, \ldots, J_k}^{(n)} = \sum d_{J_1}^{(n_1)} \otimes d_{J_2}^{(n_2)} \otimes \cdots \otimes d_{J_k}^{(n_k)}, \tag{3.39a}$$

with

$$\sum_{i=1}^{k} n_i = n. \tag{3.39b}$$

IV. BOSON EXPANSIONS FOR A GENERAL SYSTEM IN A SINGLE j SHELL

The boson-expansion method can also be generalized to include systems with odd number of nucleons.[3-6] In that case the operator algebra should be extended to include the single-fermion operators a_m and $a_m^\dagger$ as well as the pairing and multipole operators. The commutation relations

$$[a_m^\dagger, A_M^{(J)\dagger}] = 0, \tag{4.1a}$$

$$[a_m, A_M^{(J)\dagger}] = \sqrt{2}\begin{bmatrix} j & j & J \\ m & M-m & M \end{bmatrix} a_{M-m}^\dagger, \tag{4.1b}$$

$$[a_m^\dagger, B_M^{(J)}] = -\frac{1}{(2\Omega)^{1/2}}\begin{bmatrix} j & J & j \\ m & M & m+M \end{bmatrix} a_{m+M}^\dagger, \tag{4.1c}$$

and

$$[a_m, B_M^{(J)}] = \frac{1}{(2\Omega)^{1/2}}\begin{bmatrix} j & J & j \\ m-M & M & m \end{bmatrix} a_{m-M}^\dagger, \tag{4.1d}$$

together with Eqs. (3.7) form the complete algebra.

To study the expansions for the general system we will follow the same steps as those in Sec. III.

A. Fermion States and the Physical Boson States

The fermion space now contains states with an odd number of particles; these states can be generated by operating $a_m^\dagger$ on those even states defined in the last section. In the seniority scheme they have the general form

$$|Nvkq\mu\rangle \equiv S_{Nvkq\mu}[a^\dagger \times 0_{N-1,v-1,IM\omega}^\dagger(A^\dagger)]_{Nvkq\mu}|0\rangle, \tag{4.2}$$

where $0_{N-1,v-1,IM\omega}^\dagger(A^\dagger)$ is a polynomial of $A^\dagger$ generating the even state,

$$|NvIM\omega\rangle \equiv 0_{NvIM\omega}(A^\dagger)|0\rangle, \tag{4.3}$$

and $S_{Nvkq\mu}$ is a normalization constant.

The one-particle state is $a_m^\dagger|0\rangle$. The $N=3$, $v=1$ states are

$$|3|jm\rangle = S_j^{(0)} a_m^\dagger A^{(0)\dagger}|0\rangle, \tag{4.4a}$$

with the normalization $S_j^{(0)}$ given by

$$\begin{aligned} S_j^{(0)} &= [\tilde{g}_j(0,0)]^{-1/2} \\ &= (1-1/\Omega)^{-1/2}, \end{aligned} \tag{4.4b}$$

where the function $\tilde{g}_k(J_1, J_2)$ is defined in Eq. (B11) of Appendix B.

The $N=v=3$ states can be constructed in a way similar to that of $N=v=4$ states. First one forms a state

$$|kq(J)\rangle \equiv S_k^{(J)}[a^\dagger \times A^{(J)\dagger}]_q^{(k)}|0\rangle, \tag{4.5a}$$

with $J \neq 0$ and $S_k^{(J)}$ given by

$$\begin{aligned} S_k^{(J)} &= [\langle 0|a \times A^{(J)}]_q^{(k)}[a^\dagger \times A^{(J)\dagger}]_q^{(k)}|0\rangle]^{-1/2} \\ &= [\tilde{g}_k(J,J)]^{-1/2} \\ &= \left[1 + 2\hat{J}\begin{Bmatrix} j & j & J \\ j & k & J \end{Bmatrix}\right]^{-1/2}. \end{aligned} \tag{4.5b}$$

Next, one projects out the $v=3$ part and obtains

$$|33jq(J)\rangle = R_k^{(J)}\{|kq(J)\rangle - |31jq\rangle\langle 3|jq1kq(J)\rangle\}, \tag{4.6a}$$

with

$$\begin{aligned} \langle 31jq(J)\rangle &= S_j^{(0)} S_k^{(J)} \delta_{jk} \tilde{g}_j(0,J) \\ &= \left[-\delta_{jk}\frac{(\hat{J})^{1/2}}{\Omega}\right]\left(1-\frac{1}{\Omega}\right)^{-1/2} \\ &\quad \times \left(1 + 2\hat{J}\begin{Bmatrix} j & j & J \\ j & j & J \end{Bmatrix}\right)^{-1/2} \end{aligned} \tag{4.6b}$$

and

$$R_k^{(J)} = [1 - |\langle 31jq|kq(J)\rangle|^2]^{-1/2}. \tag{4.6c}$$

Finally, in the situation when there exist several states with the same angular momentum k, we orthogonalize them with the Schmidt process such that

$$|kq(J)\rangle \to |kq[J]\rangle . \tag{4.7}$$

In the image space, we now define a set of ideal fermion operators $\tilde{a}_m^\dagger$ with the properties

$$\{\tilde{a}_m, \tilde{a}_{m'}^\dagger\}=0 , \tag{4.8a}$$

$$\{\tilde{a}_m, \tilde{a}_{m'}^\dagger\}=\delta_{mm'} , \tag{4.8b}$$

$$[\tilde{a}_m, \mathfrak{A}_M^{(J)\dagger}]=0 , \tag{4.8c}$$

where $\mathfrak{A}_M^{(J)\dagger}$ are the boson operators defined previously. We shall still call this space the boson space, but the physical boson states now include all the even states, plus those states which are generated from the even states by a single ideal fermion operator $\tilde{a}_m^\dagger$, i.e., the states of the form

$$|Nvkq\mu)_B \equiv S_{Nvkq\mu}[\tilde{a}^\dagger \times \tilde{O}^\dagger_{N-1,v-1,IM\omega}(\mathfrak{A}^\dagger)]_{Nvkq\mu}|0)_B , \tag{4.9}$$

where $\tilde{O}^\dagger_{N-1,v-1,IM\omega}(\mathfrak{A}^\dagger)$ generates the boson states $|N-1,\, v-1,\, IM\omega)_B$ of the even system and $\tilde{S}_{Nvkq\mu}$ is a normalization constant. Note that we have explicitly excluded any boson state with more than one ideal fermion in it.

The criterion discussed in Sec. IIC for selecting proper physical boson states is still valid for the odd system with the obvious modification to include ideal fermions as well as bosons. To prove this criterion, we only have to show that as $\Omega\to\infty$, $S_{Nvkq\mu}\to\tilde{S}_{Nvkq\mu}$, which in turn is equivalent to establishing that

$$\langle 0|a(AA\cdots A)(A^\dagger A^\dagger\cdots A^\dagger)a^\dagger|0\rangle \to {}_B(0|(\mathrm{A}\mathrm{A}\cdots\mathrm{A})(\mathrm{A}^\dagger\mathrm{A}^\dagger\cdots\mathrm{A}^\dagger)\tilde{a}\tilde{a}^\dagger|0)_B , \tag{4.10}$$

where the individual subscripts of the operators have been omitted. This can be proved with the help of the commutation relations and of the condition

$$\langle 0|aA^\dagger=0 , \tag{4.11}$$

but further details will be omitted.[18]

We can proceed then to construct the boson expansions.

B. Boson Expansions and Their Asymptotic Behavior

With the physical boson states chosen according to the criterion of the last section, the expansion of the projection operator $|0)_{B\,B}(0|$ becomes

$$|0)_{BB}(0|=1-\sum_m \tilde{a}_m^\dagger\tilde{a}_m-\sum_{JM}\mathrm{A}_M^{(J)\dagger}\mathrm{A}_M^{(J)}+0(4) . \tag{4.12}$$

The boson expansions are obtained as before by using Marumori's method.

1. *Expansion of* $a^\dagger$

We have

$$Ua_m^\dagger U^\dagger=\sum_{\substack{NvIM\omega\\ v'kq\mu}}\tilde{O}^\dagger_{N+1,v'kq\mu}|0)_B\langle N+1,v'kq\mu|a_m^\dagger|NvIM\omega\rangle_B(0|\tilde{O}_{NvIM\omega}+\sum_{\substack{Nvkq\mu\\ v'IM\omega}}\tilde{O}^\dagger_{Nv'IM\omega}|0)_B$$
$$\times\langle Nv'IM\omega|a_m^\dagger|N-1,vkq\mu\rangle_B(0|\tilde{O}_{N-1,vkq\mu} . \tag{4.13a}$$

Keeping terms up to the third order, we get

$$Ua_m^\dagger U^\dagger=\langle 11jm|a_m^\dagger|0\rangle\tilde{a}_m^\dagger|0)_{B\,B}(0|+\sum_{vJM}\langle 2vJM|a_m^\dagger|11jm'\rangle\mathfrak{A}_M^{(J)\dagger}|0)_{B\,B}(0|\tilde{a}_{m'}$$
$$+\sum_{\substack{v'k[J_1]\\ vJM}}\langle 3v'kq[J_1]|a_m^\dagger|2vJM\rangle[\tilde{a}^\dagger\times A^{(J_1)\dagger}]_q^{(k)}|0)_{B\,B}(0|A_M^{(J)}+0(4) . \tag{4.13b}$$

Substituting Eq. (4.12) into this formula, we notice however that terms like $-\tilde{a}_m^\dagger\tilde{a}_{m'}^\dagger\tilde{a}_{m'}$ always have vanishing matrix elements in the physical boson space, because all physical boson states have the form (4.9). Therefore, we can omit terms creating or annihilating more than one ideal fermion, and the above expansion becomes

$$Ua_m^\dagger U^\dagger=\tilde{a}_m^\dagger+\frac{1}{\sqrt{\Omega}}\sum_J(\hat{J})^{1/2}[\mathfrak{A}^{(J)\dagger}\times\tilde{a}]_m^{(j)}+p_{j00}\tilde{a}_m^\dagger A^{(0)\dagger}A^{(0)}+\sum_{J=0}p_{j0J}\mathfrak{A}^{(0)\dagger}[\tilde{a}^\dagger\times\mathfrak{A}^{(J)}]_m^{(j)}+K_1+K_2+0(4) , \tag{4.14}$$

with

$$[\mathfrak{a}^{(J)\dagger}\times\tilde{a}]^{(j)}_{m}\equiv\sum_{M(m')}\begin{bmatrix} J & j & j\\ M & m' & m\end{bmatrix}(-)^{j+m'}A^{(J)\dagger}_{M}\tilde{a}_{-m'} \tag{4.15a}$$

and

$$[\tilde{a}^{\dagger}\times\mathfrak{a}^{(J)}]^{(j)}_{m}\equiv\sum_{M(m')}\begin{bmatrix} j & J & j\\ m' & M & m\end{bmatrix}(-)^{M}\tilde{a}^{\dagger}_{m'}A^{(J)}_{-M}, \tag{4.15b}$$

where

$$\begin{aligned}p_{j00}&=-1+\langle 31jm\,|a^{\dagger}_{m}|2000\rangle\\&=-1+\left(1-\frac{1}{\Omega}\right)^{1/2},\end{aligned} \tag{4.16a}$$

$$\begin{aligned}p_{j0j}&=\begin{bmatrix} j & J & j\\ m' & -M & m\end{bmatrix}(-)^{M}\langle 31jm'\,|a^{\dagger}_{m}|22JM\rangle\\&=-\frac{(\hat{J})^{1/2}}{\Omega}\left(1-\frac{1}{\Omega}\right)^{-1/2}\quad (J\neq 0),\end{aligned} \tag{4.16b}$$

$$\begin{aligned}K_1&=-\tilde{a}^{\dagger}_{m}\sum_{J\neq 0,M}\mathfrak{a}^{(J)\dagger}_{M}\mathfrak{a}^{(J)}_{M}\\&=-\sum_{J\neq 0,M}\sum_{k}\begin{bmatrix} j & J & k\\ m & M & q\end{bmatrix}[\tilde{a}^{\dagger}\times\mathfrak{a}^{(J)\dagger}]^{(k)}_{M},\end{aligned} \tag{4.17a}$$

and

$$K_2\equiv\sum_{J\neq 0,M}\sum_{k[J_1]}\begin{bmatrix} j & J & k\\ m & M & q\end{bmatrix}p_{jk[J_1]J}[\tilde{a}^{\dagger}\times\mathfrak{a}^{(J_1)\dagger}]^{(k)}_{q}\mathfrak{a}^{(J)}_{M} \tag{4.17b}$$

with

$$p_{jk[J_1]J}\begin{bmatrix} j & J & k\\ m & M & q\end{bmatrix}\equiv\langle 33kq[J_1]\,|a^{\dagger}_{m}|22JM\rangle\,. \tag{4.17c}$$

2. *Expansion of* $A^{(0)\dagger}$

For $A^{(0)\dagger}$ we find after some calculation:

$$\begin{aligned}UA^{(0)\dagger}U^{\dagger}=&\sum_{NvIM\omega,\omega'}\tilde{O}_{NvIM\omega}|0)_B\langle NvIM\omega\,|A^{(0)\dagger}|N-2,IM\omega'\\&\times(0|\tilde{O}_{N-2,IM\omega'}+\sum_{Nvkqu,\mu'}\tilde{O}^{\dagger}_{Nvkq\mu}|0)_B\langle Nvkq\mu\,|A^{(0)\dagger}|N-2,vkq\mu\omega\rangle_B(0|\tilde{O}_{N-2,vkq\mu'}\\&=VA^{(0)\dagger}V^{\dagger}+\left[-1+\left(1-\frac{1}{\Omega}\right)^{1/2}\right]\mathfrak{a}^{(0)\dagger}\sum_{m}\tilde{a}^{\dagger}_{m}\tilde{a}_{m}+0(5),\end{aligned} \tag{4.18}$$

where $VA^{(0)\dagger}V^{\dagger}$ is the expansion given in Eq. (3.19).

3. *Expansion of* $A^{(J)\dagger}$ *(*$J\neq 0$*)*

In this case the new expansion is

$$\begin{aligned}UA^{(J_1)\dagger}_{M_1}=&\sum_{\substack{NvIM\omega\\ v'I'M'\omega'}}\tilde{O}^{\dagger}_{Nvkq\omega}|0)_B\langle NvIM\omega\,|A^{(J_1)\dagger}_{M_1}|N-2,v'I'M'\omega'\rangle_B(0|\tilde{O}_{N-2,v'I'M'\omega'}\\&+\sum_{\substack{Nvkq\mu\\ v'k'q'\mu'}}\tilde{O}^{\dagger}_{NvIM\mu}|0)_B\langle Nvkq\mu\,|A^{(J_1)\dagger}_{M_1}|N-2,v'k'q'\mu'\rangle_B(0|\tilde{O}_{N-2,v'k'q'\mu'}\\&=VA^{(J_1)\dagger}_{M_1}V^{\dagger}+q^{(J_1)}_{0jj}\mathfrak{a}^{(0)\dagger}[\tilde{a}^{\dagger}\times\tilde{a}]^{(J_1)}_{M_1}+R_1'+R_2'+0(5),\end{aligned} \tag{4.19}$$

with

$$[\tilde{a}^\dagger \times \tilde{a}]^{(J_1)}_{M_1} \equiv \sum_m \begin{bmatrix} j & j & J_1 \\ m' & m & M_1 \end{bmatrix} (-)^{j+m} \tilde{a}^\dagger_{m'} \tilde{a}_{-m} ,$$

$$q^{(J_1)}_{0jj} = \begin{bmatrix} j & j & J_1 \\ m' & -m & M_1 \end{bmatrix} (-)^{j-m} \langle 31jm' | A^{(J_1)\dagger}_{M_1} | 11jm \rangle$$

$$= S^{(0)}_j \left(\frac{\hat{j}}{\hat{J}_1} \right)^{1/2} \tilde{g}_j(0, J_1)$$

$$= -\left(\frac{2}{\Omega} \right)^{1/2} \left(1 - \frac{1}{\Omega} \right)^{-1/2} , \tag{4.20}$$

$$R_1' = -\mathfrak{A}^{(J_1)\dagger}_{M_1} \sum_m \tilde{a}^\dagger_m \tilde{a}_m$$

$$= -\sum_m \sum_k \begin{bmatrix} J_1 & j & k \\ M_1 & m & q \end{bmatrix} [\mathfrak{A}^{(J_1)\dagger} \times \tilde{a}^\dagger]^{(k)}_q \tilde{a}_m , \tag{4.21}$$

and

$$R_2' = \sum_m \sum_{k[J]} q^{(J_1)}_{k[J]jj} \begin{bmatrix} J & j & k \\ M & m & q \end{bmatrix} [\mathfrak{A}^{(J)\dagger} \times \tilde{a}^\dagger]^{(k)}_q \tilde{a}_m , \tag{4.22}$$

with

$$q^{(J_1)}_{k[J]jj} \begin{bmatrix} J & j & k \\ M & m & q \end{bmatrix} \equiv \langle 33kq[J] | A^{(J_1)\dagger}_{M_1} | 31jm \rangle . \tag{4.23}$$

4. *Expansion of* $B^{(J)}$

Finally the expansion of $B^{(J_1)}_{M_1}$ is generalized to

$$UB^{(J_1)}_{M_1} U^\dagger = VB^{(J_1)}_{M_1} V^\dagger + \frac{1}{(\hat{J}_1)^{1/2}} [\tilde{a}^\dagger \times \tilde{a}]^{(J_1)}_{M_1} + \gamma^{(J_1)}_{0jj0} \mathfrak{A}^{(0)\dagger} [\tilde{a}^\dagger \times \tilde{a}]^{(J_1)}_{M_1} \mathfrak{A}^{(0)} + T_1' + T_2' + T_3' + T_4' + 0(6) , \tag{4.24}$$

where

$$\gamma^{(J_1)}_{0jj0} = -\delta_{J_1 0} d^{(J_1)}_{00} (\hat{j})^{1/2} - \frac{1}{(\hat{J}_1)^{1/2}} + (-)^{j-m'} \begin{bmatrix} j & j & J_1 \\ m & -m' & M_1 \end{bmatrix} \langle 31jm | B^{(J_1)}_{M_1} | 31jm' \rangle$$

$$= -2\delta_{J_1 0} - \frac{1}{(\hat{J}_1)^{1/2}} [S^{(0)}_j]^2 \tilde{f}_{jj}(0 | J_1 | 0)$$

$$= \begin{cases} 0 \quad (J_1 \text{ odd}) \\ 2\delta_{J_1 0} \left[-1 + \left(1 - \frac{1}{\Omega} \right)^{-1} \right] - \frac{2(-)^{J_1}}{\Omega (\hat{J}_1)^{1/2}} \left(1 - \frac{1}{\Omega} \right)^{-1} \quad (J_1 \text{ even}) \end{cases}$$

$$= \begin{cases} 0 \quad (J_1 = 0 \text{ or } J_1 \text{ odd}) \\ -\frac{2(-)^{J_1}}{\Omega (\hat{J}_1)^{1/2}} \left(1 - \frac{1}{\Omega} \right)^{-1} \quad (J_1 \neq 0 \text{ even}) , \end{cases} \tag{4.25}$$

$$T_1' = -\sum_m \sum_k d^{(J_1)}_{J_1 0} \begin{bmatrix} j & J_1 & k \\ m & M_1 & q \end{bmatrix} \{ [\tilde{a}^\dagger \times \mathfrak{A}^{(J_1)\dagger}]^{(k)}_q \mathfrak{A}^{(0)} \tilde{a}_m + (-)^{M_1} \tilde{a}^\dagger_{-m} \mathfrak{A}^{(0)\dagger} [\mathfrak{A}^{(J_1)} \times \tilde{a}]^{(k)}_{-q} \} , \tag{4.26a}$$

$$T_2' = \sum_m \sum_{k[J_2]} \gamma^{(J_1)}_{jk[J_2]0j} \begin{bmatrix} j & J_2 & k \\ m & M_2 & q \end{bmatrix} [\tilde{a}^\dagger \times \mathfrak{A}^{(J_2)\dagger}]^{(k)}_q \mathfrak{A}^{(0)} \tilde{a}_m + (-)^{M_1} \tilde{a}^\dagger_{-m} \mathfrak{A}^{(0)\dagger} [\mathfrak{A}^{(J)} \times \tilde{a}]^{(k)}_{-q} , \tag{4.26b}$$

with

$$\gamma^{(J_1)}_{jk[J_2]0j}\begin{bmatrix} j & J_2 & k \\ M & M_2 & q \end{bmatrix} \equiv \langle 33kq[J_2]|B^{(J_1)}_{M_1}|31jm\rangle \,, \tag{4.26c}$$

and

$$T_3' = -\sum_{J_2,J_3\neq 0}\sum_{kk'q'}(-)^{j+k+J_1}(\hat{j}\hat{k}\hat{k}')^{1/2}\begin{bmatrix} J_1 & J_2 & J_3 \\ j & k & k' \end{bmatrix} d^{(J_1)}_{J_2J_3}(-)^{k'-q'}\begin{bmatrix} k & k' & J_1 \\ q & -q' & M_1 \end{bmatrix}[\tilde{a}^\dagger \times \mathfrak{a}^{(J_2)\dagger}]^{(k)}_q[\mathfrak{a}^{(J_3)}\times\tilde{a}]^{(k')}_{q'} \,, \tag{4.27a}$$

$$T_4' = \sum_{k[J_2]k'[J_3]q'}\gamma^{(J_1)}_{jk[J_2]k'[J_3]j}\begin{bmatrix} k & k' & J_1 \\ q & -q' & M_1 \end{bmatrix}(-)^{k'-q'}[\tilde{a}^\dagger \times \mathfrak{a}^{(J_3)\dagger}]^{(k)}_q[\mathfrak{a}^{(J_3)\dagger}\times\tilde{a}]^{(k')}_{q'} \,, \tag{4.27b}$$

with

$$\gamma^{(J_1)}_{jk[J_2]k'[J_3]j}\begin{bmatrix} k & k' & J_1 \\ q & -q' & M_1 \end{bmatrix}(-)^{k'-q'} \equiv \langle 33kq[J_2]|B^{(J_1)}_{M_1}|33k'q'[J_3]\rangle \,. \tag{4.27c}$$

We have considered the boson expansions in a system with even or odd number of nucleons occupying a single j shell. In this general case when odd systems are included, we define an ideal fermion operator $\tilde{a}^\dagger_m$ in addition to the boson operators $\mathfrak{a}^{(J)\dagger}_M$ in the image space. The criterion (2.39) for selecting physical boson states is easily generalized, and the modified Marumori method can still be used to construct the boson expansions to the lowest-order anharmonic terms. For the fermion pair and multipole operators we have seen that their expansions turn out to be just generalizations of the expansions obtained in the last chapter.

With the coefficients of these expansions given, it is easily seen that the asymptotic behavior of these expansions as $\Omega\to\infty$ is exactly the same as those given in the last chapter, and the general behavior and properties of the boson expansions discussed before are still true here.

In the next section we shall consider the construction of subspaces of the image space such that the expansions can be made convergent in them.

C. Convergent Boson Expansions Within Subspaces of the Image Space

For an odd number of nucleons in a single j shell, the physical boson space is a direct product of the space which contains the single ideal fermion and the boson space which contains the even states. Therefore, we can define a sequence of subspaces

$$F^{(n+1)}_J \equiv \tilde{a}^\dagger_j \otimes D^{(n)}_J \,, \tag{4.28}$$

with $D^{(n)}_J$ defined in Sec. III D such that $F^{(n+1)}_J$ contains all the even states within $D^{(n)}_J$, plus the odd states formed by a single ideal fermion operator $\tilde{a}^\dagger_m$ and every state in $D^{(n)}_J$. As an example, we have

$$F^{(3)}_2 \equiv D^{(2)}_2 U\{\tilde{a}^\dagger|0)_B, \tilde{a}^\dagger\mathfrak{a}^{(0)\dagger}|0)_B, [\tilde{a}^\dagger\times\mathfrak{a}^{(2)\dagger}]^{(|j-2|,\,j,\,j+2)}|0)_B\} \,. \tag{4.29}$$

From the definition we have $F^{(n+1)}_J \supset F^{(n'+1)}_{J'}$, if, and only if, $n \geq n'$ and $J \geq J'$.

If for a given j we can find n_0 and J_0 such that the physical boson space F_P contains $F^{(n_0+1)}_{J_0}$ as a subspace,

$$F^{(n_0+1)}_{J_0} \subset F_P \,, \tag{4.30}$$

then within $F^{(n_0+1)}_{J_0}$ the boson expansions are convergent. The reason is exactly the same as that discussed in the last chapter and we do not have to repeat it here.

It is perhaps unnecessary to mention here that instead of $D^{(n)}_J$, we can also use the subspaces $d^{(n)}_J$ or $D^{(n)}$, defined at the end of the last chapter, to construct the convergent subspaces for the odd system.

In summary, for a general system in a single j shell, we can always find a certain subspace of the physical boson space in which the boson expansions can be made convergent. The size of this subspace depends on the value of j.

APPENDIX A.

RECOUPLING IDENTITIES IN A SINGLE j SHELL

Some useful identities are proved in this Appendix. We shall always turn to Edmonds' book[19] for reference. First, using Edmonds' (6.4.3), we get

$$(-)^I\begin{Bmatrix} j & j & J_1 \\ j & j & J_2 \\ J_3 & J_4 & I \end{Bmatrix}=\begin{Bmatrix} J_1 & j & j \\ J_2 & j & j \\ I & J_4 & J_3 \end{Bmatrix}$$

$$=\sum_K \hat{K}\begin{Bmatrix} J_1 & J_3 & K \\ j & j & j \end{Bmatrix}\begin{Bmatrix} J_2 & J_4 & K \\ j & j & j \end{Bmatrix}\begin{Bmatrix} J_1 & J_3 & K \\ J_4 & J_2 & I \end{Bmatrix}, \tag{A1}$$

in which j is half-integral and all other angular momenta are integral.

Another identity involving the recoupling of 6-j symbols can be obtained by using Edmonds' (6.2.12):

$$\begin{Bmatrix} J_1 & J_2 & I \\ j & j & j \end{Bmatrix}\begin{Bmatrix} J_3 & J_4 & I \\ j & j & j \end{Bmatrix}=\begin{Bmatrix} I & J_1 & J_2 \\ j & j & j \end{Bmatrix}\begin{Bmatrix} j & I & j \\ J_4 & j & J_3 \end{Bmatrix}$$

$$=\sum_K (-)^{J_1+J_2+J_3+J_4+I+K}\hat{K}\begin{Bmatrix} J_1 & J_3 & K \\ j & j & j \end{Bmatrix}\begin{Bmatrix} J_2 & J_4 & K \\ j & j & j \end{Bmatrix}\begin{Bmatrix} J_1 & J_3 & K \\ J_4 & J_2 & I \end{Bmatrix}, \tag{A2}$$

where again j is half-integral and all others are integral.

Next, the identity

$$\sum_5 Y(125)Y(345)=\sum_5 Y(135)Y(245) \tag{A3}$$

for even J_1, J_2, J_3, and J_4 can be proved straightforwardly by expanding the left-hand side using the definition of Y functions, Eqs. (3.8), and then recoupling by means of Eq. (A2). We shall, however, omit the details.

Finally, with Eq. (A1) we can get an expression for the invariant sum

$$\sum_{M_1(M_4)}\sum_{M_2(M_3)}\begin{bmatrix} J_1 & J_4 & I \\ M_1 & M_4 & M \end{bmatrix}\begin{bmatrix} J_2 & J_3 & I \\ M_2 & M_3 & M \end{bmatrix}\sum_5 Y(125)Y(345)=(\hat{J}_1\hat{J}_2\hat{J}_3\hat{J}_4)^{1/2}\begin{Bmatrix} j & j & J_1 \\ j & j & J_4 \\ J_2 & J_3 & I \end{Bmatrix}. \tag{A4}$$

APPENDIX B. INVARIANT FUNCTIONS IN A SINGLE j SHELL

We will calculate some matrix elements which can be very useful in the evaluation of boson expansion coefficients. The first one we condider is

$$G(12;34)\equiv\langle 0|A_{M_1}^{(J_1)}A_{M_2}^{(J_2)}A_{M_3}^{(J_3)\dagger}A_{M_4}^{(J_4)\dagger}|0\rangle$$

$$\equiv\langle 0|A_1A_2A_3^\dagger A_4^\dagger|0\rangle, \tag{B1a}$$

which can be evaluated by using the commutation relations Eqs. (3.7): We have

$$G(12;34)=\langle 0|A_1A_3^\dagger A_2A_4^\dagger|0\rangle+\delta_{23}\langle 0|A_1A_4^\dagger|0\rangle-2\sum_5 Y(235)(\hat{J}_5)^{1/2}\langle 0|A_1B_5^\dagger A_4^\dagger|0\rangle$$

$$=\delta_{13}\delta_{24}+\delta_{14}\delta_{23}-4\sum_5 Y(235)Y(415)$$

$$=\delta_{13}\delta_{24}+\delta_{14}\delta_{23}-4\sum_5 Y(315)Y(245), \tag{B1b}$$

where we have used Eq. (A3).

Next, utilizing Eq. (A4), one has for the invariant matrix element

$$\tilde{G}_I(12;34)=\sum_{M_1(M_2)}\sum_{M_3(M_4)}\begin{bmatrix} J_1 & J_2 & I \\ M_1 & M_2 & M \end{bmatrix}\begin{bmatrix} J_3 & J_4 & I \\ M_3 & M_4 & M \end{bmatrix}G(12;34)$$

$$=\delta_{J_1J_3}\delta_{J_2J_4}+(-)^I\delta_{J_1J_4}\delta_{J_2J_3}-4\sum_{M_1M_3}\begin{bmatrix} J_1 & J_2 & I \\ M_1 & M-M_1 & M \end{bmatrix}\begin{bmatrix} J_3 & J_4 & I \\ M_3 & M-M_3 & M \end{bmatrix}\sum_{J_5}Y(315)Y(245)$$

$$=\delta_{J_1J_3}\delta_{J_2J_4}+(-)^I\delta_{J_1J_4}\delta_{J_2J_3}-4(\hat{J}_1\hat{J}_2\hat{J}_3\hat{J}_4)^{1/2}\begin{Bmatrix} j & j & J_1 \\ j & j & J_2 \\ J_3 & J_4 & I \end{Bmatrix}, \tag{B2}$$

which is a very useful identity. We can now calculate the quantity

$$[S_I^{(J_1J_2)}]^{-2} \equiv \langle 0|[A^{(J_1)}\times A^{(J_2)}]_M^{(I)}[A^{(J_1)\dagger}\times A^{(J_2)\dagger}]_M^{(I)}|0\rangle$$
$$\equiv \sum_{M_1(M_2)}\sum_{M_1'(M_2')}\begin{bmatrix} J_1 & J_2 & I\\ M_1 & M_2 & M\end{bmatrix}\begin{bmatrix} J_1 & J_2 & I\\ M_1' & M_2' & M\end{bmatrix}\langle 0|A_{M_1}^{(J_1)}A_{M_2}^{(J_2)}A_{M_1'}^{(J_1)\dagger}A_{M_2'}^{(J_2)\dagger}|0\rangle$$
$$=\tilde G_I(J_1J_2;J_1J_2)$$
$$=1+\delta_{J_1J_2}-4\hat J_1\hat J_2\begin{Bmatrix} j & j & J_1\\ j & j & J_2\\ J_1 & J_2 & I\end{Bmatrix},$$

so that

$$S_I^{(J_1J_2)}=\left[1+\delta_{J_1J_2}-4\hat J_1\hat J_2\begin{Bmatrix} j & j & J_1\\ j & j & J_2\\ J_1 & J_2 & I\end{Bmatrix}\right]^{-1/2}. \qquad \text{(B3)}$$

Another application of Eq. (B2) is to get an expression for the overlap integral

$$\langle IM(J_1J_2)|IM(J_3J_4)\rangle \equiv S_I^{(J_1J_2)}S_I^{(J_3J_4)}\langle 0|[A^{(J_1)}\times A^{(J_2)}]_M^{(I)}[A^{(J_3)\dagger}\times A^{(J_4)\dagger}]_M^{(I)}|0\rangle$$
$$=S_I^{(J_1J_2)}S_I^{(J_3J_4)}\tilde G_I(12;34). \qquad \text{(B4)}$$

From an obvious generalization of Eq. (B2)

$$\sum_{M_1(M_2)}\sum_{M_3(M_4)}\begin{bmatrix} J_1 & J_2 & I\\ M_1 & M_2 & M\end{bmatrix}\begin{bmatrix} J_3 & J_4 & I'\\ M_3 & M_4 & M'\end{bmatrix}\tilde G(12;34)=\delta_{II'}\tilde G(12;34),$$

we get the formula

$$\sum_{M_1(M_2)}\begin{bmatrix} J_1 & J_2 & I\\ M_1 & M_2 & M\end{bmatrix}G(12;34)=\begin{bmatrix} J_3 & J_4 & I\\ M_3 & M_4 & M\end{bmatrix}\tilde G_I(12;34), \qquad \text{(B5)}$$

which shows explicitly that $\tilde G_I(12;34)$ is in fact related to the reduced matrix element.

The second matrix element we want to calculate is

$$F(12|3|45)\equiv\langle 0|A_1A_2B_3A_4^\dagger A_5^\dagger|0\rangle. \qquad \text{(B6a)}$$

With Eqs. (3.6) and (3.7), this can be reduced to

$$F(12|3|45)=(-)^{M_3}\langle 0|A_1A_2A_4^\dagger B_3^\dagger A_5^\dagger|0\rangle+(-)^{M_3}\frac{2}{(\hat J_3)^{1/2}}\sum_6{}'Y(46\bar 3)\langle 0|A_1A_2A_6^\dagger A_5^\dagger|0\rangle$$
$$=(-)^{M_3}\frac{2}{(\hat J_3)^{1/2}}\sum_6{}'Y(563)\langle 0|A_1A_2A_4^\dagger A_6^\dagger|0\rangle+\frac{2}{(\hat J_3)^{1/2}}\sum_6{}'Y(643)G(12;65)$$
$$=\frac{2}{(\hat J_3)^{1/2}}\sum_6{}'[Y(653)G(12;46)+Y(643)G(12;65)], \qquad \text{(B6b)}$$

in which $\sum'$ reminds us that we are summing over even angular momenta only.

Once again we define the invariant matrix element

$$\tilde F_{II'}(12|3|45)=\sum_{M_1(M_2)}\sum_{M_4(M_5)}\sum_{M(M')}\begin{bmatrix} J_1 & J_2 & I\\ M_1 & M_2 & M\end{bmatrix}\begin{bmatrix} J_4 & J_5 & I'\\ M_4 & M_5 & M'\end{bmatrix}\begin{bmatrix} I & I' & J_3\\ M & -M' & M_3\end{bmatrix}(-)^{M'}F(12|3|45). \qquad \text{(B7)}$$

Substituting Eq. (B6b) into the above one, and using Eq. (B5), we have for the first term

$$\sum_{M_1(M_2)}\sum_{M_4(M_5)}\sum_{M(M')}\begin{bmatrix} J_1 & J_2 & I\\ M_1 & M_2 & M\end{bmatrix}\begin{bmatrix} J_4 & J_5 & I'\\ M_4 & M_5 & M'\end{bmatrix}\begin{bmatrix} I & I' & J_3\\ M & -M' & M_3\end{bmatrix}(-)^{M'}\frac{2}{(\hat J_3)^{1/2}}\sum_6{}'Y(653)G(12;46)$$
$$=\frac{2}{(\hat J_3)^{1/2}}\sum_{M_4M}\begin{bmatrix} J_4 & J_5 & I'\\ M_4 & M_5 & M'\end{bmatrix}\begin{bmatrix} I & I' & J_3\\ M & -M' & M_3\end{bmatrix}(-)^{M'}\sum_6{}'Y(653)\begin{bmatrix} J_4 & J_6 & I\\ M_4 & M_6 & M\end{bmatrix}\tilde G_I(12;46)$$
$$=-\frac{2(-)^{I+I'}}{(\hat J_3)^{1/2}}(\hat I\hat I')^{1/2}\sum_6{}'(\hat J_5\hat J_6)^{1/2}\begin{Bmatrix} J_5 & J_6 & J_3\\ j & j & j\end{Bmatrix}\begin{Bmatrix} J_5 & J_6 & J_3\\ I & I' & J_4\end{Bmatrix}\tilde G_I(12;46),$$

while the second term can be obtained by interchanging J_4 and J_5 in the above equation and multiplying by a factor $(-)^{I'}$. With all these we finally get

$$\tilde{F}_{II'}(12|3|45) = -\frac{2}{(\hat{J}_3)^{1/2}}(-)^{I+I'}(\hat{I}\hat{I}')^{1/2}\sum_6{}'\left[\begin{Bmatrix} J_5 & J_6 & J_3 \\ j & j & j \end{Bmatrix}(\hat{J}_5\hat{J}_6)^{1/2}\begin{Bmatrix} J_5 & J_6 & J_3 \\ I & I' & J_4 \end{Bmatrix}\times\tilde{G}_I(12;46) + (-)^{I'}\begin{Bmatrix} J_4 & J_6 & J_3 \\ j & j & j \end{Bmatrix}(\hat{J}_4\hat{J}_6)^{1/2}\begin{Bmatrix} J_4 & J_6 & J_3 \\ I & I' & J_5 \end{Bmatrix}\tilde{G}_I(12;56)\right]. \tag{B8}$$

A relation similar to Eq. (B5) is

$$\sum_{M_1(M_2)}\sum_{M_4(M_5)}\begin{bmatrix} J_1 & J_2 & I \\ M_1 & M_2 & M \end{bmatrix}\begin{bmatrix} J_4 & J_5 & I' \\ M_4 & M_5 & M' \end{bmatrix}F(12|3|45) = (-)^{M'}\begin{bmatrix} I & I' & J_3 \\ M & -M' & M_3 \end{bmatrix}\tilde{F}_{II'}(12|3|45). \tag{B9}$$

For the odd system, we define the function

$$\begin{aligned} g(1,2) &\equiv \langle 0|a_m A_1 A_2^\dagger a_{m'}^\dagger|0\rangle \\ &= \langle 0|A_1(\delta_{mm'} - a_{m'}^\dagger a_m)A_2^\dagger|0\rangle \\ &= \delta_{mm'}\delta_{J_1J_2} - \langle 0|[A_{M_1}^{(J_1)}, a_{m'}][a_m, A_{M_2}^{(J_2)\dagger}]|0\rangle \\ &= \delta_{mm'}\delta_{J_1J_2} - 2\begin{bmatrix} j & j & J_1 \\ m' & m_1 - m' & M_1 \end{bmatrix}\begin{bmatrix} j & j & J_2 \\ m & M_2 - m & M_2 \end{bmatrix}, \end{aligned} \tag{B10}$$

and the invariant function

$$\begin{aligned} \tilde{g}_k(1,2) &\equiv \sum_{mm'}\begin{bmatrix} j & J_1 & k \\ m & M_1 & q \end{bmatrix}\begin{bmatrix} j & J_2 & k \\ m' & M_2 & q \end{bmatrix}g(1,2) \\ &= \delta_{J_1J_2} + 2(\hat{J}_1\hat{J}_2)^{1/2}\begin{Bmatrix} j & j & J_1 \\ j & k & J_2 \end{Bmatrix}, \end{aligned} \tag{B11}$$

which satisfies the identity

$$\sum_m \begin{bmatrix} j & J_1 & k \\ m & M_1 & q \end{bmatrix}g(1,2) = \begin{bmatrix} j & J_2 & k \\ m' & M_2 & q \end{bmatrix}\tilde{g}_k(1,2). \tag{B12}$$

We also need to evaluate the function

$$\begin{aligned} f(1|2|3) &\equiv \langle 0|a_m A_1 B_2 A_3^\dagger a_{m'}^\dagger|0\rangle \\ &= (-)^{M_2}\langle 0|a_m A_1 A_3^\dagger B_2^\dagger a_{m'}^\dagger|0\rangle + (-)^{M_2}\frac{2}{(\hat{J}_3)^{1/2}}\sum_4 Y(342)\langle 0|a_m A_1 A_4^\dagger a_{m'}^\dagger|0\rangle \\ &= \frac{1}{\sqrt{2\Omega}}\begin{bmatrix} j & J_2 & j \\ m' & M_2 & m'+M_2 \end{bmatrix}\langle 0|a_m A_1 A_3^\dagger a_{m'+M_2}^\dagger|0\rangle + \frac{2}{(\hat{J}_2)^{1/2}}\sum_4{}' Y(432)g(1,4) \\ &= \frac{1}{\sqrt{2\Omega}}\begin{bmatrix} j & J_2 & j \\ m' & M_2 & m'+M_2 \end{bmatrix}g(1,3) + \frac{2}{J_2}\sum_4{}' Y(432)g(1,4), \end{aligned} \tag{B13}$$

and the invariant function defined as

$$\begin{aligned} \tilde{f}_k(1|2|3) &\equiv \sum_{m(M_1)}\sum_{m'(M_3)}\sum_{q(q')}\begin{bmatrix} j & J_1 & k \\ m & M_1 & q \end{bmatrix}\begin{bmatrix} j & J_3 & k' \\ m' & M_3 & q' \end{bmatrix}\begin{bmatrix} k & k' & J_2 \\ q & -q' & M_2 \end{bmatrix}(-)^{k'-q'}f(1|2|3) \\ &= \left(\frac{\hat{k}\hat{k}'}{J_2}\right)^{1/2}\left[(-)^{j+k'+J_2}\begin{Bmatrix} j & j & J_2 \\ k & k' & J_3 \end{Bmatrix}\tilde{g}_k(1,3) - 2(-)^{j+k}\sum_4{}'\begin{Bmatrix} J_2 & J_3 & J_4 \\ j & j & j \end{Bmatrix}(\hat{J}_3\hat{J}_4)^{1/2}\begin{Bmatrix} J_2 & J_3 & J_4 \\ j & k & k' \end{Bmatrix}\tilde{g}_k(1,4)\right], \end{aligned} \tag{B14}$$

which has the property

$$\sum_{m(M_1)}\sum_{m'(M_3)}\begin{bmatrix} j & J_1 & k \\ m & M_1 & q \end{bmatrix}\begin{bmatrix} j & J_3 & k' \\ m' & M_3 & q' \end{bmatrix}f(1|2|3) = (-)^{k'-q'}\begin{bmatrix} k & k' & J_2 \\ q & -q' & M_2 \end{bmatrix}\tilde{f}_{kk'}(1|2|3). \tag{B15}$$

*Work supported in part by the U. S. Atomic Energy Commission.

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PHYSICAL REVIEW C VOLUME 4, NUMBER 5 NOVEMBER 1971

Effect of the 7.12-MeV Level in ^{16}O on the Alpha Spectrum from ^{16}N β Decay

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(Received May 1971; revised manuscript received 10 August 1971)

Taking into account the final-state density of the leptons, a single resonance model for the $^{12}C+\alpha$ 1^- scattering amplitude predicts a markedly asymmetric peak in the α spectrum following ^{16}N β decay into continuum states of ^{16}O. The observed peak is nearly symmetric and it is shown that this symmetry can arise from destructive interference from the subthreshold 1^- state (7.12 MeV) and/or a background from states of ~17-MeV excitation in ^{16}O. It is shown that the α width of the 7.12-MeV state must be an order of magnitude smaller than that of the 1^- state (9.58 MeV).

I. INTRODUCTION

The α spectrum from the β decay of ^{16}N to continuum states of ^{16}O has been carefully investigated[1,2] because of the possibility of observing parity violation in the decay of the 2^- (8.88 MeV) into the $^{12}C+\alpha$ channel. The α spectrum displays a single peak due to transitions to the broad 1^- state at 9.58 MeV. The peak occurs about 150 keV lower because the density of final states for the leptons rises rapidly as the excitation energy is lowered from its maximum allowed value of 10.41 MeV.

The exact shape of the spectrum is of only moderate interest to the parity-violation investigations but is of paramount interest in the present work because of the prospect of obtaining information about the α width of the subthreshold 1^- (7.12-MeV) state. From the standpoint of nuclear-structure theory, the α widths of the low-lying states in ^{16}O are quantities to be explained by cluster models of these states.[3,4] The α width of the 1^- (7.12-MeV) state is also an important parameter in fixing the rate of the reaction $^{12}C(\alpha,\gamma)^{16}O$ that synthesizes ^{16}O in stellar interiors.[5] The small value of the photocapture cross section (~50 nb at peak) has so far frustrated attempts to get the width directly from the γ experiment.[6,7]

Most of the early measurements[8-10] of the shape of the α spectrum following ^{16}N decay resulted in an asymmetric peak that fell off more slowly on the low-energy side. Such a shape can be fit quite well by a single-level expression which includes the lepton phase space. The curve marked "single level" in Fig. 1 represents a good fit to most of the data. However, even in 1961 Kaufmann and Wäffler[11] found that the peak was nearly symmetric and the experiments of the past two years[1,2,12,13] have tended in this direction. The solid lines in Fig. 1 summarize these data.

It is plain that destructive interference with the 9.58-MeV state is occurring. This interference can arise from the effects of the 7.12-MeV states, as well as from all 1^- states of higher energy. (3^- states can also contribute, but the effect should be small because of the higher centripetal barrier and the absence of a 3^- state in the 7–10-MeV