where resonance behavior is not as pronounced as in the case described here.

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Exchange Currents and Soft Photons

M. K. Liou and M. I. Sobel*
Department of Physics and Institute for Nuclear Theory,
Brooklyn College of the City University of New York, Brooklyn, New York 11210

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We verify the validity of the low-energy theorem for emission of soft photons in the case of two particles which interact via an exchange potential. To lowest order in K, the photon momentum, we define the amplitude for emission from exchange currents, and show that it cancels part of the amplitude arising from internal nucleon lines. Comparison is made with the definition of exchange currents made by other authors.

When photons are emitted from a system of interacting nucleons, a part of the probability amplitude for this process must be due to exchange currents. This may be thought of as photon emission from charged mesons exchanged between two nucleons. Such a contribution will also be found if one treats the N-N scattering in terms of a phenomenological potential

$$V_N = V_N^{(A)} + \vec{\tau}_1 \cdot \vec{\tau}_2 V_N^{(B)} \tag{1}$$

with isospin dependence. $V_N^{(A)}$ and $V_N^{(B)}$ will be allowed to have a general dependence on the momentum operators $\hat{P}_1$ and $\hat{P}_2$ of the two nucleons. Thus the process of N-N bremsstrahlung can probe the inner regions of the N-N force by allowing one to observe radiation coming from exchange currents and nucleon currents.

On the other hand, the low-energy theorem for photons,[1] based on current conservation, asserts that for soft photons the bremsstrahlung amplitude depends only on the on-energy-shell amplitude for two-nucleon scattering. In particular, if the bremsstrahlung amplitude is $\langle \vec{P}_1', \vec{P}_2', \vec{K} | T | \vec{P}_1, \vec{P}_2 \rangle$, for initial (final) proton momenta $\vec{P}_1, \vec{P}_2$ ($\vec{P}_1', \vec{P}_2'$) and photon momentum $\vec{K}$, we have

$$\langle \vec{P}_1', \vec{P}_2', \vec{K} | T | \vec{P}_1, \vec{P}_2 \rangle = N\hat{e} \cdot \vec{M} \tag{2}$$

and

$$\vec{M} = \vec{A}'/K + \vec{B}' + \vec{C}' + \cdots, \tag{3}$$

where $\vec{A}'$ and $\vec{B}'$ depend only on the on-shell two-body amplitude. Here $\hat{e}$ is the photon polarization, $N = -\epsilon/(2\pi\sqrt{K})$, and ϵ is the charge. The low-energy theorem has been proved explicitly for static-potential models,[2,3] and generalized to the case of momentum-dependent potentials.[4,5] However, these discussions neglect the possibility of photon emission from the charge-exchange effects due to the exchange potential $\vec{\tau}_1 \cdot \vec{\tau}_2 V_N^{(B)}$, and the amplitude for such processes might contribute to the $\vec{B}'$ term in Eq. (3), contrary to the low-energy theorem.

We show in this paper that there must exist an exchange contribution, T_X, to the bremsstralung amplitude, which indeed contributes to the $\vec{B}'$ term, and that this cancels part of the amplitude T_R, where T_R represents the photons emitted by the internal nucleon lines. Thus the low-energy theorem remains true for an exchange potential.

First, to define our terms, we generate the electromagnetic interaction, $V^{(1)}_{\text{e.m.}}$, by making the replacement

$$\hat{P}_j \to \hat{P}_j - \epsilon\Lambda_j \vec{A}(\vec{r}_j) \tag{4}$$

in the kinetic energy part of the Hamiltonian[6]:

$$H_N = \frac{\hat{P}_1^{\,2}}{2m} + \frac{\hat{P}_2^{\,2}}{2m} + V_N(\hat{P}_1 - \hat{P}_2, \vec{r}_1 - \vec{r}_2)\,. \tag{5}$$

Here $\Lambda_j = \frac{1}{2}(1+\tau_{jz})$, $(j=1,2)$, and $\vec{A}(\vec{r}_j) = N\hat{e}e^{-i\vec{K}\cdot\vec{r}_j}$ is the vector potential. A second part of the electromagnetic potential will be called $V^{(2)}_{\text{e.m.}}$, and will be associated with photon emission from exchange currents. The total Hamiltonian is

$$\begin{aligned} H &= H_N + V^{(1)}_{\text{e.m.}} + V^{(2)}_{\text{e.m.}} \\ &= H_0 + V_N + V^{(1)}_{\text{e.m.}} + V^{(2)}_{\text{e.m.}}\,, \end{aligned} \tag{6}$$

where

$$H_0 = \frac{\hat{P}_1^{\,2}}{2m} + \frac{\hat{P}_2^{\,2}}{2m}\,.$$

We now define

$$V^{(2)}_{\text{e.m.}} = -N\hat{e}\cdot[V_N, \vec{Q}] + O(K)\,, \tag{7}$$

where

$$\vec{Q} = -i\sum_{j=1}^{2} \vec{r}_j \Lambda_j\,. \tag{8}$$

To motivate this definition we observe that, in the case of a potential with no isospin dependence, Eq. (7) becomes[7]

$$V^{(2)}_{\text{e.m.}} = N\hat{e}\cdot\sum_j (\vec{\nabla}_{\hat{P}_j} V_N)\Lambda_j + O(K)\,. \tag{9}$$

This is just the term which would be found if we made the replacement of Eq. (4) in the momentum dependence of the potential, V_N. Hence, Eq. (7) is consistent with previous treatment of what has been called the "gauge" term.[4,5] However, in general, for a potential with isospin dependence as in Eq. (1), there are additional terms arising in Eq. (7) because of noncommutation of Λ_j and $\vec{\tau}_1 \cdot \vec{\tau}_2$ which must also be considered as part of the exchange contribution. The current is conserved if we define $V^{(2)}_{\text{e.m.}}$ by Eq. (7). To show this, we shall derive the low-energy theorem for N-N bremsstrahlung.

The amplitude for bremsstrahlung is given by a total T matrix,

$$T = T_E + T_I\,, \tag{10a}$$

$$T_I = T_R + T_X\,, \tag{10b}$$

where

$$T_E = V^{(1)}_{\text{e.m.}} G_0(E)t(E) + t(E')G_0(E')V^{(1)}_{\text{e.m.}}\,, \tag{11a}$$

$$T_R = t(E')G_0(E')V^{(1)}_{\text{e.m.}} G_0(E)t(E)\,, \tag{11b}$$

$$T_X = [1 + t(E')G_0(E')]V^{(2)}_{\text{e.m.}}[1 + G_0(E)t(E)]\,. \tag{11c}$$

Here t is the two-body transition matrix, $G_0(E) = (E - H_0 + i\epsilon)^{-1}$, and E (E') is the initial (final) two-nucleon energy. The properties of T_E, the external scattering matrix, are well known: It is the only part that contributes to $\vec{A}'$ in Eq. (3), and its contribution to $\vec{B}'$ contains some off-shell behavior. What we shall show is that the contribution of T_X to the $\vec{B}'$ term in Eq. (3) is just that necessary to cancel a portion of T_R which has previously not been isolated. And if the cancellation did not occur, the low-energy theorem would be violated. Put another way, a calculation based on the exact treatment of T_E plus T_R would be incorrect, because of the violation of current conservation.

To see this we proceed to analyze T_R by employing the operator identity

$$t(E')G_0(E')[G_0^{-1}(E')\vec{Q} - \vec{Q}G_0^{-1}(E)]G_0(E)t(E) \equiv \vec{Q}t(E) - t(E')\vec{Q} + [1 + t(E')G_0(E')][V_N, \vec{Q}][1 + G_0(E)t(E)]\,, \tag{12}$$

which holds for any operator $\vec{Q}$. We note that

$$\vec{M}_R = \sum_{j=1}^{2} \langle \vec{P}_1', \vec{P}_2' | t(E')G_0(E')\frac{\hat{P}_j}{m}\Lambda_j G_0(E)t(E)|\vec{P}_1, \vec{P}_2\rangle + O(K)\,. \tag{13}$$

In previous discussions of bremsstrahlung,[4,5] the proton and neutron have been considered as different particles, and so the projection operator Λ_j has not been included in Eq. (13). In these treatments we constructed $\vec{M}_R$ on the left side of this identity by taking $\vec{Q} = -\sum_j i\vec{r}_j$. In the present case we take $\vec{Q}$ as given in Eq. (8), and we can verify that the left side of the identity equals $\vec{M}_R$. Thus $\vec{M}_R$ becomes equal to the matrix element of the right side of Eq. (12) between plane-wave two-nucleon states:

$$\vec{M}_R = \vec{M}_{R0} + \vec{M}_{R1} + O(K) \tag{14}$$

with

$$\vec{M}_{R0} = \langle \vec{P}_1', \vec{P}_2' | \vec{Q} t(E) - t(E')\vec{Q} | \vec{P}_1, \vec{P}_2 \rangle, \tag{15a}$$

$$\vec{M}_{R1} = \langle \vec{P}_1', \vec{P}_2' | [1 + t(E')G_0(E')][V_N, \vec{Q}][1 + G_0(E)t(E)] | \vec{P}_1, \vec{P}_2 \rangle. \tag{15b}$$

We postpone discussion of $\vec{M}_{R0}$. Looking at $\vec{M}_{R1}$, we observe that it precisely cancels the corresponding amplitude $\vec{M}_X$; i.e.,

$$\vec{M}_X + \vec{M}_{R1} = 0. \tag{16}$$

Here $\vec{M}_X$ is related to T_X by

$$\langle \vec{P}_1', \vec{P}_2' | T_X | \vec{P}_1, \vec{P}_2 \rangle = N\hat{e} \cdot \vec{M}_X,$$

with T_X given by Eqs. (11c) and (7). It will be observed that the terms $\vec{M}_X$ and $\vec{M}_{R1}$ occur only in the case of neutron-proton scattering.

Using Eqs. (14) and (16), the internal scattering amplitude becomes

$$\vec{M}_I = \vec{M}_R + \vec{M}_X = \vec{M}_{R0} + O(K), \tag{17a}$$

and the total amplitude can be written as

$$\vec{M} = \vec{M}_E + \vec{M}_I = \vec{M}_E + \vec{M}_{R0} + O(K). \tag{17b}$$

The fact that the off-shell terms cancel out to order K^0 was shown in Ref. 3, and we repeat the results here for completeness, and to indicate how the isotopic projection operators, Λ_j, appear. The results for $\vec{M}_{R0}$ can be written in terms of half-off-shell elements of t by replacing $-i\vec{r}_j$ in $\vec{Q}$ by $\vec{\nabla}_\alpha e^{-i\vec{\alpha}\cdot\vec{r}_j}|_{\vec{\alpha}=0}$. Then, a term such as $-i\langle \vec{P}_1', \vec{P}_2' | t(E')\vec{r}_1 | \vec{P}_1, \vec{P}_2 \rangle$ becomes $\vec{\nabla}_\alpha \langle \vec{P}_1', \vec{P}_2' | t(E') | \vec{P}_1 - \vec{\alpha}, \vec{P}_2 \rangle |_{\vec{\alpha}=0}$. We introduce the scalar variables ν, u, and Δ_i (Δ_f), which are, respectively, the average of the initial and final kinetic energies in the center-of-mass system, the square of the momentum transfer, and the amount that the initial (final) state is off the energy shell. We obtain

$$\vec{M}_{R0} = -\frac{1}{m}\bigg[\left(-\tfrac{1}{2}\vec{q}_i\frac{\partial t}{\partial \nu} + m\vec{q}\frac{\partial t}{\partial u} + \vec{P}_1\frac{\partial t}{\partial \Delta_i}\right)\Lambda_1 + \left(\tfrac{1}{2}\vec{q}_i\frac{\partial t}{\partial \nu} - m\vec{q}\frac{\partial t}{\partial u} + \vec{P}_2\frac{\partial t}{\partial \Delta_i}\right)\Lambda_2 - \Lambda_1\left(\tfrac{1}{2}\vec{q}_f\frac{\partial t}{\partial \nu} + m\vec{q}\frac{\partial t}{\partial u} - \vec{P}_1'\frac{\partial t}{\partial \Delta_f}\right)$$
$$- \Lambda_2\left(-\tfrac{1}{2}\vec{q}_f\frac{\partial t}{\partial \nu} - m\vec{q}\frac{\partial t}{\partial u} - \vec{P}_2'\frac{\partial t}{\partial \Delta_f}\right)\bigg] + O(K). \tag{18}$$

Here $\vec{q}_i = \frac{1}{2}(\vec{P}_1 - \vec{P}_2)$, $\vec{q}_f = \frac{1}{2}(\vec{P}_1' - \vec{P}_2')$, $\vec{q} = \vec{q}_f - \vec{q}_i$, and the derivatives of the t functions are evaluated at $\nu = (q_i^2 + q_f^2)/2m$, $u = q^2$, and $\Delta_i = \Delta_f = 0$. The external scattering amplitude can similarly be written in the following way[8]:

$$\vec{M}_E = \frac{\vec{P}_1'}{P_{1\mu}'K_\mu}\Lambda_1\left(t + \frac{\vec{q}_f\cdot\vec{K}}{2m}\frac{\partial t}{\partial \nu} + \vec{q}\cdot\vec{K}\frac{\partial t}{\partial u}\right) + \frac{\vec{P}_2'}{P_{2\mu}'K_\mu}\Lambda_2\left(t - \frac{\vec{q}_f\cdot\vec{K}}{2m}\frac{\partial t}{\partial \nu} - \vec{q}\cdot\vec{K}\frac{\partial t}{\partial u}\right) - \frac{\vec{P}_1}{P_{1\mu}K_\mu}\left(t - \frac{\vec{q}_i\cdot\vec{K}}{2m}\frac{\partial t}{\partial \nu} + \vec{q}\cdot\vec{K}\frac{\partial t}{\partial u}\right)\Lambda_1$$
$$- \frac{\vec{P}_2}{P_{2\mu}K_\mu}\left(t + \frac{\vec{q}_i\cdot\vec{K}}{2m}\frac{\partial t}{\partial \nu} - \vec{q}\cdot\vec{K}\frac{\partial t}{\partial u}\right)\Lambda_2 + \frac{1}{m}(\vec{P}_1'\Lambda_1 + \vec{P}_2'\Lambda_2)\frac{\partial t}{\partial \Delta_f} + \frac{1}{m}\frac{\partial t}{\partial \Delta_i}(\vec{P}_1\Lambda_1 + \vec{P}_2\Lambda_2) + O(K). \tag{19}$$

Thus, adding $\vec{M}_E$ and $\vec{M}_{R0}$, we find the total amplitude $\vec{M}$ depends only on the on-shell t matrix and its on-shell derivatives, with the off-shell derivatives precisely canceling out. With the inclusion of contributions from exchange currents corresponding to $V^{(2)}_{\text{e.m.}}$ defined by Eq. (7), we verify that the low-energy theorem holds. Nucleon spin has not been included in our discussion, but can be included in the same way as in previous derivations.[3]

It is of interest to compare our results with the result recently found by Ohtsubo, Fujita, and Takeda.[9] These authors obtain the two-body exchange current from the low-energy theorem as derived from field theory[1,10] rather than potential theory. They show that in the static limit the exchange current can be written in terms of the static nuclear potential as

$$\vec{J}^{(2)}(\vec{r}) = -i\epsilon[\sum_j \Lambda_j\vec{r}_j\delta(\vec{r} - \vec{r}_j), V_N(\vec{r}_1 - \vec{r}_2)]. \tag{20}$$

This is the exchange current obtained by the usual Siegert theorem.[11] From the point of view of this paper, the exchange current will be related to the internal scattering amplitude $\vec{M}_I$. Using Eqs. (17a) and (15a), and making the Born approximation, $t \to V_N$, we find[12]

$$\epsilon\vec{M}_I = -i\epsilon\langle \vec{P}_1', \vec{P}_2' | [\sum_j \Lambda_j \vec{r}_j, V_N(\vec{r}_1 - \vec{r}_2)] | \vec{P}_1, \vec{P}_2\rangle + O(K).$$

Translating $\epsilon\vec{M}_I$ into an equivalent current we obtain Eq. (20).

In summary, we have shown how one can derive the electromagnetic Hamiltonian arising from a general potential which contains the exchange force and momentum dependence. Using this Hamiltonian, we have derived the low-energy theorem and the exchange current to order K^0. The exchange current is essential for conserving current. It will be important when there is a $\vec{\tau}_1 \cdot \vec{\tau}_2$ term in the potential, as in the one-pion-exchange potential.

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[7]An alternative derivation can be obtained from the Appendix of Ref. 4. It has been shown that

$$\vec{F}_0 + \vec{G} = 0,$$

where

$$[V_N, Q_0] = \vec{K}\cdot\vec{F}_0, \quad \langle \vec{K}|V^{(2)}_{\text{e.m.}}|0\rangle = N\hat{e}\cdot\vec{G}, \quad Q_0 = e^{-i\vec{K}\cdot\vec{r}_1}.$$

We can rewrite this result in the form

$$\langle \vec{K}|V^{(2)}_{\text{e.m.}}|0\rangle = -N\hat{e}\cdot\vec{\nabla}_K[V_N, Q_0]|_{\vec{K}=0} + O(K),$$

$$= -N\hat{e}\cdot[V_N, -i\vec{r}_1] + O(K).$$

If we replace $(-i\vec{r}_1)$ in the above commutator by $Q = -i\sum_{j=1}^2 \vec{r}_j\Lambda_j$, we obtain Eq. (7).

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$$N\hat{e}\cdot\vec{M}_X^{(0)} = \langle \vec{P}_1', \vec{P}_2'|V^{(2)}_{\text{e.m.}}|\vec{P}_1, \vec{P}_2\rangle.$$

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Sign of δ, the Amplitude Mixing Ratio of Gamma Transitions, and Beta-Gamma-Gamma and Gamma-Gamma-Gamma Angular-Correlation Studies

B. P. Singh, H. S. Dahiya, and U. S. Pande
Physics Department, University of Roorkee, Roorkee, India
(Received 22 December 1970)

The method of β-γ-γ and γ-γ-γ angular correlation is used to determine the sign of δ (the amplitude mixing ratio in γ-ray transitions independent of phase conventions).

If the uncertainty in the description of the sign of δ $(\equiv\langle I\| L_1' \pi_1' \| I_i\rangle / \langle I \| L_1 \pi_1 \| I_i\rangle)$,[1] the amplitude mixing ratio of γ transitions, could be eliminated, then it could be compared with theory. The sign of δ depends on whether δ is determined by the first or the second γ ray of the γ-γ cascade as reported by Ofer[2] for the first time and this has been confirmed in many cases.[3] There are many cascades of three γ rays and if the middle γ ray is a dipole-quadrupole mixture, the opposite sign of δ is obtained by doing angular-correlation studies of the first and second γ rays as compared with angular-correla-