Pion charge exchange and the optical theorem

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(Received 21 October 1988)

We give some applications of the optical theorem (and its extensions) to charge exchange reactions of pions between isobaric analog states of nuclear targets, based on isospin invariance. We then derive an inequality, valid in the strong absorption limit, to give a bound on reaction channels which may be contributing to the isotensor optical potential, at pion energies near the 3,3 resonance. We further show that coupling to pion absorption on $T=1$ pairs may provide a major component of this potential.

We consider elastic, single charge-exchange (SCX), and double charge-exchange (DCX) scattering to isobaric analog states. The main purpose of this paper is to present a discussion of the dynamical content of the newly isolated isotensor contribution to pion DCX. The arguments are based on the optical theorem, and are independent of many details of the underlying mechanisms.

Let $\boldsymbol{\phi}$, $\mathbf{T}$ denote isospin operators of the pion and target. For interactions which conserve isospin, the scattering amplitude may be written as an operator in the isospin space of the pion and target as

$$f=f_0+f_1(\boldsymbol{\phi}\cdot\mathbf{T})+f_2(\boldsymbol{\phi}\cdot\mathbf{T})^2 , \tag{1}$$

where f and f_i are functions of beam energy and scattering angle. (Unless otherwise specified, we shall use f, f_i to denote the forward scattering amplitude.) The usual elastic and charge-exchange amplitudes are obtained from Eq. (1) by taking matrix elements

$$f_{c'c}\equiv\langle c',T_3+c-c'|f|c,T_3\rangle , \tag{2}$$

with c the initial pion charge $(1,0,-1)$ and T_3 the target isospin projection. Normally, $T_3=-T=(Z-N)/2$ for the ground-state multiplet. For example, elastic scattering, SCX, and DCX amplitudes for an incident π^+ are given by

$$f_{11}=f_0-Tf_1+T(T+1)f_2 , \tag{3a}$$

$$f_{01}=\sqrt{T}(f_1-Tf_2) , \tag{3b}$$

$$f_{-11}=\sqrt{T(2T-1)}f_2 . \tag{3c}$$

The optical theorem for each pion charge state c is given by

$$\mathrm{Im}f_{cc}=\frac{k}{4\pi}\sigma_T(c) , \tag{4}$$

with $\hbar k$ the beam momentum and $\sigma_T(c)$ the total cross section for that charge channel.

Since the charge-state amplitudes $f_{c'c}$ are all determined by the isospin amplitudes f_0, f_1, and f_2, the three relations implied by Eq. (4) can be used to express the $\mathrm{Im}f_{01}$ and $\mathrm{Im}f_{-11}$ in terms of the total cross sections $\sigma_T(1)$, $\sigma_T(0)$, and $\sigma_T(-1)$. $\sigma_T(0)$ is not, however, experimentally accessible, but a useful inequality may still be obtained from Eqs. (1)–(4), which only relates (almost) measurable quantities. We first obtain the result[1]

$$\mathrm{Im}f_{01}=\frac{1}{2\sqrt{T}}\left[\frac{k}{4\pi}\right][\sigma_T(-1)-\sigma_T(1)] -\frac{\sqrt{2T-1}}{2}\mathrm{Im}f_{-11} . \tag{5a}$$

Recognizing that $|f_{-11}|\ll|f_{01}|$ (generally by a factor of 30 or so), this equation compares the forward SCX amplitude to the difference of total cross sections, along with a small correction for DCX (which is generally smaller than the experimental errors in the other quantities). The full SCX amplitude is not measured, only the modulus, so that we must use

$$\frac{d\sigma}{d\Omega}(0)_{\mathrm{SCX}}\geq|\mathrm{Im}f_{01}|^2 \tag{5b}$$

to obtain an inequality from Eq. (5a), which we write

$$\left[\frac{d\sigma}{d\Omega}(0)_{\mathrm{SCX}}\right]^{1/2}\geq\frac{1}{2\sqrt{T}}\left[\frac{k}{4\pi}\right]|\sigma_T(-1)-\sigma_T(1)| , \tag{6}$$

omitting the small correction for DCX. This relation can be compared to data, as shown in Table I, for experiments at 165 MeV. Note that the comparisons involve targets differing by a few mass units, but similar (or identical) values of T, since the SCX and σ_T have not been measured on the same targets. We neglect Coulomb and related charge-dependent corrections, which are minimized at energies near the 3,3 resonance, since they are out of phase with, and are estimated to be smaller than, the charge-independent contributions to Eq. (6). The data of Table I are consistent with the inequality Eq. (6), within the rather large experimental uncertainties.[4] Clearly these results are to be considered qualitative and are given simply to illustrate the possible use of Eq. (6). Note that the data are also consistent with $\mathrm{Im}f_{01}>0$, which should be true in the resonance region, as will be discussed later.

To proceed further with the treatment of DCX scattering, we introduce an optical potential U. Again, isospin invariance restricts the form of U to

$$U=U_0+U_1(\boldsymbol{\phi}\cdot\mathbf{T})+U_2(\boldsymbol{\phi}\cdot\mathbf{T})^2 . \tag{7}$$

The potential U appears in the equation of motion

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TABLE I. A comparison of the measured quantities (in fm) involved in the inequality given by Eq. (6).

Target (T)	$\left[\frac{d\sigma}{d\Omega}(0)_{SCX}{}^{a}\right]^{1/2}$	Target (T)	$\frac{1}{2\sqrt{T}}\left[\frac{k}{4\pi}\right][\sigma_T(-1)-\sigma_T(+1)]^{b}$
^{60}Ni (2)	0.29±0.02	^{56}Fe (2)	0.10±0.44
^{90}Zr (5)	0.30±0.02	^{93}Nb ($\frac{11}{2}$)	0.52±0.76
^{208}Pb (22)	0.42±0.11	^{209}Bi ($\frac{43}{2}$)	1.12±0.70

[a]Reference 2.
[b]Reference 3.

(Klein-Gordon) for the pion in the elastic and isobaric analog channels, and fully determines the scattering amplitudes given by Eqs. (1) or (2).

There has been considerable interest recently in determining the isospin structure of the optical potential in the resonance energy region. Multiple scattering theory predicts that to first order in the nuclear density of the target, the potential will contain isoscalar and isovector terms only. The quadratic (isotensor) term involves terms that are second order in the isovector density, and these terms also provide corrections to the first-order parts of U_0 and U_1. The second-order terms are dependent on correlations, pion absorption, and other phenomena which have made them hard to predict reliably. It will be useful for us to write $U=\overline{U}+U_2(\boldsymbol{\phi}\cdot\mathbf{T})^2$, where

$$\overline{U}=U_0+U_1(\boldsymbol{\phi}\cdot\mathbf{T}) \ . \tag{8}$$

It should be noted that both isovector and isotensor parts of U may contribute to DCX: U_1 to second order through successive SCX via analog states, and U_2 to first order directly (plus higher order terms).

Recent empirical studies of DCX have shown some interesting features of the structure of U. It has been known for some time[5] that potentials of the form of Eq. (8), with U_1 only having terms *linear* in the target isovector density, do not predict the correct angular distribution. Experiments at 164 MeV show diffraction minima at $\sim 20°$, which is a considerably smaller angle than these potentials will produce. This has been taken as evidence for the existence of an isotensor (U_2) term, to provide sufficient interference with the contribution to DCX from the U_1 term in order to give the correct angular distribution. Based on a theoretical model by Johnson and Siciliano,[6] Greene *et al.*[7] have performed fits to the parametric forms of U_2. By a systematic study of elastic, SCX and DCX data at ~ 165 MeV for a variety of targets, they have obtained a set of values for the parameters that provide reasonable agreement with existing data for analog states. In order to gain some insight into the physical origins of the resulting values of their parameters (and especially their signs), it is useful to obtain some general results based upon the optical theorem and the eikonal approximation.

It is well known that $\mathrm{Im}U$ reflects the loss of elastic flux into open reaction channels. In fact, the reaction cross section can be expressed through an extended optical theorem[8]

$$\sigma_r=-2v^{-1}\int \mathrm{Im}U(\mathbf{r})|\psi^{(+)}(\mathbf{r})|^2d^3r \ , \tag{9}$$

where $\psi^{(+)}$ is the distorted elastic (optical) wave and v is the pion velocity (assuming that U is local).[9] This allows us to interpret the isotensor potential U_2 in terms of reaction channels coupled to its imaginary part.

We write the full elastic channel optical potential $U_c=\overline{U}_c+\Delta U_c$ with $\overline{U}_c$ given by

$$\overline{U}_c=U_0+U_1\langle c|\boldsymbol{\phi}\cdot\mathbf{T}|c\rangle \ , \tag{10a}$$

and

$$\Delta U_c=U_2\langle c|(\boldsymbol{\phi}\cdot\mathbf{T})^2|c\rangle \ . \tag{10b}$$

In order to consider the contribution of ΔU_c to Eq. (9), we define the quantity $\sigma_c^{(2)}$ by

$$\sigma_c^{(2)}\equiv-2v^{-1}\int \mathrm{Im}\Delta U_c|\psi_c^{(+)}|^2d^3r \ , \tag{11}$$

which is positive for $\mathrm{Im}\Delta U_c<0$ (or $\mathrm{Im}U_2<0$). Equation (11) represents that part of the reaction cross section that couples to the $(\boldsymbol{\phi}\cdot\mathbf{T})^2$ part of the optical potential, due to dynamical processes not included in $\overline{U}$. These presumably involve two or more target nucleons, and have an isospin structure that gives the $(\boldsymbol{\phi}\cdot\mathbf{T})^2$ form. Two examples which might be expected to contribute significantly to this term are pion absorption on $T=1$ pairs and double scattering from $T=1$ correlated pairs (uncorrelated double scattering being included in $\overline{U}$). Models of these reactions are under study.

We now examine the constraints that the abovementioned relationships impose on the optical potentials used in pion-nucleus scattering and charge-exchange reactions. The empirical optical potentials have some interesting features in the present context. In the resonance region, the potentials are largely imaginary, as one would expect from simple multiple-scattering arguments involving the 3,3 πN resonance. Based on the isospin of the resonance ($T=\frac{3}{2}$), first-order theory gives a potential of the form in Eq. (8) with $\mathrm{Im}U_0<0$, and $\mathrm{Im}U_1<0$. Calculations using such first-order potentials result in a forward amplitude for SCX that turns out to have $\mathrm{Im}f_{01}>0$, which agrees with the sign given by Eq. (5a) using the data of Table I. The DCX forward amplitude for this potential is also found to have $\mathrm{Im}f_{-11}<0$. These signs can be understood from the following simple eikonal argument.

The forward amplitude in the eikonal approximation can be written [for a local $U(\mathbf{r})$] as

$$f=-ik\int_0^\infty b\,db(e^{i\chi(b)}-1) \ , \tag{12a}$$

with phase function χ defined by

$$\chi(b)=-v^{-1}\int_{-\infty}^{\infty}U(b,z)dz\ , \tag{12b}$$

where v ($=k/\omega$) is the pion velocity.

This phase may be expanded, as in Eq. (7), to give

$$\chi(b)=\chi_0+\chi_1(\boldsymbol{\phi}\cdot\mathbf{T})+\chi_2(\boldsymbol{\phi}\cdot\mathbf{T})^2\ . \tag{13}$$

For the potential Eq. (8), $\chi_2=0$, and we find to lowest order in χ_1 that

$$\overline{f_1}=k\int_0^{\infty}b\,db\,e^{i\chi_0(b)}\chi_1(b)\ , \tag{14a}$$

$$\overline{f_2}=\tfrac{1}{2}ik\int_0^{\infty}b\,db\,e^{i\chi_0(b)}\chi_1^2(b)\ . \tag{14b}$$

For purely imaginary U_1 and U_0, with $\mathrm{Im}U_1<0$, we thus find purely imaginary forward amplitudes, with $\mathrm{Im}\overline{f}_1$ (or $\mathrm{Im}\overline{f}_{01}$) >0, and $\mathrm{Im}\overline{f}_2$ (or $\mathrm{Im}\overline{f}_{-11}$) <0.

If we now include the quadratic term U_2 in U and write the full isotensor amplitude as $f_2=\overline{f}_2+\Delta f_2$, we find that no lowest order in χ_1 and χ_2,

$$\Delta f_2=k\int_0^{\infty}b\,db\,e^{i\chi_0(b)}\chi_2(b)\ . \tag{14c}$$

Again, this amplitude is purely imaginary for imaginary U_2 (and U_0). In order to provide interference and thus move the minimum to more forward angles, Δf_2 should be *opposite in phase* to $\overline{f}_2$, that is, $\mathrm{Im}\Delta f_2>0$. *This means that $\mathrm{Im}U_2$ must be negative.* This places a constraint on any specific reaction model that may be proposed as being the source of the U_2 term. This predicted sign of U_2 also agrees with the phenomenology of Greene *et al.*, who find that the best fit U_2 is largely imaginary and negative. (Actually, the fits of Greene *et al.* involve an energy shift that moves them considerably off resonance. As a result, their potential is not purely imaginary but is approximately of the form $\mathrm{Re}U_0\simeq-\mathrm{Im}U_0$ and $\mathrm{Re}U_1\simeq-\mathrm{Im}U_1$. While this complicates the algebra slightly, the final predictions concerning the DCX amplitudes and their confirmation by the eikonal analysis are not changed.)

We next show how to use the extended optical theorem Eq. (9) and the DCX amplitudes (in the eikonal approximation) to put a bound on the reaction cross section $\sigma_c^{(2)}$. For simplicity, we shall assume purely imaginary optical potentials in Eqs. (7) or (10). The eikonal wave function for this case can then be rewritten as

$$\psi_c^{(+)}(b,z)=e^{ikz}e^{-[\omega(b,z)+\kappa(b)]/2}\ , \tag{15}$$

with the limiting values $\psi_c^{(+)}\to e^{ikz}$ for $z\to-\infty$, and $\psi^{(+)}\to e^{ikz-\kappa(b)}$ for $z\to+\infty$. ω and κ are real functions and $i\kappa(b)=\chi_c(b)$. [We take $z=0$ as the symmetry plane of the target, with $\omega(b,z)=-\omega(b,-z)$, and $\omega(b,0)=0$.] Setting Eq. (15) into Eq. (11), we write

$$\begin{aligned}\sigma_c^{(2)}&=\frac{4\pi}{v}\int_0^{\infty}b\,db\int_{-\infty}^{\infty}dz\,|\Delta U_c|e^{-\omega(b,z)-\kappa(b)}\\&=\frac{4\pi}{v}\int_0^{\infty}b\,db\,e^{-\kappa(b)}\\&\qquad\times\int_0^{\infty}dz\,|\Delta U_c|(e^{-\omega(b,z)}+e^{\omega(b,z)})\\&\geq\frac{4\pi}{v}\int_0^{\infty}b\,db\,e^{-\kappa(b)}\int_{-\infty}^{\infty}dz\,|\Delta U_c|\ ,\end{aligned} \tag{16}$$

where we have assumed the symmetry $\Delta U_c(b,z)=\Delta U_c(b,-z)$.

For comparison we find

$$\begin{aligned}\mathrm{Im}\Delta f_{cc}&=\mathrm{Im}\Delta f_2\langle c|(\phi\cdot T)^2|c\rangle\\&=\frac{k}{v}\int_0^{\infty}b\,db\,e^{i\chi_0(b)}\int_{-\infty}^{\infty}dz\,|\Delta U_c|\ ,\end{aligned} \tag{17}$$

where we use Eqs. (10b), (12b), and (14c). Working to lowest order in χ_0, χ_1, and χ_2, we set $\kappa(b)\simeq-i\chi_0(b)$ in Eq. (16). Now comparing Eqs. (16) and (17), we obtain the inequality

$$\sigma_c^{(2)}\geq\frac{4\pi}{k}\mathrm{Im}\Delta f_{cc}\ , \tag{18}$$

which gives a lower bound to $\sigma_c^{(2)}$. This inequality is modified in the presence of real optical potentials, but the changes will be small for $|\mathrm{Re}U|\ll|\mathrm{Im}U|$.

As mentioned following Eq. (11), $\sigma_c^{(2)}$ depends on ΔU_c, which in turn reflects isospin-dependent coupling to nonelastic channels. Suppose we label the final channels by a, with projection operator on the channel space, Q_a. Then a possible contribution to ΔU_c could be of the form (using the Feshbach formalism)

$$\Delta U_c(a)=\langle c|V^+Q_a(E-H_a+i\eta)^{-1}Q_aV|c\rangle\ , \tag{19}$$

where V connects the elastic (and isobaric analog) channel to a: e.g., $V=v(T\cdot\phi)$, and the channel interaction H_a is assumed to be isoscalar, for simplicity. (This is not the most general form.) Then it is easy to show that $\sigma_c^{(2)}$ calculated from Eq. (11), using Eq. (19), is simply the nonelastic cross section into channel a. This cross section may be a directly measurable quantity, if the final channel a is distinct from those involved in the $\overline{U}_c$ part of the optical potential. In that case, its contribution to $\sigma_c^{(2)}$ can be directly compared to $(4\pi/k)\mathrm{Im}\Delta f_{cc}$ using optical model analysis to obtain Δf_{cc}. The inequality Eq. (18) should be satisfied for the channels giving the dominant contributions to $\sigma_c^{(2)}$.

As a test case, we discuss the cross section for pion absorption with np emission. The reaction $(\pi^{\pm},np)$, where measured, is known to have much smaller cross sections than (π^+,pp) and (π^-,nn). In a pair-absorption model, this means that absorption on $T=0$ (np) nucleon pairs in the target dominates over absorption on $T=1$ pairs, e.g. $(nn$ or $pp)$. The absorption on (nn) pairs can contribute to the isotensor optical potential. Let us see to what extent the inequality Eq. (18) is satisfied, if we identify $\sigma_c^{(2)}=\sigma(\pi^+,np)$ (with $c=1$).

Altman *et al.*[10] measure the ratio R_+ of correlated pp to np pairs for π^+ absorbed by ^{18}O, for fixed proton angles and find $R_+ = 27.0 \pm 5.0$ for 165 MeV. If we assume the integrated cross sections also scale with R_+, we estimate

$$\sigma(\pi^+, np) \simeq \sigma_{\text{abs}}/(1+R_+) \sim 9.0 \text{ mb} , \quad (20)$$

using the total absorption cross section $\sigma_{\text{abs}} = 252$ mb. The fraction of Eq. (20) attributable to the valence pair may be estimated to be $\sim \frac{1}{5}$, counting zero-range *nn* pairs in ^{18}O. Alternatively, one may use the ratios of $\sigma(\pi^+, np)$ on ^{18}O/^{16}O, measured by Altman *et al.*[11] which gives an estimate of $\frac{1}{6}$ for the fraction. The result yields

$$\sigma(\pi^+, np) \simeq 1.5 - 1.8 \text{ mb} , \quad (21)$$

where we further assume the coupling to be all isotensor, as in Eq. (19).

The optical model calculations of Greene *et al.* give us[12] numerical values for the separate amplitudes $\bar{f}_{c'c}$ and $\Delta f_{c'c}$, from which we obtain (for π^+ on ^{18}O at 165 MeV)

$$\frac{4\pi}{k} \text{Im}\Delta f_{11} = 4.8 \text{ mb} . \quad (22)$$

We find the quantities Eqs. (21) and (22) are similar in magnitude. With the identification[13] $\sigma(\pi^+, np) = \sigma_c^{(2)}$, the inequality Eq. (18) is not satisfied by (our estimate) of absorption. [Although the data are not precisely on resonance, we do not expect this to change Eq. (18) substantially.]

We conclude from this that absorption on *nn* pairs is consistent with a sizable fraction (e.g., $\frac{1}{3}$) of Eq. (22), which is contributed by the isotensor optical potential. Other inelastic channels may also contribute, notably the correlated double scattering from $T = 1$ pairs. Investigation of these mechanisms in specific models is underway and will be reported more fully elsewhere.

We thank E. Siciliano for supplying us with amplitudes from the work of Ref. 7, and M. Johnson for several discussions on the topics treated here. This research was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-88ER40425 with the University of Rochester.

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[8]See, e.g., M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), p. 786.

[9]The assumption of locality is consistent with use of the eikonal approximation [see text at Eq. (12)] and is adequate for forward scattering.

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[12]E. R. Siciliano (private communication).

[13]The coefficient need not be unity, as will be shown elsewhere.