Shape transition and dynamical symmetries in the interacting boson model

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A geometric interpretation of effective interacting boson approximation Hamiltonians describing series of isotopes is carried out. It is shown that a recently introduced SO(5) dynamical symmetry Hamiltonian, which describes the Ru region with fixed parameters for all isotopes, gives rise to a sharp shape transition from spherical to γ-unstable nuclei. Moreover, it is shown that the SO(6), SU(3), and U(5) interacting boson approximation symmetries may display transitional behavior when studied as a function of boson number. A set of constraints in the parameter space are derived, in order for the standard geometrical interpretation of these symmetries to hold.

I. INTRODUCTION

In recent years the use of dynamical symmetry Hamiltonians has brought about significant developments in nuclear structure physics. The original SU(5), SU(3), and SO(6) dynamical symmetries of the interacting boson approximation (IBM) have played a pivotal role in these developments, given their simplicity and strong appeal arising from their group theoretical properties.[1] Even if these exact solutions are seldom manifested in nuclear structure—and then only approximately—they have provided reference points for calculations involving more general Hamiltonians.[1] This has been supplemented by geometrical analyses,[2,3] which by means of a suitably defined coherent state basis, lead to potential energy surfaces (PES) associated to arbitrary IBM Hamiltonians and to the dynamical symmetries in particular, which have thus been related to certain nuclear shapes.[4]

Dynamical symmetries (DS) have also been studied and exploited in more complicated situations, where the systems under consideration often involve complex Hamiltonians with a large number of parameters, and a general solution is a formidable problem, as is the case for the proton-neutron interacting boson approximation (IBM-2) (Ref. 5) and interaction boson fermion approximation (IBFM) (Ref. 6) Hamiltonians. In these cases the DS solutions play a very significant role and have given rise to a wide range of applications.[7] In a more recent development, Wu *et al.* proposed a fermion dynamical symmetry model[8] (FDSM), where SO(8) and Sp(6) symmetries are constructed in terms of fermion pairs, and the DS concept used as the fundamental criterion for the selection of the nuclear Hamiltonian.[8] Later on, evidence was presented for the SO(7) DS in their model, in the region of the Ru and Pd isotopes, where a fit to the energies and $B(E2)$ electromagnetic transitions was carried out using a single set of parameters.[9] It was suggested that this symmetry has a unique transitional behavior, and a geometric interpretation subsequently made,[10] stressing the ability of the SO(7) Hamiltonian in describing spherical, vibrational-like nuclei for small number of valence particles and deformed, γ-unstable nuclei for large valence particle number.[10]

The purpose of this paper is twofold. The first is to give a geometrical interpretation of a recent IBM analysis of the Ru isotopes,[11,12] where an SO(5) DS Hamiltonian with fixed parameters was used. To this end we first review the concept of dynamical symmetry in order to stress that SO(5) can be viewed as a bona fide DS,[12,13] which, besides providing an excellent description of the Ru data,[11,12] involves a sharp shape transition with increasing boson number. The second objective is to review the geometric content of the traditional SO(6), SU(3), and U(5) IBM DS, in order to show that these symmetries, as mathematical entities, are able to describe a wider range of geometries than usually believed, including the ability of producing shape transitions when studied as a function of boson number N. The traditional geometric interpretation of these symmetries is shown to apply only when certain N-dependent constraints in the Hamiltonian parameters are satisfied.

II. DYNAMICAL SYMMETRIES

We start our discussion by reexamining the ideas connected with the DS concept.[12] We refer to the IBM system in order to simplify our arguments, although the same considerations apply for other cases. The dynamical group of the IBM is U(6), in the sense that all operators in the system can be expressed in terms of powers of its generators.[14] An equivalent statement is that all states in the system are spanned by a fixed irreducible representation of the U(6) dynamical group, which in this case is the totally symmetric representation $[N]$. The symmetry group of the IBM is SO(3), since only the angular momentum generators commute with a realistic IBM Hamiltonian. Thus, the IBM wave functions are in general classified by N and L only. Intermediate groups are used to construct complete bases, leading to the well-known chains, conventionally identified by the largest subgroup of U(6), i.e., the U(5), SO(6), and SU(3) chains. In the general case the labels corresponding to these intermediate groups will be mixed by the Hamiltonian. We can now define a dynamical symmetry of the system.[12] If the one- and two-body matrix elements in the IBM Hamiltonian are chosen to take some particular set of values,[15] one (or more) of the intermediate groups remains well defined in the wave functions, i.e., the labels characterizing its representations are not mixed by the Hamiltonian.[16] We then say that the Hamiltonian displays a *dynamical symmetry.* Consider the recent study of the Ru isotopes carried out in Ref. 11. A simultaneous least-squares fit to the energies of ^{102}Ru–^{108}Ru was made and the converged Hamiltonian found to be[11]

$$H=887\hat{n}_d-53\hat{N}\hat{n}_d-25.3C_2[\mathrm{SO}(6)]+30.8C_2[\mathrm{SO}(5)]+5.5L^2\ , \tag{1}$$

where the C_2 are Casimir operators of the indicated groups and the parameters are in KeV. Although the Hamiltonian is not diagonal in any of the IBM chains, but is a mixture of the U(5) and SO(6) chains, it is apparent that SO(5), being a subgroup of both U(5) and SO(6), will remain well defined[17] for all Ru isotopes. Note that, since $\hat{L}^2$ does not commute with all the generators of SO(5), the latter group is not a symmetry group of the Hamiltonian. it is, however, a bona fide dynamical symmetry, according to our previous discussion.[12,13] Of course, a higher dynamical symmetry will occur for U(5) and SO(6) diagonal Hamiltonians, which include the SO(5) symmetry as a subsymmetry. We conclude that the IBM has four distinct dynamical symmetries, characterized by the groups SO(5), SU(3), SO(6), and U(5), where we shall subsequently refer to the first one only when the last two are *not* DS of the system.

III. SHAPE TRANSITION IN THE Ru ISOTOPES

The analysis carried out in Ref. 11 concentrated on the spectroscopic properties of the Ru and Rh isotopes and on the application of nuclear supersymmetry for a simultaneous description of these nuclei. Here we shall focus on the geometrical interpretation of Hamiltonian (1) following the well-known prescription of Gilmore, Ginocchio and Kirson, and others.[2–4] The coherent state method can be applied to an arbitrary IBM Hamiltonian and information extracted about the potential energy surface associated to it.[3] An energy surface may be defined as given by

$$E_N(\beta,\gamma)=\langle N;\beta\gamma|H|N;\beta\gamma\rangle\ , \tag{2}$$

where H is the IBM Hamiltonian and

$$|N;\beta\gamma\rangle=\frac{1}{(N!)^{1/2}(1+\beta^2)^N}\left\{s^\dagger+\beta\left[\cos\gamma\, d_0^\dagger+\frac{\sin\gamma}{\sqrt{2}}(d_2^\dagger+d_{-2}^\dagger)\right]\right\}^N|0\rangle\ , \tag{3}$$

is the intrinsic state in the notation of Ginocchio and Kirson.[3] The evaluation of (2) involves simple manipulations, and for the details we refer the reader to the original papers.[3,4] To assess the role of the different interactions in (1), we first write it in the general form

$$H=\epsilon\hat{n}_d-\alpha\hat{N}\hat{n}_d-k_1C_2[\mathrm{SO}(6)]+k_2C_2[\mathrm{SO}(5)]+k_3\hat{L}^2\ , \tag{4}$$

and by application to (3), find the result

$$E_N(\beta)=(\epsilon-\alpha N+4k_2+6k_3)N\beta^2/(1+\beta^2)+k_1N(N-1)(1-\beta^2)^2/(1+\beta^2)^2\ , \tag{5}$$

where a term $[-k_1N(N+4)]$ has been deleted from (5), since it only displaces the zero of energy.

This energy surface involves the competition of a "spherical" term $[\beta^2/(1+\beta^2)]$ and a "deformed" one $[(1-\beta^2)^2/(1+\beta^2)^2]$. There is a characteristic N dependence of these terms, which go as N and $N(N-1)$, respectively. Since it does not depend on γ, it corresponds to a "γ-unstable" geometry. We note that the U(5) terms[11,18]

$$(\epsilon-\alpha\hat{N})\hat{n}_d=\sum_m(\epsilon-\alpha)d_m^\dagger d_m-\alpha\sum_{L=0,2,4}(2L+1)^{1/2}[(d^\dagger d^\dagger)^{(L)}(\tilde{d}\tilde{d})^L]^{(0)}-\alpha\sqrt{5}[(d^\dagger s^\dagger)^{(2)}(\tilde{d}s)^{(2)}]^{(0)}\ , \tag{6}$$

together with the SO(5) and SO(3) interactions, contribute to the spherical shape, while the SO(6) term alone pulls towards deformation. As can be seen from (6) [and was remarked upon in Refs. (18), (11), and (12)] the $\hat{N}\hat{n}_d$ interaction is a standard IBM term in which N does not appear explicitly. The role of this operator in (5) is to produce a sharper shape transition as a function of N when compared with the case $\alpha=0$, as we shall proceed to show.

Calculating the β derivative of (5) we find that the energy surface has always a minimum, located at $\beta=0$, if either

$$(\epsilon-\alpha N+4k_2+6k_3)>4(N-1)k_1\ ,$$

or

$$(\epsilon-\alpha N+4k_2+6k_3)=\pm4(N-1)k_1>0\ ,$$

or at a deformed

$$\beta_0=\{[4(N-1)k_1-(\epsilon-\alpha N+4k_2+6k_3)]/[4(N-1)k_1+\epsilon-\alpha N+4k_2+6k_3]\}^{1/2}\ ,$$

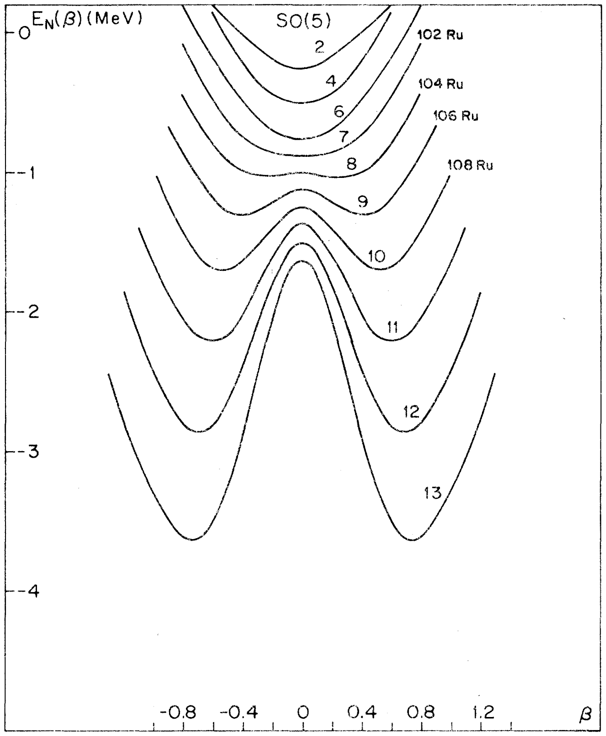

FIG. 1. Potential energy surface corresponding to Hamiltonian (1). On top of each curve the boson number N is indicated.

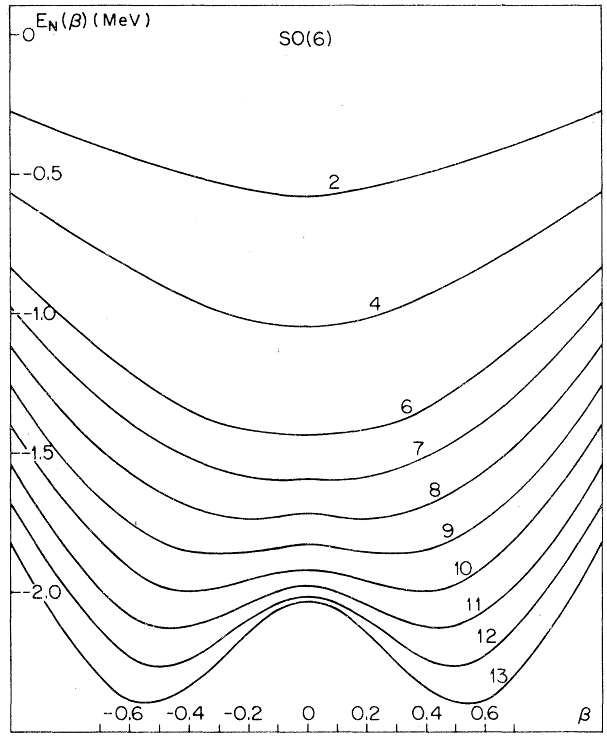

FIG. 2. Potential energy surface corresponding to the SO(6) Hamiltonian (7), with $k_1=0.012$, $k_2=0.065$, $k_3=0.010$, in MeV units. On top of each curve the boson number N is indicated.

if

$$-4(N-1)k_1 < (\epsilon-\alpha N+4k_2+6k_3) < 4(N-1)k_1 \ .$$

In addition, there is a minimum at $\beta \to \infty$, when

$$(\epsilon-\alpha N+4k_2+6k_3) < -4k_1(N-1) \ .$$

Substituting the value of the parameters as in (1) we arrive at the results shown in Fig. 1. The condition for the shape transition from spherical to deformed (γ unstable) turns out to be $N>7$, which is clearly seen in the figure for $N=8$, which corresponds to the PES associated to ^{104}Ru. The minimum value of β denoted by β_0, moves slowly as a function of N, from $\beta_0=0.26$ for $N=8$ to $\beta_0=0.74$ for $N=13$ and tending to $\beta=1$ for $N\to\infty$, which is the "classical limit" quoted for the SO(6) symmetry in the literature.[4] The SO(5) DS is hence an intrinsically transitional symmetry, able to reproduce the spectroscopic properties and the shape transition occurring in the Ru isotopes, solely as a function of the boson number N, with no change in the parameters in the Hamiltonian (1). The shape transition predicted by this study for the Ru isotopes is consistent with the spectroscopic data for these nuclei. In fact, ^{104}Ru is the first isotope in the group which displays a clear departure from vibrational behavior.[11,18,19] I believe that the geometrical interpretation presented here constitutes additional evidence for the consistency of the IBM analysis of this region.[11,12,18,19]

Returning to the general question regarding the geometrical content of dynamical symmetries, it could be argued that SO(5) is a somewhat special symmetry of the IBM, since the other three DS, namely SO(6), SU(3), and U(5), are diagonal and perhaps more restrictive in their geometrical content. In fact, it will be shown in the next section that without imposing restrictions on the parameter space this is not so, and that they too may display transitional behavior when studied as a function of N.

IV. SHAPE TRANSITION IN THE SO(6), SU(3), AND U(5) DYNAMICAL SYMMETRIES

From the results of the last section for the SO(5) symmetry, it becomes immediately clear that the SO(6) DS has also a complex structure when studied as a function of its parameters, and includes the capability of producing a shape transition. This is so because the SO(6) Hamiltonian corresponds to $\epsilon=\alpha=0$ in (4), which in turn implies that the minimum in the PES occurs at $\beta_0=0$ for either

$$4k_2+6k_3 > 4(N-1)k_1 \ ,$$

or

$$4k_2+6k_3 = \pm 4(N-1)k_1 > 0 \ ,$$

and at

$$\beta_0=\{[4(N-1)k_1-4k_2-6k_3]/[4(N-1)k_1+4k_2+6k_3]\}^{1/2}$$

for

$$-4(N-1)k_1<4k_2+6k_3<4(N-1)k_1 \ .$$

Likewise, there is a minimum at $\beta\to\infty$ for $4k_2+6k_3<-4k_1(N-1)$.

Again there is an N dependence, which is less pronounced than the one occurring for the SO(5) DS, but sufficient to cause a shape transition for particular values of the parameters. In order to illustrate this point we choose a slightly modified SO(6) Hamiltonian of the form

$$H=k_1\hat{P}^{\dagger}\cdot\hat{P}+k_2C_2[\mathrm{SO}(5)]+k_3\hat{L}^2 \ , \quad (7)$$

where the operator $\hat{P}^{\dagger}\cdot\hat{P}$ corresponds to a pairing operator,[1] related to the SO(6) Casimir operator through

$$4\hat{P}^{\dagger}\cdot\hat{P}=\hat{N}(\hat{N}+4)-C_2[\mathrm{SO}(6)] \ . \quad (8)$$

In Fig. 2 we show the results of a schematic calculation, where the parameters were chosen to produce a shape transition at $N=7$ and where the potential curves are displaced for the sake of clarity. As remarked before, and can be seen by comparing Figs. (1) and (2), SO(6) gives rise to a smoother shape transition. One should keep in mind that the parameters chosen in Fig. (2) give rise to peculiar spectra and that no claim is made here of physical relevance of the corresponding SO(6) Hamiltonian. Rather, the inequalities

$$-4(N-1)k_1<4k_2+6k_3<4(N-1)k_1$$

should be viewed as an additional N-dependent constraint on the Hamiltonian, for the standard interpretation of SO(6) as corresponding to a γ-unstable geometry to be valid.

Consider next the SU(3) Hamiltonian

$$H=-K_d\hat{Q}\cdot\hat{Q}+K_s\hat{L}^2 \ , \quad (9)$$

where

$$\hat{Q}_\mu=(s^{\dagger}\tilde{d}+d^{\dagger}s)^{(2)}_\mu-\frac{\sqrt{7}}{2}(d^{\dagger}\tilde{d})^{(2)}_\mu \ . \quad (10)$$

The calculation of the energy surface (2) leads to

$$E_N(\beta,\gamma)=-K_d\{N(5+11\beta^2/4)/(1+\beta^2)+N(N-1)[(\beta^4/2+2\sqrt{2}\beta^3\cos3\gamma+4\beta^2)/(1+\beta^2)^2]\}+K_s6N\beta^2/(1+\beta^2) \ , \quad (11)$$

which in the physical range $\beta\geq0$, $0\leq\gamma\leq\pi/3$, gives minima in $\gamma=0$ and $\gamma=\pi/3$, for $K_d>0$ and $K_d<0$, respectively. These minima correspond to prolate and oblate configurations. With no loss of generality, we choose $K_d>0$ in what follows. To find the extrema in β one needs to solve a cubic equation of the form

$$\beta^3+\frac{3}{\sqrt{2}}\alpha_N\beta^2-3\beta+\frac{3}{\sqrt{2}}\alpha_N-\frac{7\sqrt{2}}{2}=0 \ , \quad (12a)$$

where

$$\alpha_N=1+(2K_s/K_d+\tfrac{3}{4})/(N-1) \ . \quad (12b)$$

The cubic equation (12a) has no positive roots for values of α_N greater than a critical one given approximately by $\alpha_c=2.522\,244\,05$. We find the following results.

(a) There is a unique minimum of the energy surface (11), situated at $\beta=0$ for $\alpha_N\geq\alpha_c$.

(b) For $\frac{7}{3}<\alpha_N<\alpha_c$, there are two minima, one at $\beta=0$ and one at $\beta\neq0$. The spherical minimum is the global minimum (the lower of the two) for $\frac{5}{2}<\alpha_N<\alpha_c$, while the deformed minimum is the lower one for $\frac{7}{3}<\alpha_N<\frac{5}{2}$. The minima become equal at $\alpha_N=\frac{5}{2}$.

(c) For $\alpha_N\leq\frac{7}{3}$, thee is a unique, deformed minimum. We note that for $N\to\infty$ the minimum occurs at $\beta=\sqrt{2}$.

Since $K_s/K_d=(\alpha_N-1)N/2-\alpha_N/2+\frac{1}{8}$, the above inequalities give rise to N-dependent conditions on the ratio of the SU(3) Hamiltonian parameters, in order for a deformed minimum to exist. In particular, for a given K_s/K_d a shape transition occurs (from spherical to deformed shape), for a critical value of N arising for $\alpha_N=\frac{5}{2}$, or

$$N=\tfrac{3}{2}+\tfrac{4}{3}(K_s/K_d) \ . \quad (13)$$

It is interesting to note that for $K_s/K_d=-\frac{3}{8}$, which corresponds to Hamiltonian (9) being proportional to the SU(3) Casimir operator, relation (13) gives $N_c=1$, which implies deformed shapes for all N.

In Fig. (3) we show a schematic calculation with parameters chosen to give $N_c=7$. In this case, it is evident from (13) that for the boson numbers and typical SU(3) parameters used in physical applications, the nucleus is well within the deformed region. Equation (13) gives a constraint of the form $K_s/K_d<3N/4-\frac{9}{8}$, for Hamiltonian (9) to correspond to a deformed rotor. Again, the SU(3) spectra associated to spherical nuclei do not have a physical counterpart.

Finally, we look at the U(5) DS. The most general Hamiltonian in this limit has the form[18]

$$H=\epsilon\hat{n}_d-\alpha\hat{N}\hat{n}_d+k_1\hat{n}_d^2+k_2C_2[\mathrm{SO}(5)]+k_3\hat{L}^2 \ . \quad (14)$$

The energy surface turns out to be

$$E_N(\beta,\gamma)=(K-\alpha N)N\beta^2/(1+\beta^2)$$
$$+k_1N(N-1)\beta^4/(1+\beta^2)^2 \ , \quad (15)$$

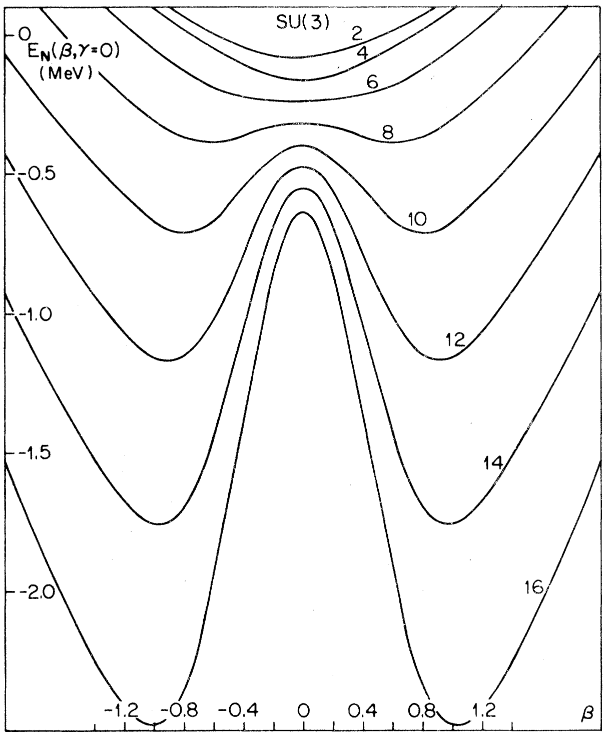

FIG. 3. Potential energy surface corresponding to the SU(3) Hamiltonian (9), with $K_d=0.008$ MeV, $K_s=0.032$ MeV. N is indicated on top of each curve.

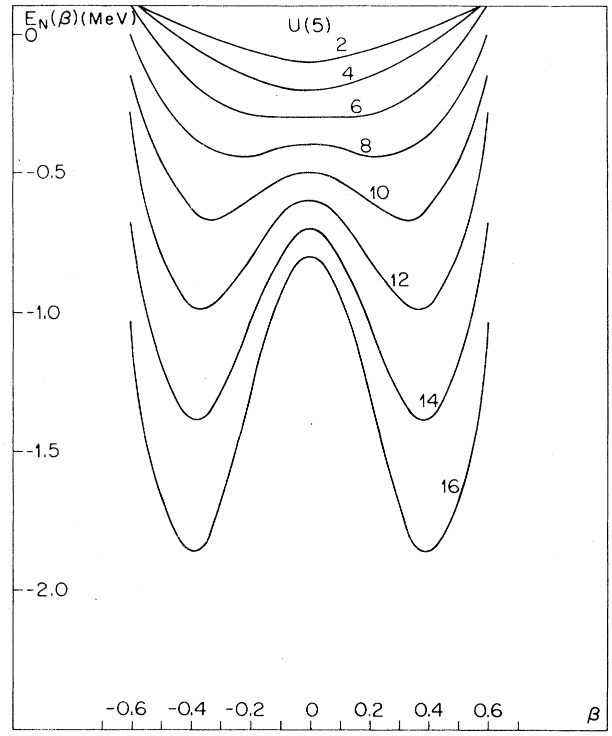

FIG. 4. Potential energy surface corresponding to the U(5) Hamiltonian (14), with $K=0.660$, $\alpha=0.100$, $k_1=0.160$, in MeV units. N is indicated on top of each curve.

where

$$K \equiv \epsilon + 4k_2 + 6k_3 + k_1 .$$

In this case one can easily show that $\beta_0=0$ is a minimum if $K-\alpha N \geq 0$, while it is a maximum otherwise. In the latter case a (γ-unstable) deformed minimum occurs at

$$\beta_0 = \{-N(K-\alpha N)/[N(K-\alpha N)+2N(N-1)k_1]\}^{1/2}$$

for positive arguments of the square root, which occur if $2k_1(N-1) > \alpha N - K > 0$. There is an additional minimum at $\beta \rightarrow \infty$, when the conditions $k - \alpha N < 0$, $2k_1(N-1) < \alpha N - K$ are satisfied. In Fig. 4 we show the energy surface for parameters chosen to give a shape transition at $N=7$, where again the potential curves are displaced for clarity. From these conditions it is clear that the $\hat{N}\hat{n}_d$ term plays a fundamental role in producing the transition; without its inclusion $K-\alpha N = K$ is either positive or negative. Again, for small boson numbers and the usually large values of K arising in typical applications, the minimum is always at $\beta_0=0$. In general, the constraint $K-\alpha N \geq 0$ should be satisfied for Hamiltonian (14) to correspond to an anharmonic vibrator. The U(5) spectrum arising from a Hamiltonian with parameters such that the minimum is at $\beta > 0$, will exhibit a spectrum with no resemblance to that of an anharmonic vibrator.

V. CONCLUSIONS

In this paper we have considered the geometrical behavior of dynamical symmetry Hamiltonians in the IBM as a function of boson number. It has been shown that these symmetries, as mathematical structures, are not associated to definite geometrical configurations, giving rise in general to strongly N-dependent energy surfaces and to shape transitions. An N-dependent equation of constraint in the parameter space for each symmetry has been derived, in order for them to correspond to the standard shapes invoked in the literature. These results in no way contradict the usual interpretation of spherical, γ-unstable and axially deformed shapes for U(5), SO(6), and SU(3) symmetries, since these shapes are indeed found for boson numbers and parameter values where these symmetries have been identified. The parameter values producting unconventional shapes generally produce peculiar spectra. On the other hand, the SO(5) dynamical symmetry of the model has been shown to constitute a simple, physically meaningful symmetry, which incorporates shape transition in a natural way. The geomerical interpretation of the SO(5) Hamiltonian (1) for the Ru isotopes, constitutes a remarkable manifestation of the versatility and consistency of the model.

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