Causality and the Coulomb sum rule in nuclei

R. Rosenfelder
Institut für Physik, University of Mainz, D-6500 Mainz, Federal Republic of Germany
*and Paul Scherrer Institute, Würenlingen und Villigen, CH 5232 Villigen PSI, Switzerland**
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The spectral function in the Jost-Lehmann-Dyson representation of causal commutators is determined for the nonrelativistic limit of inclusive lepton scattering from nuclei. From this an extrapolation of the Coulomb sum rule to higher-momentum transfers is performed which is consistent with the requirement of causality.

I. INTRODUCTION

Deep-inelastic scattering of leptons from nuclear targets has been discussed extensively in the last years in conjunction with a modified quark structure of nucleons inside the nucleus.[1] Even in the kinematical region where the scattering takes place predominantly on individual nucleons, basic features seem not to be understood: The energy integral over the longitudinal cross section at constant momentum transfer (the Coulomb sum rule) does not approach experimentally the number of protons[2] as predicted by nonrelativistic models. This may be due to experimental limitations, correlations,[3] final-state interactions of the ejected nucleons,[4] or relativistic effects.[5] Actually, relativistic Fermi gas models show a totally different behavior of the Coulomb sum rule at high q (Ref. 6) than the nonrelativistic descriptions.

However, these models are unsatisfactory as they neglect recoil effects and do not obey general principles of relativistic quantum field theory. One of these principles is *causality,* i.e., the requirement that two points in space-time cannot communicate with each other if they are separated by a spacelike distance. It is the purpose of this paper to investigate these effects on the Coulomb sum rule by using a causal representation of the inclusive cross section. I will use the representation derived by Jost, Lehmann, and Dryson (JLD),[7] although a similar one due to Deser, Gilbert, and Sudarshan (DGS) (Ref. 8) has also been discussed in the literature.[9,10]

II. THE JOST-LEHMANN-DYSON REPRESENTATION

Consider the inclusive scattering process of a lepton whereby energy ν and four-momentum q_μ is transferred to a target with four-momentum P_μ. As is well known,[1] the hadronic tensor $W_{\mu\nu}$ for this process can be written as the Fourier transform of the commutator of two electromagnetic currents j_μ, viz.,

$$W_{\mu\nu}=\frac{1}{4\pi}\int d^4x\, e^{-iq\cdot x}\langle P|[j_\mu(x),j_\nu(0)]|P\rangle \ . \quad (1)$$

From this the structure functions W_1, W_2 measured in an inclusive experiment may be obtained by a suitable projection.[11]

The commutator appearing in Eq. (1) has an important property: If x_μ is spacelike, the operators commute since the points x and 0 cannot communicate with each other,

$$\langle P|[j_\mu(x),j_\nu(0)]|P\rangle=0 \text{ for } x_\mu^2<0 \ . \quad (2)$$

Jost, Lehmann and Dyson (JLD)[7] have shown that this causal requirement puts certain restrictions on the possible form of the matrix element in Eq. (1). In particular, in the rest frame of the target $P_\mu=(M,0)$ the structure functions W_i $(i=1,2)$ can be written in the form

$$W_i(\nu,q^2)=\text{sgn}(\nu)\int d^3u\int d\lambda^2\delta[\nu^2-(\mathbf{u}-\mathbf{q})^2-\lambda^2]\times\Psi_i(\mathbf{u},\lambda^2) \ . \quad (3)$$

The spectral functions $\Psi(\mathbf{u},\lambda^2)$ are only nonzero in the region

$$|\mathbf{u}|\le M,\ \lambda^2\ge[M-(M^2-\mathbf{u}^2)^{1/2}]^2\equiv g(u) \ . \quad (4)$$

That the restriction of causality (2) is embodied in the JLD representation can be easily seen by transforming Eq. (3) into[12]

$$W_i(\nu,q^2)=\int d^4x\, e^{iq\cdot x}\left[-\frac{i}{2\pi}\right]\times\int d\lambda^2\tilde{\Psi}_i(\mathbf{x},\lambda^2)\Delta(x,\lambda^2) \ . \quad (5)$$

Here $\tilde{\Psi}_i(\mathbf{x},\lambda^2)$ is the Fourier transform of $\Psi_i(\mathbf{u},\lambda^2)$ with respect to $\mathbf{u}$ and Δ is the causal commutator for *free* fields. From its explicit form[13]

$$\Delta(x,\lambda^2)=\frac{\text{sgn}(x_0)}{2\pi}\frac{\partial}{\partial x_\mu^2}\{\Theta(x_\mu^2)J_0[\lambda(x_\mu^2)^{1/2}]\} \quad (6)$$

$[J_0(z)$ is a Bessel function of order zero], we see that it indeed vanishes for $x_\mu^2\le 0$. Thus the spectral function Ψ can be viewed as the weight function for the superposition of contributions from free fields with mass λ.

In addition, the JLD representation incorporates the threshold condition

$$W_i=0 \quad \text{for } |\nu|\le -q_\mu^2/2M \ ,$$

and the leading singularity in Eq. (6) gives rise to the scaling behavior

$$\nu W_2 \to \tfrac{1}{2}\int d^3u\,\delta(\mathbf{u}\cdot\hat{\mathbf{q}}-Mx_B)\int d\lambda^2\Psi_2(\mathbf{u},\lambda^2)\equiv F_2(x_B) \tag{7}$$

for

$$q_\mu^2 \to -\infty\,,\quad \nu\to\infty\,,\quad x_B=-q_\mu^2/2M=\text{fixed}\,,$$

provided the integrals exist.[14]

III. DETERMINATION OF THE SPECTRAL FUNCTION IN THE NONRELATIVISTIC LIMIT

Despite the above-mentioned advantages, the JLD (or DGS) representations have not been used extensively. This is mostly due to the belief that the spectral function "... is not a function that has much direct physical significance and is purely an artificial expression in a certain mathematical form of the fact that the commutator vanishes outside the light cone."[10] Actually, the spectral function may contain singularities[15] or even be a distribution.

My basic assumption is that this does *not* happen in the *nonrelativistic* limit or, more precisely, that in this limit the longitudinal spectral function associated with $S_L=W_{00}$ does not vary too much in one of its arguments. Then the following simplifications can be used: First, for a spin-zero or unpolarized target $\Psi_L(\mathbf{u},\lambda^2)$ can only depend on $u=|\mathbf{u}|$. Second, for $\nu,|\mathbf{q}|\ll M$ the integration limits (4) are well approximated by $g(u)\approx 0$, $q-\nu<u<q+\nu$ (see Fig. 1). Thus Eq. (3) becomes

$$S_L^{\rm nr}(\nu,q)\simeq\frac{\pi}{q}\int_{q-\nu}^{q+\nu}du\,u\int_0^{\nu^2-(u-q)^2}d\lambda^2\Psi_L(u,\lambda^2)\,. \tag{8}$$

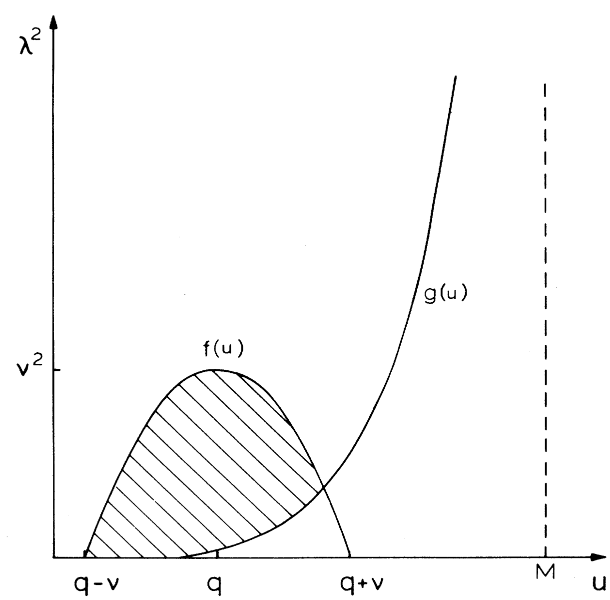

FIG. 1. Support of the spectral function $\Psi(u,\lambda^2)$ for the case $q+\nu<M$ (target mass). The region in which Ψ is nonzero is bounded by the functions $g(u)$ [see Eq. (5)] and $f(u)=\nu^2-(u-q)^2$.

where nr stands for nonrelativistic. Furthermore, nonrelativistically the three-momentum transfer $q=|\mathbf{q}|$ is usually much larger than the energy transfer (recall that for a quasifree process $\nu=q^2/2m$, where m is the mass of the constituent) and according to the basic assumption stated above we may replace $\Psi_L(u,\lambda^2)$ by $\Psi_L(q,\lambda^2)$ in Eq. (8). After an integration by parts we then have the following integral equation for the spectral function,

$$S_L^{\rm nr}(\nu,q^2)\simeq 4\pi\int_0^\nu dt\,t^2\Psi_L(q,\nu^2-t^2)\,. \tag{9}$$

This is of Abel's type and can be solved analytically to give[16]

$$\Psi_L(q,\lambda^2)\simeq\frac{1}{\pi^2}\frac{\partial}{\partial\lambda^2}\int_0^\lambda d\nu\frac{1}{(\lambda^2-\nu^2)^{1/2}}\frac{\partial S_L^{\rm nr}(\nu,q)}{\partial\nu}\,. \tag{10}$$

Note that by using Eq. (10) we can *always* determine a spectral function to a given nonrelativistic structure function, since causality is no restriction if the velocity of light is allowed to go to infinity. As expected, the spectral function (10) lacks an intuitive interpretation. However, it will be the starting point for the "causalization" of the Coulomb sum rule in the next section.

IV. THE CAUSAL COULOMB SUM RULE

The Coulomb sum rule at constant three-momentum transfer is defined as[17]

$$C(q)=\int_0^\infty d\nu\frac{S_L(\nu,q)}{F^2(q)}\,, \tag{11}$$

where

$$F(q)\simeq\frac{1}{(1+q^2/\Lambda^2)^2},\quad \Lambda^2=0.71\ \text{GeV}^2 \tag{12}$$

is the single nucleon form factor.

Take now the spectral function (10) determined in the nonrelativistic limit, insert it into the *exact* JLD representation and evaluate the Coulomb sum rule (11). One may object that this procedure can only yield as much information as the input, i.e., the nonrelativistic structure function it contains. However, it is always safer to extrapolate a function in a form which respects the general constraints. A well-known example is the use of Padé approximants for elementary and special functions: Although the coefficients are determined from the power-series expansion (i.e., from the behavior of the function at small values of the argument), some information about the asymptotic behavior is built into the specific form of the approximant and leads to a vastly improved description. In the same way, the causality constraint built into the JLD representation of the structure function may lead to a better extrapolation into the relativistic domain.

Interchanging the order of integration, one then obtains for the "causalized" Coulomb sum rule

$$C(q)=\frac{1}{F^2(q)\pi q}\int_0^\infty d\nu\,\nu\int_0^\infty du\,u\,S_L^{\rm nr}(\nu,u)\left[\frac{1}{(u-q)^2+\nu^2}-\frac{1}{(u+q)^2+\nu^2}\right]. \tag{13}$$

Equation (13) is the main result of this paper. It is worthwhile to note several points.

(a) "Causalization" in the present approach is a folding of the nonrelativistic structure function with a Breit-Wigner function of width 2ν in momentum space.

(b) If this width is small compared to the variations of the nonrelativistic structure function, then $u \simeq q$ gives the dominant contribution in Eq. (13) and one obtains

$$C(q)\simeq\frac{1}{F^2(q)}\int_0^\infty d\nu\,\nu S_L^{\rm nr}(\nu,q)\frac{1}{\pi q}\int_0^\infty du\,u\left[\frac{1}{(u-q)^2+\nu^2}-\frac{1}{(u+q)^2+\nu^2}\right]\simeq C^{\rm nr}(q)$$

because the last integral just gives $\pi q/\nu$.

(c) One may estimate when these conditions are met: The variations of $S_L^{\rm nr}$ are due to the Fermi momentum p_F and the proton form factor (12). Since most of the excitation strength lies in the quasielastic region where $\nu \simeq q^2/2m$ (m is the nucleon mass), we see that Eq. (13) reduces to the nonrelativistic description if $q \ll (mp_F)^{1/2}$ or $q \ll (m\Lambda)^{1/2}$. An alternative way of looking at these causality corrections is to introduce[18]

$$\Phi(t,q)=\int_{-\infty}^{+\infty} d\nu\, S(\nu,q)e^{i\nu t},$$

and keeping c (the velocity of light) explicitly. Then Eq. (13) can be written as

$$C(q)=\frac{1}{F^2(q)}\frac{1}{\pi q}\int_0^\infty dt\int_0^\infty du\,u\,{\rm Im}\Phi^{\rm nr}(t,u)(e^{-ct|u-q|}-e^{-ct(u+q)}), \tag{14}$$

and exhibits clearly the retardation effects due to the finite velocity of signals. Furthermore, using the energy-time and momentum-position uncertainty relations it follows from Eq. (14) that causality corrections are small if the interaction time t is large compared to the extension of the object probed.

(d) From Eq. (13) it follows that the causal sum rule always remains positive. This is a nontrivial property of the approximate spectral function derived in Eq. (10).

(e) The folding in Eqs. (13) and (14) also involves the single nucleon form factor $F^2(u)$ which multiplies the nonrelativistic structure function in order to describe the extension of the constituents. In this respect the causalized Coulomb sum rule "knows" about the internal structure of the nucleon although it does not contain explicit quark dynamics.[19]

(f) The high-momentum behavior of the causal Coulomb sum rule can be obtained by expanding Eq. (13) for large q. One obtains

$$C(q)\to\frac{1}{F^2(q)\pi q}\int_0^\infty d\nu\,\nu\int_0^\infty du\,u\,S_L^{\rm nr}(\nu,u)\frac{4u}{q^3}.$$

Using the energy-weighted sum rule

$$\int_0^\infty d\nu\,\nu S_L^{\rm nr}(\nu,u)=ZF^2(u)\frac{u^2}{2m}$$

(Z is the number of protons) and the dipole parametrization (12) of the single nucleon form factor, the asymptotic behavior becomes

$$C(q)\to\frac{Z}{16}\frac{\Lambda}{m}\frac{\Lambda^4}{q^4}\left[1+\frac{q^2}{\Lambda^2}\right]^4. \tag{15}$$

This has a minimum at $q=\Lambda$ and grows rapidly at large q leading finally to a divergence of the Coulomb sum rule. Such a divergence is expected both from the increasing number of nucleonic resonances, which can be excited, and from the observed small x_B behavior of the structure functions in the scaling limit. Note that the size parameter Λ of the nucleon determines the high q behavior in Eq. (15).

V. NUMERICAL RESULTS AND DISCUSSION

I have evaluated Eq. (13) numerically for the simple Fermi gas model

$$S_L^{\rm nr}(\nu,q)=F^2(q)\int d^3p\,n(p)\delta\left[\nu-\frac{(\mathbf{p}+\mathbf{q})^2}{2m}+\frac{\mathbf{p}^2}{2m}\right]-(\nu\to-\nu), \tag{16}$$

which describes approximately the inclusive scattering of electrons from nuclei. The momentum distribution of the protons is taken to be

$$n(p)=\frac{Z}{\pi^{3/2}p_0^3}e^{-p^2/p_0^2}. \tag{17}$$

Figure (2) shows the result for the causal sum rule together with the corresponding nonrelativistic expression

$$C^{\rm nr}(q)=Z\,{\rm erf}\left[\frac{q}{2p_0}\right], \tag{18}$$

where erf(x) is the error function. It is well known that the Fermi gas is unrealistic at low-momentum transfer

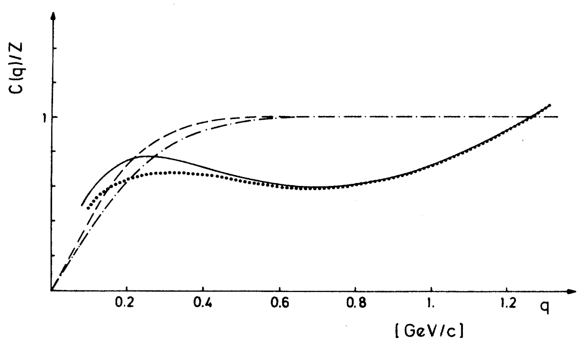

FIG. 2. Causal sum rule [Eq. (13), solid line] and nonrelativistic Coulomb sum rule (dashed line) as function of the momentum transfer q. The parameter p_0 in the Gaussian momentum distribution has been taken as 130 MeV. Also shown are the results for $p_0=160$ MeV as dotted and dashed-dotted curves, respectively.

because it implies a spatially infinite system. An indication for this failure is the linear growth of Eq. (18) with small q, whereas for finite systems C^{nr} starts quadratically. Consequently, the causal folding on Eq. (13) produces large effects which would be absent in a more realistic model.[20] However, at higher-momentum transfer the model describes the essential features of the spectrum, and effects of imposing causality can be discussed more reliably. It is seen that at intermediate momenta the Coulomb sum rule is reduced compared to the nonrelativistic description (in agreement with the experimental observation[2]), reaches a minimum around $q \simeq 700$ MeV/c, and starts to increase again. The latter behavior is expected from the high-q behavior of $C(q)$ derived in Eq. (15). Furthermore, it can be seen from Fig. 2 that the reduction at intermediate momenta increases with the fall-off parameter p_0, i.e., the nonrelativistic Coulomb sum rule is less fulfilled in heavier nuclei.[17] This is again qualitatively in agreement with the experimental observation.[2]

To summarize, I have determined approximately the spectral function in the JLD representation in the nonrelativistic region and used it to extrapolate the nonrelativistic Coulomb sum rule into the relativistic domain without violating the fundamental principle of causality. The result [Eq (13) or (14)] exhibits some features which are in qualitative agreement with the measured data.

It is evident that the crucial assumption which allowed the determination of the spectral function in the nonrelativistic limit is the smoothness hypothesis introduced to obtain Eq. (9). Corrections to that assumption can be calculated systematically by expanding

$$\Psi(u,\lambda^2)=\Psi(q,\lambda^2)+(u-q)\frac{\partial\Psi}{\partial\lambda^2}+\cdots$$

starting with the solution (10). The results presented in this work suggest that already the lowest-order term provides us with a reasonable spectral function for inclusive scattering.

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*Present address.

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[16]See, e.g., I. N. Sneddon, *The Use of Integral Transforms* (McGraw-Hill, New York, 1972), p. 207. The solution requires $\partial S_L^{nr}/\partial\nu^2=0$ at $\nu=0$, which is always fulfilled if the excitation spectrum contains a gap.

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[19]The spectral function (10) could even be used to predict the scaling function $F_2(x_B)$ from Eq. (7). However, such an extrapolation from the nonrelativistic to the ultrarelativistic region probably is not reliable.

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