PHYSICAL REVIEW C
NUCLEAR PHYSICS

THIRD SERIES, VOLUME 39, NUMBER 6 JUNE 1989

Tests of factorization in strong-interaction few-body problems

Z.-J. Cao, B. D. Keister, and H. Stumpf

Department of Physics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213

(Received 10 February 1989)

A frequently employed approximation in momentum integrals in few-body problems is to assume that the integrand is sharply peaked in regions where the bound particles have low internal momentum. If justified, this allows one to remove portions of the integrand and evaluate them at their peak-value momentum points. In the extreme case, the only remaining term in the integral is the momentum wave function, whose integral corresponds to a position-space wave function evaluated at zero interparticle separation. The validity of this approximation is examined for systems of two and three strongly interacting particles.

I. INTRODUCTION

Factorization is a highly desirable procedure for simplifying calculations involving strongly interacting particles. The basic idea can be examined by considering the integral

$$\int d^3p f(\mathbf{p})\Psi(\mathbf{p}) , \qquad (1.1)$$

where $\Psi(\mathbf{p})$ is a two-body momentum wave function, and $f(\mathbf{p})$ describes other aspects of the momentum dynamics of the problem at hand. If the momentum scale of $f(\mathbf{p})$ is much greater than that of $\Psi(\mathbf{p})$, that is, the wave function characterizes weak binding, then it is tempting to simplify the integral by considering $\Psi(\mathbf{p})$ to be sharply peaked at $\mathbf{p}=0$, and then removing $f(\mathbf{p})$ from the integral, leaving

$$\int d^3p f(\mathbf{p})\Psi(\mathbf{p}) \approx f(\mathbf{p}=0)\int d^3p \Psi(\mathbf{p}) = [f(\mathbf{p}=0)][\Psi(\mathbf{r}=0)] , \qquad (1.2)$$

where $\Psi(\mathbf{r}=0)$ is the wave function evaluated at zero interparticle separation.

Factorization thus provides a great simplification of typical integrals encountered in strong-interaction problems, and gives an appealing picture of a process in terms of one or two momentum scales. It has been utilized frequently and in many ways, from nuclear transfer reactions[1] to high-momentum transfer processes involving quarks in hadrons.[2]

Given the complexity of exact calculations, the validity of factorization as a consistent approximation has had relatively few tests. For the case of electromagnetic form factors of composite systems, there have been several studies of their behavior in the limit of very large momentum transfer. Most recent studies have concentrated on studies of quarks in hadrons, including the development of perturbative QCD methods,[3] but there are also earlier works which address this question within the context of nonrelativistic particles interacting via potentials.[4-7] These latter papers primarily address themselves to obtaining analytic limits as functions of "well behaved" potentials. One of them[4] provides a quantitative calculation which illustrates the approach to an asymptotic limit for the case of many particles interacting via δ-function potentials.

Our purpose in this paper is to examine the quantitative range of validity of the factorization approximation for two simple models which we hope are representative of realistic applications. The first is the problem of multiple scattering of scalar nucleons from a scalar deuteron, and the second involves the calculation of form factors of two-body bound states, be they deuterons composed of nucleons or hadrons composed of quarks. In general, we find that, up to momentum transfers of $q^2=200\ \mathrm{fm}^{-2}$, factorization provides at best a *qualitative* picture of the integrals which they replace, and becomes quantitatively accurate only under certain specialized conditions.

II. MULTIPLE SCATTERING OF SCALAR NUCLEONS

We begin by considering a model of elastic scattering of a scalar nucleon from a bound state of two scalar nucleons, which is designed to mimic the proton-deuteron problem. Since our goal is to study factorization, all consequences of identical particle symmetries are ignored. At particle energies in the GeV region, a multiple-scattering series can converge quite rapidly. The rate of convergence of such a series is a separate study—our concern here is to see whether the individual terms can be factorized.

As illustrated in Fig. 1, the multiple-scattering series for the elastic scattering amplitude $\langle \mathbf{P}'\mathbf{p}'|T|\mathbf{P}\mathbf{p}\rangle$ can be written in terms of the free three-body Green's function G_0 and the two-body scattering operator t as

$$\langle \mathbf{P}'\mathbf{p}'|t+tG_0t+tG_0tG_0t+\cdots|\mathbf{P}\mathbf{p}\rangle , \tag{2.1}$$

where $\mathbf{P}$ and $\mathbf{P}'$ are the initial and final momenta of the bound state, and $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final momenta of the projectile. The first term in the series represents single scattering, sometimes called the impulse approximation. We shall return to the role of factorization in single scattering later in this paper. It is the double-scattering term which is most convenient for examining factorization

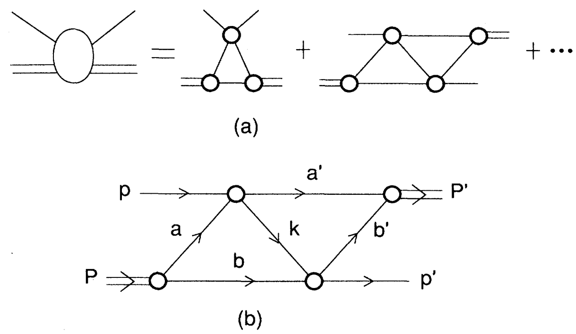

FIG. 1. The multiple-scattering series for scattering from a composite two-body target (a), along with definitions of momentum variables used in the text (b).

$$\langle \mathbf{P}'\mathbf{p}'|T|\mathbf{P}\mathbf{p}\rangle_2=\int d^3l\int d^3n\,\Psi^\dagger(l)t(\mathbf{k},\mathbf{a}';\mathbf{p},\mathbf{a})\frac{1}{e}t(\mathbf{p}',\mathbf{b}';\mathbf{k},\mathbf{b})\Psi(\mathbf{n}) , \tag{2.2}$$

where the variables

$$\mathbf{a}=\tfrac{1}{2}\mathbf{P}+l;\ \mathbf{a}'=\tfrac{1}{2}\mathbf{P}'+\mathbf{n};\ \mathbf{b}=\tfrac{1}{2}\mathbf{P}-l;\ \mathbf{b}'=\tfrac{1}{2}\mathbf{P}'-\mathbf{n};\ \mathbf{k}=\mathbf{p}+\mathbf{a}-\mathbf{a}' \tag{2.3}$$

are illustrated in Fig. 1, and the energy denominator is

$$e=\frac{\mathbf{p}^2}{2m}+\frac{\mathbf{P}^2}{4m}-\frac{\mathbf{a}'^2}{2m}-\frac{\mathbf{k}^2}{2m}-\frac{\mathbf{b}^2}{2m} . \tag{2.4}$$

Nonrelativistic kinematics are used throughout this paper.

The double-scattering contribution is difficult to evaluate because the integral has six dimensions, and because the integrand is singular at points where the energy denominator vanishes, corresponding to kinematically allowed physical intermediate states. The calculation is further complicated by the fact that the two-body t matrices must be evaluated at off-energy-shell kinematic values. If valid, the factorization approximation offers a tremendous simplification of the double-scattering integral:

$$\langle \mathbf{P}'\mathbf{p}'|T|\mathbf{P}\mathbf{p}\rangle_{2(\text{factorized})}=|\Psi(\mathbf{r}=0)|^2t(\mathbf{p}-\tfrac{1}{2}\mathbf{q},\tfrac{1}{2}\mathbf{P}';\mathbf{p},\tfrac{1}{2}\mathbf{P})\frac{1}{e}t(\mathbf{p}',\tfrac{1}{2}\mathbf{P}';\mathbf{p}-\tfrac{1}{2}\mathbf{q},\tfrac{1}{2}\mathbf{P}) , \tag{2.5}$$

where $\mathbf{q}$ is the momentum transferred to the bound state.

In order to study the factorization approximation to double scattering, we make a further simplifying assumption, namely that the momentum scale of the off-shell extrapolation of the t matrix greatly exceeds that of all other momentum scales in the problem. The t matrix can therefore be set equal to a constant and removed from the integral. The "test" of factorization then becomes one of determining whether the energy denominator can also be removed from the integral. For realistic models of nucleon-nucleon scattering, this assumption is highly questionable.[8] Nevertheless, it is an assumption which is used regularly in medium-energy nucleon-nucleus studies. Its effect is to make the integral *more* likely to be factorizable. Conversely, if the momentum scale of the off-shell t matrix is low enough that it cannot be ignored, then the range of validity of the factorization approximation will be even smaller than what we will find it to be.

Factorization as described here can be tested in at least two ways. One is simply to compare the results of the six-dimensional integration with the factorized approximation. The other is to examine the size of the contribution from the region surrounding the singularity of the Green's function. For the purposes of a test, the t matrices can be set to unity (or any other real number to set the scale). The Green's function can be written as

$$\left[\frac{1}{e}\right]=\mathcal{P}\left[\frac{1}{e}\right]-i\pi\delta[e] . \tag{2.6}$$

At best, the factorized approximation can predict only the contribution from the principal value of the Green's function, since the wave function peaks do not satisfy the kinematics of the delta function. Thus, factorization assumes that the delta-function contribution is small compared to the principal-value part. Comparing real and imaginary parts of the exact integral therefore provides a good indicator of the validity of factorization.

The double scattering contribution involves two three-dimensional internal momenta. The energy denominator can vanish when the internal momenta are configured to correspond to a physically allowed three-particle breakup channel. The real part of the energy denominator therefore requires a principal-value integration over the singularity. The intuition behind factorization suggests that the dominant contributions to the integral come from the region $l \approx \mathbf{n} \approx 0$. Except for very small values of momentum transfer, the energy denominator is not small in this region, so one might expect the contribution from the singularity to be small. However, for the actual numerical calculation, it was necessary to use a mapping of momentum points which were actually *centered* on the singular point. This was done as follows. For every momenta l and $\mathbf{n}$ correspond to a two-particle internal momentum $\mathbf{k}$ and cluster momentum $\mathbf{K}$. In the three-body center of mass, the energy denominator is

$$e=\frac{\mathbf{p}^2}{2m}+\frac{\mathbf{P}^2}{4m}-\epsilon_k-\Omega_K \ , \tag{2.7}$$

where $\epsilon_k \equiv \mathbf{k}^2/m$ and

$$\Omega_K \equiv \mathbf{K}^2/2m+\mathbf{K}^2/4m \ .$$

In place of the radial variables $|\mathbf{k}|$ and $|\mathbf{K}|$, we use $\mathcal{E}$ and ϵ, where $0 \le \mathcal{E} < \infty$ and $0 \le \epsilon \le \mathcal{E}$, and therefore $|\mathbf{k}|=\sqrt{m\epsilon}$ and $|\mathbf{K}|=\sqrt{\frac{4}{3}m(\mathcal{E}-\epsilon)}$. The energy denominator can be written as

$$e=\frac{\mathbf{p}^2}{2m}+\frac{\mathbf{P}^2}{4m}-\mathcal{E} \ . \tag{2.8}$$

All of the singular behavior is therefore contained in the integration over $\mathcal{E}$. By choosing integration points for $\mathcal{E}$ which are symmetric about the singular point, the integral becomes stable enough that it is more efficient and in fact more accurate per integration point to use a Monte Carlo algorithm. The most efficient algorithm we were able to find was the routine VEGAS.[9]

For this test of factorization, different model wave functions were used. These included the S-state wave function of the deuteron using the Reid soft core or the Paris potential, a momentum dipole form

$$\psi(\mathbf{p})=\frac{1}{(\mathbf{p}^2+\Lambda_1^2)(\mathbf{p}^2+\Lambda_2^2)} \ , \tag{2.9}$$

and a Gaussian

$$\psi(\mathbf{p})=e^{-p^2\alpha^2} \ . \tag{2.10}$$

Note that the normalization is arbitrary, since we are only interested in making comparisons between approximate and exact calculations.

A wave function based upon a short-range repulsive core vanishes at the origin. Thus, strict factorization implies a result of zero. Of course, what this really means is that the momentum scale of the wave function is not that characterized by weak binding, but rather that of the size of the repulsive core. If the remainder of the integral has a momentum scale much greater than that of the core dynamics, then a modified factorization approximation can be obtained by expanding the remainder of the integrand about the point $l=\mathbf{n}=0$, and keeping the first nonvanishing term

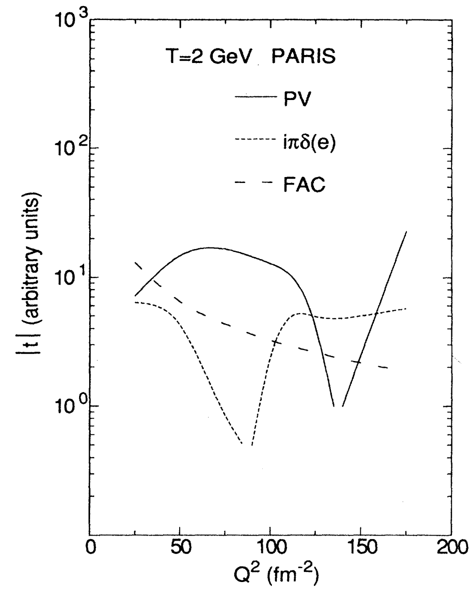

FIG. 2. Factorized versus exact double scattering for the Paris potential.

$$\frac{1}{e}=\frac{1}{e}\bigg|_{l=n=0}+l^2\mathbf{n}^2\nabla_l^2\nabla_n^2\frac{1}{e}\bigg|_{l=n=0}+\cdots \ . \tag{2.11}$$

Setting the t matrices to unity, this yields

$$\langle \mathbf{P}'\mathbf{p}'|T|\mathbf{P}\mathbf{p}\rangle_{2(\text{factorized})}=|\nabla^2\Psi(\mathbf{r})|_{r=0}^2\nabla_l^2\nabla_n^2\left[\frac{1}{e}\right]_{l=n=0} . \tag{2.12}$$

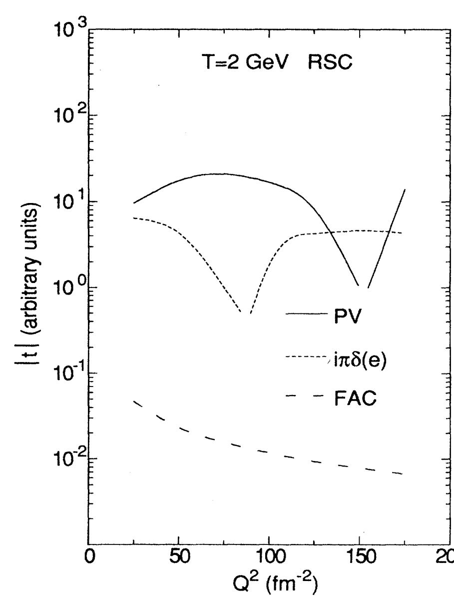

FIG. 3. Factorized versus exact double scattering for the Reid soft core potential.

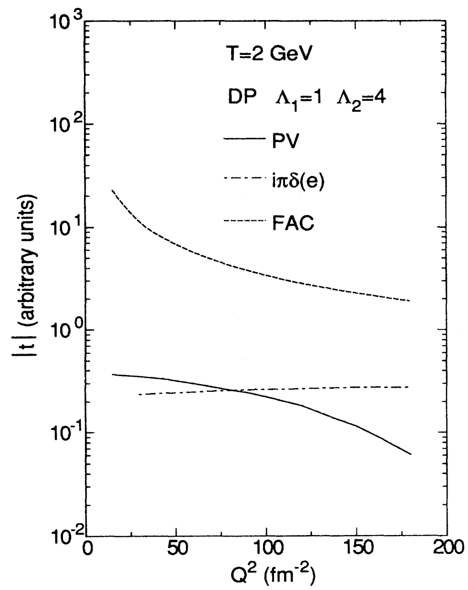

FIG. 4. Factorized versus exact double scattering for the dipole wave function discussed in the text.

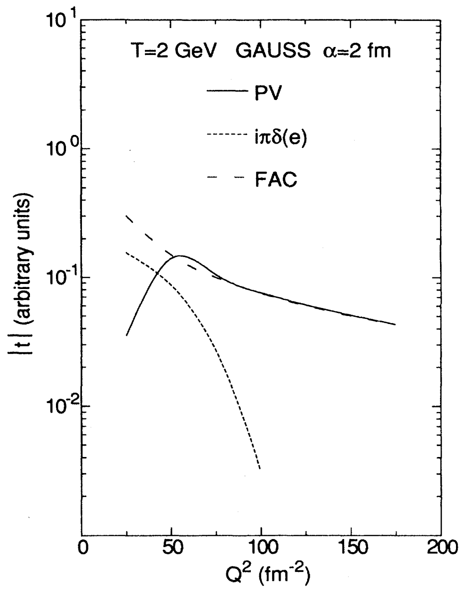

FIG. 6. Factorized versus exact double scattering for a Gaussian wave function with scale $\alpha=2.0$ fm.

We now present our numerical findings for the case of 2-GeV proton energy. Results for the S-state Paris potential wave function are shown in Fig. 2. Since the wave function does not vanish at $\mathbf{r}=0$, the form (2.5) was used. For the S-state Reid soft-core wave function (Fig. 3), the modified form (2.12) was used. In both cases, the factorized and nonfactorized results bear little relation to each other. The presence of nodes in both real and imaginary parts of the exact integrals suggests that asymptotic kinematics are far from being reached. The conclusions using momentum dipole wave functions (Fig. 4) are similar to these. Note that the contribution from the singularity is comparable to the principal value integral. For Gaussian wave functions (Figs. 5 and 6), only very low values of the wave function momentum scale yield a peak in the momentum integrals which is sharp enough to justify factorization.

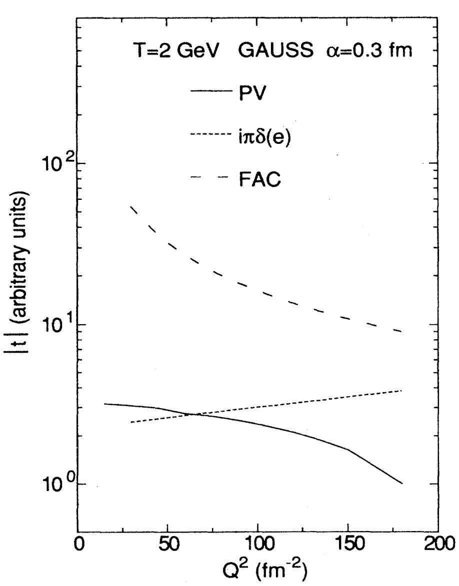

FIG. 5. Factorized versus exact double scattering for a Gaussian wave function with scale $\alpha=0.3$ fm.

III. TWO-BODY FORM FACTORS

Arguments very similar to those discussed above for hadronic multiple scattering can be found in studies of form factors of composite objects. One such application is the calculation of electromagnetic form factors of hadrons at very large momentum transfer using perturbative QCD.[2]

Consider the form factor for two scalar particles in a spinless bound state:

$$F(q^2)=\int d^3n\,\Psi^\dagger(\mathbf{n}+\tfrac{1}{2}\mathbf{q})\Psi(\mathbf{n})\ , \tag{3.1}$$

where $\mathbf{q}$ is the momentum transferred to the bound state. If the composite state is weakly bound, one might expect the integral over $\mathbf{n}$ to be peaked near $\mathbf{n}\approx 0$ or else $\mathbf{n}+\frac{1}{2}\mathbf{q}\approx 0$. If $\mathbf{n}\approx 0$, then we still need to know the wave function $\Psi(\mathbf{n}+\frac{1}{2}\mathbf{q})$, i.e., at large momentum values. In the spirit of factorization, this high momentum behavior can be exposed by iterating the bound state wave equation:[2]

$$\Psi=\frac{1}{e}V\Psi\ , \tag{3.2}$$

which yields an expression for the form factor

$$F(q^2)=\int d^3l \int d^3n \Psi^\dagger(l) \left[\frac{1}{e}\right] V\Psi(\mathbf{n}) , \tag{3.3}$$

as shown in Fig. 7. In the Breit frame,

$$e=\frac{\mathbf{q}^2}{16m}-\frac{\mathbf{b}^2}{2m}-\frac{\mathbf{c}^2}{2m} , \tag{3.4}$$

where

$$\mathbf{b}=-\frac{\mathbf{q}}{4}-l ;\ \mathbf{c}=\frac{3\mathbf{q}}{4}+l . \tag{3.5}$$

The strong interaction has the form

$$V=V(\mathbf{q})=\frac{V_0}{(\frac{1}{2}\mathbf{q}+l-\mathbf{n})^2+\mu^2} , \tag{3.6}$$

corresponding to the exchange of a "meson" of mass μ. Following the treatment of the previous section, a factorized approximation to Eq. (3.3) is therefore

$$F(q^2)_{(\text{factorized})}=|\Psi(\mathbf{r}=0)|^2 \frac{1}{e} V \bigg|_{l=n=0} . \tag{3.7}$$

Equation (3.7) is consistent with the high momentum behavior found in other studies[4–6] of form factors of composite systems of particles interacting via nonrelativistic potentials. Amado and Woloshyn[4] examined a model of many particles interacting via δ-function potentials, for which wave functions can be obtained analytically.[10] Our study of two-body composite systems involves wave functions whose potentials have nonzero range which can be comparable to that of the wave functions themselves.

We also examine the concept of a "hard-soft" separation, employed frequently in calculations with perturbative QCD.[2] The basic idea is that the high-momentum ("hard") component of the wave function is best calculated by iterating the wave equation to expose the momentum dependence of the interaction itself. What remains is then an *integral* over the wave function, which is assumed to be dominated by low-momentum ("soft") components that may be only remotely related to the high-momentum behavior of the potential.

The parametrization of the strong interaction in Eq. (3.7) should be directly related to the wave function for the bound state. We have not made such a connection in

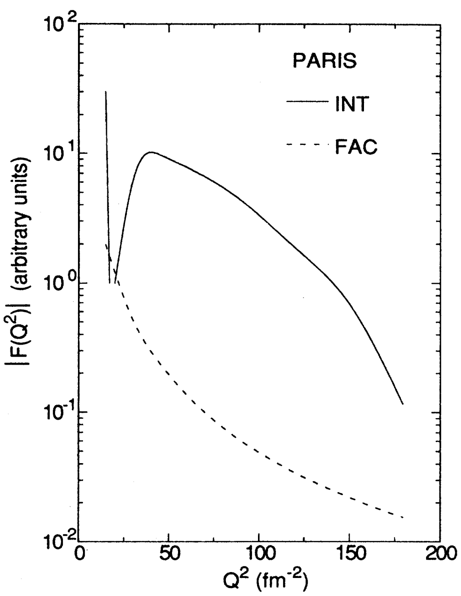

FIG. 8. Factorized versus exact form factor for the Paris potential.

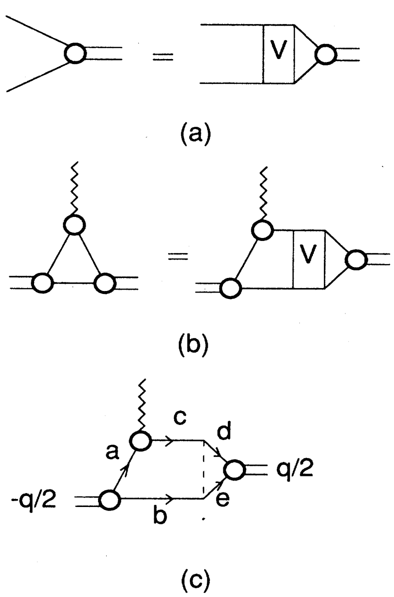

FIG. 7. Iteration of the wave equation in evaluating a two-body form factor (a) and (b), along with definitions of momentum variables used in the text (c).

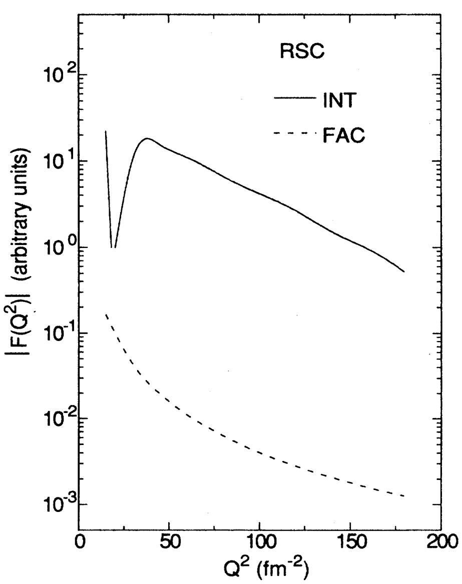

FIG. 9. Factorized versus exact form factor for the Reid soft core potential.

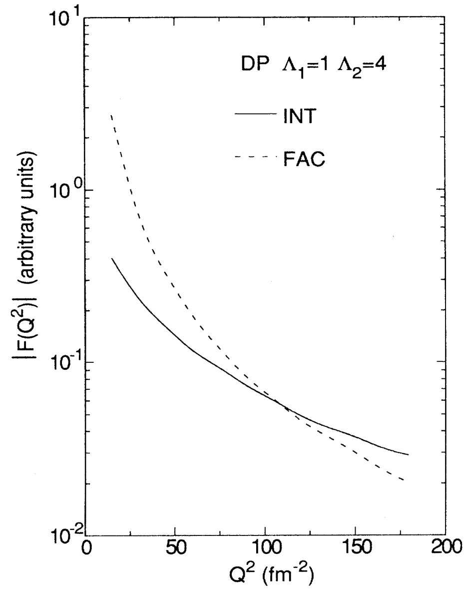

FIG. 10. Factorized versus exact form factor for the dipole wave function discussed in the text.

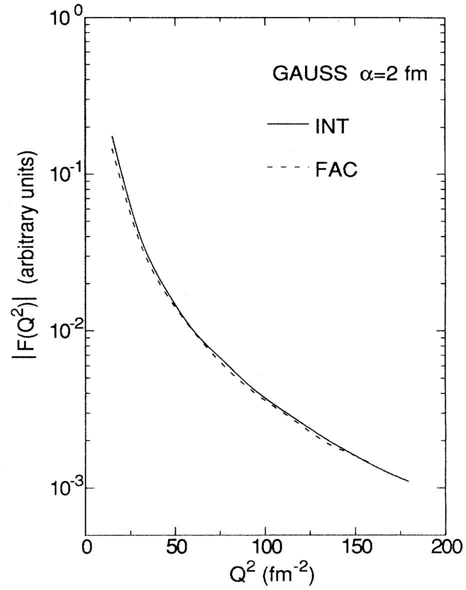

FIG. 12. Factorized versus exact form factor for a Gaussian wave function with scale $\alpha=2.0$ fm.

these model studies. Instead, we take the point of view that the wave function may describe well the low-momentum properties of the bound state, while the high-momentum properties are better characterized using the properties of the interaction itself. This idea is consistent with the original spirit behind iterating the wave equation to expose high-momentum properties.[2]

Our calculations for the form factor are shown by analogy to the previous section. Results for the S-state Paris and Reid wave functions are shown in Figs. 8 and 9. Once again, the factorized expression is a poor approximation to the exact integral. The factorized approximation for the momentum dipole wave function (Fig. 10) is qualitatively similar to the exact form factor, but the momentum dependence is still not reproduced at the highest momentum transfers shown. For Gaussian wave functions (Figs. 11 and 12), factorization is again successful only for very low momentum scales in the wave function.

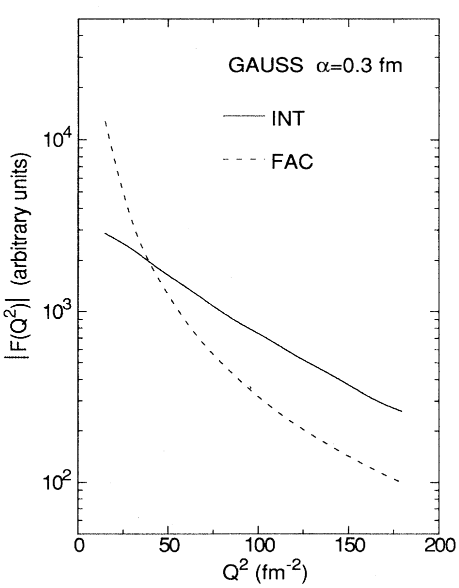

FIG. 11. Factorized versus exact form factor for a Gaussian wave function with scale $\alpha=0.3$ fm.

IV. SUMMARY

The idea of factorization of integrals describing processes with composite systems at high momentum transfer provides a simple physical picture, can be proven for asymptotically large values of q^2, and greatly simplifies the nature of explicit calculation. However, our studies suggest that for many applications, factorization is at best a qualitative approximation. The notion of separating pieces of an integral with widely differing momentum scales is clear enough, but when the two relevant pieces are inverse polynomials (as is the case for many applications in strong-interaction physics), that separation is rarely sharp enough to justify the approximation. If the wave function can be approximately characterized as a Gaussian with very low momentum content, then factorization can be achieved for realistically attainable momentum transfers. However, for the particularly interesting case of the deuteron, there are important features which make factorization a questionable procedure, including the typically polynomial behavior of its wave function and the short-range repulsion in the potential.

We conclude that the factorization of integrals into low- and high-momentum contributions, while justified in an asymptotic sense, must be carefully tested in the actual momentum range of interest.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant No. PHY-8610134. The computations were made possible by a grant of time on the Cray X-MP at the Pittsburgh Supercomputing Center.

[1]J. R. Taylor, *Scattering Theory* (Wiley, New York, 1972). See especially Sec. 21a.

[2]S. J. Brodsky and G. R. Farrar, Phys. Rev. D **11**, 1309 (1975).

[3]A. Duncan and A. H. Mueller, Phys. Lett. **90B**, 159 (1980); Phys. Rev. D **21**, 1636 (1980); G. P. Lepage and S. J. Brodsky, *ibid.* **22**, 2157 (1980).

[4]R. D. Amado, Phys. Rev. C **14**, 1264 (1976); R. D. Amado and R. M. Woloshyn, *ibid.* **15**, 2200 (1977).

[5]I. M. Narodetsky, Yu. A. Simonov, and F. Palumbo, Phys. Lett. **58B**, 125 (1975).

[6]S. D. Drell, A. C. Finn, and M. H. Goldhaber, Phys. Rev. **157**, 1402 (1967).

[7]J. D. Stack, Phys. Rev. **164**, 1904 (1967).

[8]Z.-J. Cao, B. D. Keister, and H. Stumpf (unpublished).

[9]G. P. Lepage, J. Comp. Phys. **27**, 192 (1978).

[10]F. Calogero and A. Degasperis, Phys. Rev. A **11**, 265 (1975).