Unitary transformation from color-spin to isospin-spin coupling schemes for six-quark color singlet states

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We derive a unitary transformation relating color singlet six-quark states constructed in the color-spin (*CS*) and isospin-spin (*TS*) schemes. The chosen orbital symmetries are [42] in the $TS=(01)$ sector and [33] in the $TS=(00)$ sector. They are important in the description of the *NN* system.

I. INTRODUCTION

There are two classification schemes which have been used in the literature to construct totally antisymmetric states of six quarks incorporating orbital (O), color (C), isospin (T), and spin (S) degrees of freedom. In the one called the *CS* coupling scheme one first combines the SU_3-color singlet [222] and the SU_2-spin $[f]_S$ representations to a given SU_6 symmetry $[f]_{CS}$. Subsequently one can couple[1,2] this to an SU_2-isospin representation $[f]_T$ to obtain an SU_{12} representation $[f]_{CST}$ adjoint to the orbital symmetry $[f]_O$ chosen. In constructing a totally antisymmetric state as above one can also permute the orbital and the isospin symmetries, i.e., couple first *CS* to

TABLE I. The $\bar{K}([42]pq\,[33]p'q'|[f'']p''q'')$ matrices used in the $S\times T$ coupling for $S=1$, $T=0$.

		[42] [33]			
		$\overline{11}\,\overline{22}$	$\overline{22}\,\overline{22}$	$\overline{12}\,\overline{22}$	$\widetilde{12}\,\widetilde{12}$
[33]	$\overline{22}$	$\sqrt{\frac{10}{108}}$	$\sqrt{\frac{8}{108}}$	$\sqrt{\frac{60}{108}}$	$\sqrt{\frac{30}{108}}$
[51]	$\overline{11}$	$-\sqrt{\frac{4}{27}}$	$\sqrt{\frac{5}{27}}$	$\sqrt{\frac{6}{27}}$	$-\sqrt{\frac{12}{27}}$
	$\overline{12}$	0	0	1	0
[411]	$\overline{11}$	$\sqrt{\frac{1}{6}}$	0	$\sqrt{\frac{2}{6}}$	$-\sqrt{\frac{3}{6}}$
	$\overline{13}$	$-\sqrt{\frac{25}{54}}$	$-\sqrt{\frac{20}{54}}$	$\sqrt{\frac{6}{54}}$	$\sqrt{\frac{3}{54}}$
[2211]	$\overline{24}$	$-\sqrt{\frac{3}{10}}$	0	$\sqrt{\frac{6}{10}}$	$\sqrt{\frac{1}{10}}$
[321]	$\overline{12}$	$\sqrt{\frac{8}{15}}$	0	$\sqrt{\frac{1}{15}}$	$\sqrt{\frac{6}{15}}$
	$\overline{13}$	0	0	-1	0
	$\overline{23}$	$-\sqrt{\frac{8}{27}}$	$\sqrt{\frac{10}{27}}$	$-\sqrt{\frac{3}{27}}$	$\sqrt{\frac{6}{27}}$

		[42] [33]			
		$\overline{11}\,\widetilde{12}$	$\overline{22}\,\widetilde{12}$	$\overline{12}\,\widetilde{12}$	$\widetilde{12}\,\overline{22}$
[33]	$\widetilde{12}$	$\sqrt{\frac{5}{12}}$	$-\sqrt{\frac{2}{12}}$	0	$\sqrt{\frac{5}{12}}$
[51]	$\widetilde{12}$	$-\sqrt{\frac{2}{3}}$	0	0	$-\sqrt{\frac{1}{3}}$
[411]	$\widetilde{23}$	$-\sqrt{\frac{1}{3}}$	0	0	$\sqrt{\frac{2}{3}}$
	$\widetilde{13}$	0	0	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$
[2211]	$\widetilde{12}$	1	0	0	0
	$\widetilde{24}$	0	0	$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
	$\widetilde{34}$	$-\sqrt{\frac{1}{20}}$	$\sqrt{\frac{10}{20}}$	0	$\sqrt{\frac{9}{20}}$
[321]	$\widetilde{12}$	0	0	$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{2}{5}}$
	$\widetilde{13}$	$\sqrt{\frac{8}{15}}$	$\sqrt{\frac{5}{15}}$	0	$-\sqrt{\frac{2}{15}}$
	$\widetilde{23}$	0	0	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$

TABLE II. The $\bar{K}([33]pq\,[33]p'q'|[f'']p''q'')$ matrices used in the $S\times T$ coupling for $S=0$, $T=0$.

		[33] [33] $\overline{22}\,\overline{22}$	[33] [33] $\widetilde{12}\,\widetilde{12}$
[6]	$\overline{11}$	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{5}}$
[42]	$\overline{11}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$
	$\overline{12}$	-1	0
	$\overline{22}$	$-\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
[222]	$\overline{33}$	$-\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$
[3111]	$\overline{11}$	0	-1
	$\overline{14}$	-1	0

		[33] [33] $\overline{22}\,\widetilde{12}$	[33] [33] $\widetilde{12}\,\overline{22}$
[42]	$\widetilde{12}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$
[222]	$\widetilde{12}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
[3111]	$\widetilde{14}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
	$\widetilde{34}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

O to get a function of dual symmetry, $[\tilde{f}]_{CSO}=[f]_T$ as, for example, in Ref. 3.

One can also build totally antisymmetric states starting from an intermediate representation $[f]_{TS}$ of the SU_4 isospin-spin group. The list of all contributing SU_4 representations associated with the orbital symmetries $[6]_O$, $[51]_O$, $[42]_O$, and $[33]_O$ can be found in Table I of Ref. 4. We shall call this classification scheme the TS scheme.

Each of these two schemes has its advantages. In the CS scheme the expectation value of the color magnetic operator

$$\Gamma_{\text{c.m.}}=-\sum_{i<j}\hat{\lambda}_i\cdot\hat{\lambda}_j\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\ , \tag{1.1}$$

with $\hat{\lambda}_i$ the SU_3-color generators, can be easily calculated.[2] In the TS scheme one has the advantage of being able to interchange the values of T and S at the group theory level of calculation of the matrix elements. For example, results for $TS=(01)$ and (10) are identical before implementing the spin-spin matrix elements. Moreover, in calculating the matrix elements of the six-quark Hamiltonian, one can use[5,6] fractional parentage coefficients available in the literature of the nuclear shell model.[7]

To understand the relationship between results obtained in the CS and TS schemes it is necessary to know the unitary transformation between bases for various sectors. This question has been raised, for example, in Ref. 8.

To our knowledge, the present work is the first to accomplish such a task. Our results concern specific orbital symmetries, the $[42]_O$ in the $TS=(01)$ sector and $[33]_O$ in the $TS=(00)$ sector. They play a very important role in describing the NN interaction at short separation distances (see, e.g., Refs. 5, 6, and 8). The other symmetries contributing to the NN state (Sec. III), i.e., the $[6]_O$ for $TS=(01)$ and $[51]_O$ for $TS=(00)$ produce only one state, so they are identical in either scheme.

In Sec. II we shall sketch the method used in obtaining

TABLE III. The $\bar{K}([f]pq\,[33]p'q'|[2211]p''q'')$ matrices used in the $CS\times T$ coupling for $S=1$, $T=0$.

		[321] [33] $\overline{12}\,\overline{22}$	$\overline{13}\,\overline{22}$	$\overline{23}\,\overline{22}$	$\widetilde{12}\,\widetilde{12}$	$\widetilde{13}\,\widetilde{12}$	$\widetilde{23}\,\widetilde{12}$
[2211]	$\overline{24}$	$\sqrt{\frac{15}{40}}$	$-\sqrt{\frac{2}{40}}$	$-\sqrt{\frac{9}{40}}$	$-\sqrt{\frac{5}{40}}$	0	$\sqrt{\frac{9}{40}}$

		[321] [33] $\overline{12}\,\widetilde{12}$	$\overline{13}\,\widetilde{12}$	$\overline{23}\,\widetilde{12}$	$\widetilde{12}\,\overline{22}$	$\widetilde{13}\,\overline{22}$	$\widetilde{23}\,\overline{22}$
	$\widetilde{12}$	0	$-\sqrt{\frac{3}{8}}$	0	$-\sqrt{\frac{5}{8}}$	0	0
[2211]	$\widetilde{24}$	$-\sqrt{\frac{15}{80}}$	0	$\sqrt{\frac{27}{80}}$	$-\sqrt{\frac{5}{80}}$	$-\sqrt{\frac{30}{80}}$	$\sqrt{\frac{3}{80}}$
	$\widetilde{34}$	0	$-\sqrt{\frac{6}{20}}$	0	$\sqrt{\frac{5}{20}}$	0	$-\sqrt{\frac{9}{20}}$

		[3111] [33] $\overline{11}\,\overline{22}$	$\overline{14}\,\overline{22}$	$\widetilde{14}\,\widetilde{12}$	$\widetilde{34}\,\widetilde{12}$
[2211]	$\overline{24}$	$\sqrt{\frac{4}{20}}$	$-\sqrt{\frac{6}{20}}$	$-\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{9}{20}}$

		[3111] [33] $\overline{11}\,\widetilde{12}$	$\overline{14}\,\widetilde{12}$	$\widetilde{14}\,\overline{22}$	$\widetilde{34}\,\overline{22}$
	$\widetilde{12}$	0	0	1	0
[2211]	$\widetilde{24}$	0	$-\sqrt{\frac{6}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{3}{10}}$
	$\widetilde{34}$	$\sqrt{\frac{6}{40}}$	0	$\sqrt{\frac{25}{40}}$	$-\sqrt{\frac{9}{40}}$

		[21111] [33] $\overline{15}\,\overline{22}$	$\widetilde{15}\,\widetilde{12}$	$\widetilde{45}\,\widetilde{12}$
[2211]	$\overline{24}$	$\sqrt{\frac{1}{5}}$	0	$-\sqrt{\frac{4}{5}}$

		[21111] [33] $\overline{15}\,\widetilde{12}$	$\widetilde{15}\,\overline{22}$	$\widetilde{45}\,\overline{22}$
	$\widetilde{12}$	0	0	1
[2211]	$\widetilde{24}$	0	$-\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{2}{5}}$
	$\widetilde{34}$	$-\sqrt{\frac{3}{5}}$	0	$-\sqrt{\frac{2}{5}}$

		[222] [33] $\overline{33}\,\overline{22}$	$\widetilde{23}\,\widetilde{12}$
[2211]	$\overline{24}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

		[222] [33] $\widetilde{23}\,\overline{22}$	$\overline{33}\,\widetilde{12}$
	$\widetilde{12}$	$-\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
[2211]	$\widetilde{24}$	1	0
	$\widetilde{34}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$

TABLE IV. The $\bar{K}([f]pq\,[33]p'q'|[222]p''q'')$ matrices used in the $CS\times T$ coupling for $S=0$, $T=0$.

[33] [33]

		$\overline{22}\,\overline{22}$	$\widetilde{12}\,\widetilde{12}$
[222]	$\overline{33}$	$\sqrt{\frac{3}{4}}$	$-\sqrt{\frac{1}{4}}$

		$\overline{22}\,\widetilde{12}$	$\widetilde{12}\,\overline{22}$
[222]	$\widetilde{23}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

[411] [33]

		$\overline{11}\,\overline{22}$	$\overline{13}\,\overline{22}$	$\widetilde{13}\,\widetilde{12}$	$\widetilde{23}\,\widetilde{12}$
[222]	$\overline{33}$	$\sqrt{\frac{3}{8}}$	$\sqrt{\frac{3}{8}}$	0	$-\sqrt{\frac{2}{8}}$

		$\overline{11}\,\widetilde{12}$	$\overline{13}\,\widetilde{12}$	$\widetilde{13}\,\overline{22}$	$\widetilde{23}\,\overline{22}$
[222]	$\widetilde{23}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	0

[2211] [33]

		$\overline{24}\,\overline{22}$	$\widetilde{12}\,\widetilde{12}$	$\widetilde{34}\,\widetilde{12}$
[222]	$\overline{33}$	$-\sqrt{\frac{5}{12}}$	$-\sqrt{\frac{2}{12}}$	$\sqrt{\frac{5}{12}}$

		$\widetilde{24}\,\overline{22}$	$\widetilde{12}\,\overline{22}$	$\widetilde{34}\,\overline{22}$	$\overline{24}\,\widetilde{12}$
[222]	$\widetilde{23}$	$\sqrt{\frac{30}{54}}$	$-\sqrt{\frac{4}{54}}$	$-\sqrt{\frac{5}{54}}$	$-\sqrt{\frac{15}{54}}$

$[1^6]$ [33]

		$\widetilde{56}\,\widetilde{12}$
[222]	$\overline{33}$	−1

		$\widetilde{56}\,\overline{22}$
[222]	$\widetilde{23}$	1

the transformation and shall give results of intermediate steps consisting of tables of $\bar{K}$ matrices associated with both schemes. They can be used subsequently in the calculation of matrix elements of the six-quark Hamiltonians chosen for *NN* studies. Section III contains the two unitary transformations obtained in this work and a discussion.

II. THE METHOD

The states of interest for $TS=(01)$ and (00) in the *TS* schemes are listed in Table I of Ref. 4. The corresponding color-spin symmetries $[f]_{CS}$ can be obtained from the inner product $[222]_C\times[f]_S$. This yields

TABLE V. The $\bar{K}([42]pq\,[222]p'q'|[f'']p''q'')$ matrices used in the $S\times C$ coupling for $S=1$, $O\times C$ coupling for $[42]_O$ or $CS\times T$ coupling for $[222]_{CS}$, $T=0$.

[42] [222]

		$\overline{11}\,\overline{33}$	$\overline{12}\,\overline{33}$	$\overline{22}\,\overline{32}$	$\widetilde{12}\,\widetilde{23}$
[222]	$\overline{33}$	$-\sqrt{\frac{5}{12}}$	0	$-\sqrt{\frac{2}{12}}$	$\sqrt{\frac{5}{12}}$
[21111]	$\overline{15}$	$\sqrt{\frac{2}{3}}$	0	0	$-\sqrt{\frac{1}{3}}$

[42] [222]

		$\overline{11}\,\widetilde{23}$	$\overline{12}\,\widetilde{23}$	$\overline{22}\,\widetilde{23}$	$\widetilde{12}\,\overline{33}$
[222]	$\widetilde{23}$	$\sqrt{\frac{5}{54}}$	$\sqrt{\frac{30}{54}}$	$\sqrt{\frac{4}{54}}$	$\sqrt{\frac{15}{54}}$
[21111]	$\widetilde{15}$	0	1	0	0
	$\widetilde{45}$	$-\sqrt{\frac{4}{27}}$	$\sqrt{\frac{6}{27}}$	$\sqrt{\frac{5}{27}}$	$-\sqrt{\frac{12}{27}}$

TABLE VI. The $\bar{K}([f]pq\,[222]p'q'|[f'']p''q'')$ matrices used in the $S\times C$ coupling for $S=0$ or $O\times C$ coupling for $[51]_O$ or $[33]_O$.

[51] [222]		$\overline{11}\,\overline{33}$	$\overline{12}\,\overline{33}$	$\widetilde{12}\,\widetilde{23}$
[2211]	$\overline{24}$	$\sqrt{\frac{4}{5}}$	0	$\sqrt{\frac{1}{5}}$

[51] [222]		$\overline{11}\,\widetilde{23}$	$\overline{12}\,\widetilde{23}$	$\widetilde{12}\,\overline{33}$
	$\widetilde{12}$	−1	0	0
[2211]	$\widetilde{24}$	$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$	0
	$\widetilde{34}$	$\sqrt{\frac{2}{5}}$	0	$\sqrt{\frac{3}{5}}$

[33] [222]		$\overline{22}\,\overline{33}$	$\widetilde{12}\,\widetilde{23}$
[2211]	$\overline{24}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

[33] [222]		$\overline{22}\,\widetilde{23}$	$\widetilde{12}\,\overline{33}$
	$\widetilde{12}$	$-\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
[2211]	$\widetilde{24}$	1	0
	$\widetilde{34}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$
$[1^6]$	$\widetilde{56}$	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{5}}$

$$[f]_{CS}=[42],[321],[222],[3111],\text{ and }[21111]\quad \text{for } T=0,\ S=1\ , \tag{2.1}$$

$$[f]_{CS}=[33],[411],[2211],\text{ and }[1^6]\quad \text{for } T=0,\ S=0\ . \tag{2.2}$$

All symmetries of (2.1) can be combined with the orbital symmetry $[42]_O$ to produce a totally antisymmetric $TS=(01)$ state, and the same holds for (2.2) in obtaining a $TS=(00)$ state of orbital symmetry $[33]_O$. We note that $[6]_O$ selects only $[222]_{CS}$ for $TS=(01)$ states and $[51]_O$ only $[2211]_{CS}$ for $TS=(00)$ states. Accordingly, the unitary matrices we are looking for have to be 5×5 for $[42]_O$ and 4×4 for $[33]_O$.

Our procedure is straightforward. We write each basis vector in the *TS* scheme as a linear combination of basis vectors in the *CS* scheme. Then we make explicit the content of each basis state in terms of Young diagrams associated with orbital, color, isospin, and spin degrees of freedom and by identification of these diagrams we obtain equations for the coefficients entering the linear combinations. In all cases the system of equations is overdetermined.

TABLE VII. The unitary transformation between the *CS* and *TS* basis vectors of orbital symmetry $[42]_O$, isospin $T=0$, and spin $S=1$.

	ψ_1^{CS}	ψ_2^{CS}	ψ_3^{CS}	ψ_4^{CS}	ψ_5^{CS}
$[42]_O\{33\}_{TS}$	$\frac{9\sqrt{5}}{36}$	$-\frac{8\sqrt{5}}{36}$	$-\frac{5\sqrt{2}}{36}$	$\frac{11}{36}$	$-\frac{20}{36}$
$[42]_O\{51\}_{TS}$	$\frac{9\sqrt{5}}{45}$	$-\frac{8\sqrt{5}}{45}$	$-\frac{5\sqrt{2}}{45}$	$-\frac{25}{45}$	$\frac{25}{45}$
$[42]_O\{411\}_{TS}$	$\frac{9\sqrt{10}}{180}$	$-\frac{8\sqrt{10}}{180}$	$\frac{170}{180}$	$-\frac{25\sqrt{2}}{180}$	$-\frac{20\sqrt{2}}{180}$
$[42]_O\{2211\}_{TS}$	$\frac{11}{20}$	$\frac{8}{20}$	$\frac{\sqrt{10}}{20}$	$\frac{5\sqrt{5}}{20}$	$\frac{4\sqrt{5}}{20}$
$[42]_O\{321\}_{TS}$	$-\frac{18}{45}$	$-\frac{29}{45}$	$\frac{2\sqrt{10}}{45}$	$\frac{10\sqrt{5}}{45}$	$\frac{8\sqrt{5}}{45}$

TABLE VIII. The unitary transformation between the *CS* and *TS* basis vectors of orbital symmetry $[33]_O$, isospin $T=0$, and spin $S=0$.

	ψ_1^{CS}	ψ_2^{CS}	ψ_3^{CS}	ψ_4^{CS}
$[33]_O\{6\}_{TS}$	$-\frac{\sqrt{5}}{5}$	$\frac{\sqrt{10}}{5}$	$\frac{3}{5}$	$\frac{1}{5}$
$[33]_O\{42\}_{TS}$	$-\frac{3\sqrt{5}}{20}$	$\frac{3\sqrt{10}}{20}$	$-\frac{11}{20}$	$-\frac{12}{20}$
$[33]_O\{222\}_{TS}$	$-\frac{15}{20}$	$-\frac{5\sqrt{2}}{20}$	$-\frac{3\sqrt{5}}{20}$	$\frac{4\sqrt{5}}{20}$
$[33]_O\{3111\}_{TS}$	$-\frac{5\sqrt{2}}{20}$	$-\frac{10}{20}$	$\frac{3\sqrt{10}}{20}$	$-\frac{4\sqrt{10}}{20}$

Each antisymmetric state can be expanded as

$$\psi_{[1^6]}=\frac{1}{\sqrt{\eta_f}}\sum_Y(-)^{P_Y}|fY\rangle|\tilde{f}\,\tilde{Y}\rangle\ , \tag{2.3}$$

where Y is the Young diagram associated with the representation $[f]$ of the symmetric group S_6 and $[\tilde{f}]$ is the adjoint representation of $[f]$, i.e., $\tilde{Y}$ is obtained from Y by interchanging rows and columns. The phase $(-)^{P_Y}$ depends on the number P of transpositions to restore the normal order in the Young diagram Y. Together with the factor $\eta_f^{-1/2}$, where η_f is the dimension of $[f]$, it represents the Clebsch-Gordon coefficient of the inner product of $[f]\times[\tilde{f}]$ leading to $[1^6]$. Equation (2.3) is subsequently rewritten in terms of the diagonalized Young-Yamanouchi-Rutherford representation[7] where the last pair of particles has definite symmetry. The next step is to make explicit the content of $[f]$ and $[\tilde{f}]$ as inner products of various S_6 representations. In the *TS* scheme $[f]=[f]_O\times[f]_C$ and $[\tilde{f}]=[f]_T\times[f]_S$. In the *CS* scheme $[f]=[f]_O$ and $[\tilde{f}]=\{[f]_C\times[f]_S\}\times[f]_T$. An inner product of two S_6 representations $[f']$ and $[f'']$ can be expanded by using the corresponding S_6 Clebsch-Gordan coefficients. We use the factorization property[9] of S_n coefficients into a K matrix and a S_{n-1} coefficient and write them as in Ref. 4 as a product of a $\bar{K}$ matrix resulting from two sequent factorizations and an S_4 Clebsch-Gordan coefficient. Following the notations given in Appendix A of Ref. 4 we have

$$\begin{aligned}|fY\rangle=\sum\bar{K}([f']p'q'[f'']p''q''|[f]pq)\\ \times S(f_{p'q'}y'f''_{p''q''}y''|f_{pq}y)\\ \times|[f']p'q'y'\rangle|[f'']p''q''y''\rangle\ ,\end{aligned} \tag{2.4}$$

where the summation runs over $p'q'$, $p''q''$, and y' and y''. Here p labels the row of the 6th particle, q the row of the 5th particle, and y is the remainder of the Young diagram Y after the removal of the 6th and 5th particle, i.e., a diagram of the S_4 representation f_{pq}. In the diagonalized Young-Yamanouchi-Rutherford representation the indices pq appear either in a symmetric $\overline{pq}$ or in an antisymmetric $\tilde{pq}$ combination only. The factor S in (2.4) is the Clebsch-Gordan coefficient needed in the inner product $[f_{pq}]=[f'_{p'q'}]\times[f''_{p''q''}]$ of S_4.

In this work a consistent phase convention of the six-particle states is absolutely necessary. We have constructed the S_4 Clebsch-Gordan coefficients and the $\bar{K}$ matrices following conventions and using symmetry properties of the Clebsch-Gordan coefficients as introduced in Ref. 9. In calculating matrix elements of the six-quark Hamiltonian the problem reduces to the calculation of two-body matrix elements, and only the knowledge of the $\bar{K}$ matrices is necessary because the S_4 Clebsch-Gordan coefficients sum up in orthogonality relations.[4] In view of this practical aspect we give in Tables I–VI the $\bar{K}$ matrices resulting from this study. They complement[10] Table III of Ref. 6. Our Tables V and VI recover Table V of Ref. 4. One can see that in a few cases we disagree on the phase. Up to a phase convention we also obtain agreement with Tables I and II of Ref. 1 and parts of Table III of Ref. 4 where they appear under the name of two-body fractional parentage coefficients. The correspondence with Ref. 1 is obvious and that with Table III of Ref. 4 can also be established easily by using the notations of the head columns of Ref. 1.

III. THE UNITARY TRANSFORMATION

In Tables VII and VIII we exhibit the two unitary matrices obtained through the procedure described in the preceding section. The rows contain the coefficients c_i of the linear combinations of ψ_i^{CS} states defining a TS scheme state as

$$[f]_O\{f\}_{TS}=\sum c_i\psi_i^{CS}\ , \tag{3.1}$$

where the left-hand side (lhs) denotes a TS scheme state in the compressed notation[4] used in the literature. In the CS scheme the notation for the $TS=(01)$ basis vectors ψ_i^{CS} $(i=1,\ldots,5)$ is as follows:

$$\begin{aligned}
\psi_1^{CS}&=[42]_O\{42\}_{CS}=\{[42]_O\times\{[42]_{CS}\times[33]_T\}_{[2211]}\}_{[1^6]}\ ,\\
\psi_2^{CS}&=[42]_O\{321\}_{CS}=\{[42]_O\times\{[321]_{CS}\times[33]_T\}_{[2211]}\}_{[1^6]}\ ,\\
\psi_3^{CS}&=[42]_O\{3111\}_{CS}=\{[42]_O\times\{[3111]_{CS}\times[33]_T\}_{[2211]}\}_{[1^6]}\ ,\\
\psi_4^{CS}&=[42]_O\{222\}_{CS}=\{[42]_O\times\{[222]_{CS}\times[33]_T\}_{[2211]}\}_{[1^6]}\ ,\\
\psi_5^{CS}&=[42]_O\{21111\}_{CS}=\{[42]_O\times\{21111]_{CS}\times[33]_T\}_{[2211]}\}_{[1^6]}\ ,
\end{aligned}\tag{3.2}$$

and for the $TS=(00)$ sector the ψ_i^{CS} $(i=1,\ldots,4)$ are as follows:

$$\begin{aligned}
\psi_1^{CS}&=[33]_O\{33\}_{CS}=\{[33]_O\times\{[33]_{CS}\times[33]_T\}_{[222]}\}_{[1^6]}\ ,\\
\psi_2^{CS}&=[33]_O\{411\}_{CS}=\{[33]_O\times\{[411]_{CS}\times[33]_T\}_{[222]}\}_{[1^6]}\ ,\\
\psi_3^{CS}&=[33]_O\{2211\}_{CS}=\{[33]_O\times\{[2211]_{CS}\times[33]_T\}_{[222]}\}_{[1^6]}\ ,\\
\psi_4^{CS}&=[33]_O\{1^6\}_{CS}=\{[33]_O\times\{[1^6]_{CS}\times[33]_T\}_{[222]}\}_{[1^6]}\ .
\end{aligned}\tag{3.3}$$

TABLE IX. The content of NN, $\Delta\Delta$, and CC states in the CS scheme for $TS=(01)$.

	$[6]_O\{222\}_{CS}$	$[42]_O\{42\}_{CS}$	$[42]_O\{321\}_{CS}$	$[42]_O\{3111\}_{CS}$	$[42]_O\{222\}_{CS}$	$[42]_O\{21111\}_{CS}$
NN	$\frac{1}{3}$	$\frac{1}{6\sqrt{5}}$	$-\frac{4}{27\sqrt{5}}$	$-\frac{1}{27\sqrt{2}}$	$\frac{31}{54}$	$-\frac{20}{27}$
ΔΔ	$\frac{2}{3\sqrt{5}}$	$\frac{2}{3}$	$-\frac{16}{27}$	$-\frac{2\sqrt{10}}{27}$	$-\frac{14}{27\sqrt{5}}$	$\frac{\sqrt{5}}{27}$
CC	$\frac{2}{\sqrt{5}}$	$-\frac{1}{4}$	$\frac{2}{9}$	$\frac{\sqrt{10}}{36}$	$-\frac{11}{36\sqrt{5}}$	$\frac{\sqrt{5}}{9}$

In view of the application of this transformation it is useful to recall the separation given by Harvey[4] into "asterisked" and "nonasterisked" SU_4 symmetries. The first can lead to di-baryon states NN, $N\Delta$, and $\Delta\Delta$, the latter do not, but, as hidden color states, can contribute to the three-quark clusters energy at short separations. For $TS=(01)$ the asterisked symmetries are $\{33\}_{TS}$ and $\{51\}_{TS}$. They define the NN, $\Delta\Delta$, and CC states by the linear combinations[4]

$$\begin{aligned} NN &= \sqrt{\tfrac{1}{9}}([6]_O\{33\}_{TS}) + \sqrt{\tfrac{4}{9}}([42]_O\{33\}_{TS}) \\ &\quad - \sqrt{\tfrac{4}{9}}([42]_O\{51\}_{TS}) , \\ \Delta\Delta &= \sqrt{\tfrac{4}{45}}([6]_O\{33\}_{TS}) + \sqrt{\tfrac{16}{45}}([42]_O\{33\}_{TS}) \\ &\quad + \sqrt{\tfrac{25}{45}}([42]_O\{51\}_{TS}) , \\ CC &= \sqrt{\tfrac{4}{5}}([6]_O\{33\}_{TS}) - \sqrt{\tfrac{1}{5}}([42]_O\{33\}_{TS}) . \end{aligned} \tag{3.4}$$

Using Table VII we rewrite these linear combinations in terms of the CS basis vectors. The result is exhibited in Table IX, where we have used the identity

$$[6]_O\{33\}_{TS} = [6]_O\{222\}_{CS} . \tag{3.5}$$

One can see that among the $[42]_O$ states the largest contributions to NN come from $\{21111\}_{CS}$ (54.9%) and $\{222\}_{CS}$ (33%) symmetries, the rest of the other $[42]_O$ states contributing together with 1%.

The important role of the $[42]_O$ state in the NN problem comes from the fact that the hyperfine interaction compensates for the extra kinetic energy of the two p-shell quarks and brings it nearly degenerate to the $[6]_O$ state.[5,8] The hyperfine interaction which is given approximately by the operator (1.1) has its lowest (negative) expectation value for the symmetry $\{42\}_{CS}$. This can justify the choice of Ref. 1 to study the contribution of this state only. But the CS composition of the NN state given above shows that the symmetries $\{222\}_{CS}$ and $\{21111\}_{CS}$ cannot *a priori* be neglected.

For the $TS=(00)$ sector the NN, $\Delta\Delta$, and CC states are as follows:[4,11]

TABLE X. Same as Table IX, but for $TS=(00)$.

	$[51]_O\{2211\}_{CS}$	$[33]_O\{33\}_{CS}$	$[33]_O\{411\}_{CS}$	$[33]_O\{2211\}_{CS}$	$[33]_O\{1^6\}_{CS}$
NN	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{5}}{6}$	0
ΔΔ	$-\frac{2}{3}$	0	0	$\frac{4}{3\sqrt{5}}$	$\frac{1}{\sqrt{5}}$
CC	$-\frac{2}{3}$	$-\frac{1}{4}$	$\frac{\sqrt{2}}{4}$	$-\frac{11}{12\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$

$$\begin{aligned} NN &= -\sqrt{\tfrac{5}{45}}([51]_O\{42\}_{TS}) - \sqrt{\tfrac{4}{45}}([33]_O\{42\}_{TS}) \\ &\quad - \sqrt{\tfrac{36}{45}}([33]_O\{6\}_{TS}) , \\ \Delta\Delta &= -\sqrt{\tfrac{20}{45}}([51]_O\{42\}_{TS}) - \sqrt{\tfrac{16}{45}}([33]_O\{42\}_{TS}) \\ &\quad + \sqrt{\tfrac{9}{45}}([33]_O\{6\}_{TS}) , \\ CC &= -\sqrt{\tfrac{4}{9}}([51]_O\{42\}_{TS}) + \sqrt{\tfrac{5}{9}}([33]_O\{42\}_{TS}) . \end{aligned} \tag{3.6}$$

Their content in CS states obtained using Table VIII is presented in Table X where

$$[51]_O\{2211\}_{CS} = [51]_O\{42\}_{TS} .$$

This shows that the $\{33\}_{CS}$, $\{411\}_{CS}$, and $\{2211\}_{CS}$ symmetries contribute with 25%, 50%, and 14%, respectively, while the amplitude of the $\{1^6\}_{CS}$ symmetry cancels identically for NN.

In both cases when calculating expectation values with respect to NN for a model Hamiltonian one should include both the contributions of the diagonal and nondiagonal matrix elements between CS states. On the other hand, the NN state has a strong coupling to other physical $\Delta\Delta$, $N\Delta$, or hidden color CC states at short separation distances.[5,6] The final conclusion about the role of various CS symmetries has then to be given by a full dynamical analysis.

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