Three- and N-body systems: An approach in terms of relative coordinates and momenta

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The dynamics of three- and n-body systems is frequently described by central two-body interactins, functions of the moduli of the relative coordinate vectors. The center-of-momentum closure relation allows one to transform the kinetic energy into a pure quadratic form in the two-body relative momenta. Defining generalized two-body relative momenta the Hamiltonian acquires a simple form, albeit in redundant variables. Symmetric systems with harmonic and short-range interactions are discussed, as well as exotic atomic systems, such as kaonic ^{3}He, antiprotonic ^{3}He, and others. A solution in terms of symmetric coordinates is also constructed.

I. INTRODUCTION

The traditional approach to three-body systems proceeds by expressing the kinetic energy in terms of Jacobi coordinates.[1,2] However, position-dependent form factors and potentials describing the two-body interaction are in their simplest expression functions of the relative distance between the two-bodies of subsystems ($r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$, central interactions). It is possible to express the kinetic energy as a pure quadratic form in the two-body relative momenta, thus allowing an approach closely patterned along the actual dynamics of the three-body system. The same method is applicable to n-particle systems. The approach described here allows us to reveal simply some interesting features of three- and n-particle systems, and to interpret certain terms of the Hamiltonian as momentum-dependent (three-body) "potentials." Such terms are similar to the so-called main component of the three-body force in three-nucleon systems.

II. NONRELATIVISTIC SYSTEMS

It is well known that physical particle systems are constructed basically from two-body interactions. There is no universal center of attraction for systems of particles, except as an approximation. Two-body systems can be sometimes described by Lagrangians and Hamiltonians with a potential, depending on the distance between the centers of mass of the bodies: $V_{ij}=V_{ij}(r_{ij})$, where $r_{ij}=|\mathbf{r}_{ij}|=|\mathbf{r}_i-\mathbf{r}_j|$. Given a three-body system the Hamiltonian in the three-body center-of-mass system is

$$H=\sum_{i=1}^{3}\frac{p_i^2}{2m_i}+\sum_{3}V_{ij}\ , \tag{1}$$

where the $\sum_3$ implies three terms V_{ij} obtained from the cyclic permutation of 1,2,3. This notation is preferred for applications to n-body systems. The center-of-mass momenta satisfy the closure relation

$$\sum_i \mathbf{p}_i=0\ . \tag{2}$$

The Hamiltonian (1) is expressed with center-of-mass momenta, a function of the velocities in the three-body center-of-mass system, and central potentials, expressed in terms of moduli of relative coordinate vectors. Thus (1) is not expressed in a canonical system of coordinates. It is possible to transform it into a Hamiltonian containing a pure quadratic form of the relative momenta, given by

$$\mathbf{p}_{ij}=\mu_{ij}\dot{\mathbf{r}}_{ij}\ , \tag{3}$$

where $\mu_{ij}=m_j m_i/m_i+m_j$. Expressing $\mathbf{p}_{ij}$ in terms of center-of-mass momenta yields

$$\mathbf{p}_{ij}=\frac{m_j\mathbf{p}_i-m_i\mathbf{p}_j}{m_i+m_j}\ . \tag{4}$$

It follows from (2) that

$$\sum_{i=1}^{3}p_i^2=-2\sum_{3}\mathbf{p}_i\cdot\mathbf{p}_j\ , \tag{5}$$

where $\sum_3$ has the same meaning as before. From (4) it follows that

$$p_{ij}^2=(m_j^2p_i^2+m_i^2p_j^2-2m_im_j\mathbf{p}_i\cdot\mathbf{p}_j)m_{ij}^{-2}\ , \tag{6}$$

where $m_{ij}=m_i+m_j$. Thus it follows that

$$m_{ij}\frac{p_{ij}^2}{2\mu_{ij}}=m_j\frac{p_i^2}{2m_i}+m_i\frac{p_j^2}{2m_j}-\mathbf{p}_i\cdot\mathbf{p}_j\ , \tag{7}$$

summing the three equations of type (7) corresponding to the three-body system and using (5) one obtains for the kinetic energy

$$T=\sum_{i=1}^{3}\frac{p_i^2}{2m_i}=\sum_{3}\frac{m_{ij}}{M}\frac{p_{ij}^2}{2\mu_{ij}}\ , \tag{8}$$

where $M=\sum_{i=1}^{3}m_i$. The kinetic energy is thus expressed as a function of a pure quadratic form of the three relative momenta. Indeed, it is a linear sum of the kinetic energies of the two-body subsystems multiplied by scaling factors depending on the three masses $f_{ij}=m_{ij}/M<1$. These factors reduce the two-body relative motion kinetic energies of the two-body subsystems (considered in isola-

tion), which feeds into the other two-body subsystems. The following equivalent Hamiltonian is obtained instead of (1):

$$H=\sum_3\left[\frac{m_{ij}}{M}\frac{p_{ij}^2}{2\mu_{ij}}+V_{ij}\right] . \tag{9}$$

It is convenient to define the momenta $\mathcal{P}_{ij}=(m_{ij}/M)p_{ij}$ and the generalized reduced masses $\mathcal{M}_{ij}=m_i m_j/M$. The Hamiltonian becomes

$$\mathcal{H}=\sum_3\left[\frac{\mathcal{P}_{ij}^2}{2\mathcal{M}_{ij}}+V_{ij}\right] . \tag{10}$$

The momenta

$$\mathcal{P}_{ij}=\mathcal{M}_{ij}\dot{\mathbf{r}}_{ij} , \tag{11}$$

thus corresponding closely to definition (3) of the two-body relative momenta, replacing the two-body reduced masses by the generalized ones. The Hamiltonian (10) is expressed in terms of the relative position vectors $\mathbf{r}_{ij}$ and the momenta $\mathcal{P}_{ij}$. They obey the following closure relations:

$$\sum_3\mathbf{r}_{ij}=0 \tag{12}$$

and

$$\sum_3\frac{\mathcal{P}_{ij}}{\mathcal{M}_{ij}}=0 , \tag{13}$$

respectively. The relation (13) is a direct consequence of (12) and (11), taking the time derivative of (12). Expression (10) possesses a high degree of symmetry, the same as the Eqs. (12) and (13). Clearly (10) has redundant coordinates. It is possible to write it eliminating one of the relative momenta:

$$\mathcal{H}_{ij,jk}=\left[1+\frac{\mathcal{M}_{ki}}{\mathcal{M}_{ij}}\right]\frac{\mathcal{P}_{ij}^2}{2\mathcal{M}_{ij}}+\left[1+\frac{\mathcal{M}_{ki}}{\mathcal{M}_{jk}}\right]\frac{\mathcal{P}_{jk}^2}{2\mathcal{M}_{jk}}+\frac{\mathcal{M}_{ki}}{\mathcal{M}_{ij}\mathcal{M}_{jk}}\mathcal{P}_{ij}\cdot\mathcal{P}_{jk}+V_{ij}+V_{jk}+V_{ki} . \tag{14}$$

The symmetry is partially but not totally lost. Equation (14) yields three equivalent Hamiltonians for the three-body problem.

III. HARMONIC POTENTIALS

For the class of harmonic potentials,

$$V_{ij}+V_{jk}+V_{ki}=\tfrac{1}{2}(K_{ij}r_{ij}^2+K_{jk}r_{jk}^2+K_{ki}r_{ki}^2) . \tag{15}$$

An exact solution of both the classical-mechanical system and the quantum-mechanical one is straightforward. The closure relation of the relative coordinates allow us to obtain

$$V=\tfrac{1}{2}[(K_{ij}+K_{ki})r_{ij}^2+\tfrac{1}{2}(K_{jk}+K_{ji})r_{jk}^2+2K_{ki}\mathbf{r}_{ij}\cdot\mathbf{r}_{jk}] . \tag{16}$$

We give here an outline of the exact solution, triggered by the expression (14) in terms of relative coordinates and generalized relative momenta. In Cartesian coordinates,

$$\mathcal{P}_{ij}=\mathcal{M}_{ij}(\dot{x}_{ij}\breve{i}+\dot{y}_{ij}\breve{j}+\dot{z}_{ij}\breve{k}) ; \tag{17}$$

designating generically by $X,\dot{X}$ the projections of $\mathbf{r}_{ij},\mathbf{r}_{jk}$ and $\dot{\mathbf{r}}_{ij},\dot{\mathbf{r}}_{jk}$ upon the X axis one obtains

$$T_x=\tfrac{1}{2}(\alpha_{ij}\dot{X}_{ij}^2+\alpha_{ik}\dot{X}_{jk}^2+2\beta_{ij,jk}\dot{X}_{ij}\dot{X}_{jk}) , \tag{18}$$

$$V_x=\tfrac{1}{2}[(K_{ij}+K_{ki})X_{ij}^2+(K_{jk}+K_{ki})X_{jk}^2+2K_{ki}X_{ij}X_{jk}] . \tag{19}$$

The components X acquire the status of generalized coordinates for the system. It is always possible to find a linear transformation of the coordinates such that the kinetic and potential energies are of the form[3]

$$T_x=\tfrac{1}{2}\sum_l\dot{X}_l^2 , \tag{20}$$

$$V_x=\tfrac{1}{2}\sum_l\omega_l^2X_l^2 , \tag{21}$$

to describe the motion projected on the x axis $l=1,2$. Similar equations are obtained for the motion projected on the y and z axes. Correspondingly, for the bosons the Schrödinger equations are given by[4]

$$-\frac{\hbar^2}{2}\left[\frac{\partial^2\Psi}{\partial X_1^2}+\frac{\partial^2\Psi}{\partial X_2^2}\right]+\tfrac{1}{2}(\omega_1^2X_1^2+\omega_2^2X_2^2)\Psi=E\Psi . \tag{22}$$

In this form the problem yields an equation separable into equations for X_1 and X_2:

$$-\frac{\hbar^2}{2}\frac{d^2\Psi_i}{dX_i}+\tfrac{1}{2}\omega_i^2X_i^2\Psi_i=E_{X_i}\Psi_i . \tag{23}$$

The solutions of (22) and (23) are the well-known oscillator wave functions and energy eigenvalues:

$$E_X=E_{X_1}+E_{X_2}=\hbar[\omega_1(n_{X_1}+\tfrac{1}{2})+\omega_2(n_{X_2}+\tfrac{1}{2})] , \tag{24}$$

$$\Psi_{n_X}(X)=\left[\frac{2\omega}{\hbar}\right]^{1/4}\frac{1}{(2^n n!)^{1/2}}e^{-(2\omega/\hbar)X^2}H_n(\sqrt{\omega/\hbar}X) . \tag{25}$$

The linear transformation is similar for the two other axes. Hence the general solution has the form

$$E=\hbar[\omega_1(n_{X_1}+n_{Y_1}+n_{Z_1}+3/2)+\omega_2(n_{X_2}+n_{Y_2}+n_{Z_2}+3/2)] , \tag{26}$$

$$\Psi(X,Y,Z)=\Psi_{n_X}(X_1)\Psi_{n_X}(X_2)\Psi_{n_Y}(Y_1)\Psi_{n_Y}(Y_2)\times\Psi_{n_Z}(Z_1)\Psi_{n_Z}(Z_2) . \tag{27}$$

Oscillator potentials allow a closed-form solution of the three-body problem in its most general form with different masses (and oscillator strengths),[5] without recurring to Lippmann-Schwinger[6] or Faddeev[7] equations. It would be most interesting to compare solutions[8] obtained by such methods with the closed-form solutions given here.

IV. GENERAL CENTRAL POTENTIALS

As mentioned before, Eq. (14) yields three equivalent Hamiltonians, privileging pairs of relative coordinates. Writing explicitly one of them, for r_{12}, r_{23}:

$$\mathcal{H}_{12,23}=\left[1+\frac{\mathcal{M}_{31}}{\mathcal{M}_{12}}\right]\frac{\mathcal{P}_{12}^2}{\mathcal{M}_{12}}+\left[1+\frac{\mathcal{M}_{31}}{\mathcal{M}_{23}}\right]\frac{\mathcal{P}_{23}^2}{2\mathcal{M}_{23}}+\frac{\mathcal{M}_{31}}{\mathcal{M}_{12}\mathcal{M}_{23}}\mathcal{P}_{12}\cdot\mathcal{P}_{23}+V_{12}(r_{12})+(V_{23})(r_{23})+V_{31}(r_{31}) , \tag{28}$$

where

$$V_{31}(r_{31})=V_{31}(|\mathbf{r}_{12}+\mathbf{r}_{23}|)=V_{31}(|r_{12}\check{e}_{12}+r_{23}\check{e}_{23}|) . \tag{29}$$

Thus, $\mathcal{H}_{12,23}$ is a function of r_{12}, r_{23} and of the associated momenta (kinetic momenta with redefined masses); the directions of relative position and momentum vectors are also involved in (28). It is reasonable to split the Hamiltonian in two:

$$\mathcal{H}^0_{12,23}=\left[1+\frac{\mathcal{M}_{31}}{\mathcal{M}_{12}}\right]\frac{\mathcal{P}_{12}^2}{2\mathcal{M}_{12}}+\left[1+\frac{\mathcal{M}_{31}}{\mathcal{M}_{23}}\right]\frac{\mathcal{P}_{23}^2}{2\mathcal{M}_{23}}+V_{12}(r_{12})+V_{23}(r_{23}) , \tag{30}$$

$$\mathcal{H}^i_{12,23}=\frac{\mathcal{M}_{31}}{\mathcal{M}_{12}\mathcal{M}_{23}}\mathcal{P}_{12}\cdot\mathcal{P}_{23}+V_{31}(|r_{12}\check{e}+r_{23}\check{e}_{23}|) . \tag{31}$$

The Schrödinger equation for Hamiltonian $\mathcal{H}^0_{12,23}$ may have closed form solutions for certain potentials. It *is separable* in r_{12} and r_{23}. Hence it is possible with the orthonormal basis of (30) to approximate the solutions of the full Hamiltonian using (31) as a perturbation. As the full Hamiltonian possesses, in general, solutions, the convergence of a perturbation calculation seems assured. Equations (30) and (31) provide solutions for exotic atomic systems, such as kaonic heliumlike systems, explored below. It is to be noted that $\mathcal{H}^0_{12,23}$ is a mathematically correct formulation for the zeroth-order approximation of three-body systems in congruent coordinates and momenta, i.e., where momenta correspond to the relative coordinates of the interaction potentials.

It is worth noticing the structure of the first term of $\mathcal{H}^i_{12,23}$ in Eq. (31). It corresponds quite strikingly to the momentum dependence of the main component of the two-pion exchange three-body force of Fujita and Miyazawa.[9] It is easy to treat such a term in the case of three-body final states with asymptotically free particles.[10] Correlation spectra of nucleons in the deuteron breakup induced by nucleons seem to confirm effects steming from such terms, which can be reinterpreted as momentum-dependent potentials.[11,12] The approach in terms of relative momenta and coordinates is suitable both for bound systems and for scattering states.

V. NUMERICAL TESTS OF THE FORMULATION

A. Symmetric three-particle oscillator system in a plane

The general three-body problem with harmonic potentials is amenable to a closed-form solution, as detailed in Sec. IV. However, it is instructive to carry out some numerical tests. In the particular case when the three masses (m) and coupling constants (k) are identical, the ground state will correspond to the totally symmetric configuration, i.e., the system will be in the shape of an equilateral triangle, one mass on each vertex. It is easy to see from this symmetry that the motion is equivalent to that of three oscillators, each moving radially with respect to the three-body center of mass under a force derived from a potential $V=\frac{3}{2}kr^2$. Hence the exact ground-state energy of the system is (three times the energy of a "linear" system)

$$E_0=\tfrac{3}{2}\sqrt{3}\hbar\omega . \tag{32}$$

A system of symmetric states can be derived, having energies $E_{n,s}=\sqrt{3}3(n+1/2)\hbar\omega$, where $\omega=\sqrt{k/m}$.

From (14) and (16) we can write the Hamiltonian $H_{12,23}$, with $\mathcal{M}_{12}=\mathcal{M}_{23}=\mathcal{M}$,

$$H_{12,23}=\frac{\mathcal{P}_{12}^2}{\mathcal{M}}+\frac{\mathcal{P}_{23}^2}{\mathcal{M}}+\frac{\mathcal{P}_{12}\cdot\mathcal{P}_{23}}{\mathcal{M}}+kr_{12}^2+kr_{23}^2+k\mathbf{r}_{12}\cdot\mathbf{r}_{23} ; \tag{33}$$

thus

$$H^0_{12,23}=\frac{\mathcal{P}_{12}^2}{\mathcal{M}}+\frac{\mathcal{P}_{23}^2}{\mathcal{M}}+kr_{12}^2+kr_{23}^2 , \tag{34}$$

$$H^i_{12,23}=\frac{\mathcal{P}_{12}\cdot\mathcal{P}_{23}}{\mathcal{M}}+k\mathbf{r}_{12}\cdot\mathbf{r}_{23} . \tag{35}$$

Clearly, according to our definitions $\mathcal{M}=m/3$ and the kinetic energy mass factor *is* $\frac{2}{3}m$. Solving the corresponding eigenvalue equations one obtains

$$E^{(0)}_{0,12,23}=\sqrt{3}\hbar\omega . \tag{36}$$

Here $\omega=\sqrt{k/m}$, as in Eq. (32) above. The ratio of (32), exact value, and (36) zeroth-order calculation is 1.5. It is fairly evident that due to the symmetry of the dynamical system, $H^i_{12,23}$, given in (35) is equivalent to each of the two terms $(\mathcal{P}_{ij}^2\mathcal{M}^{-1}+kr_{ij}^2)$ in (34). Hence the exact contribution to the energy of $H^i_{12,23}$ will be $\sqrt{3}/2\hbar\omega$.

It is relevant to ascertain the possible effectiveness of an approach where the redundancy of coordinates and momenta in (10) is neglected as a first step towards a solution, then the eigenvalue equation will provide for the ground-state energy

$$E_0=\tfrac{3}{2}\sqrt{3}\hbar\omega . \tag{37}$$

That is, *the exact value.* This result is not unexpected, because the symmetric oscillator system considered here is reducible to three independent linear oscillators, as discussed above. A similar result is obtained for the sequence of symmetric states, not only for oscillator potentials, but for any (well-behaved) potential. The full set of

states of (10) is a suitable orthonormal basis to treat the Hamiltonians in (14). It is of course possible to eliminate a (chosen) redundant variable in the solutions of (10), r_{kl}, for example, keeping r_{ij} and r_{jk}, using (12),

B. Linear symmetric three-particle oscillator

In this example the purpose is to carry out a first-order perturbation correction to the Hamiltonian $\mathcal{H}^0_{12,23}$. The system consists of three equal masses (m), acted upon by two potentials $V_{12}(r_{12})$ and $V_{23}(r_{23})$ [$V_{31}(r_{31})=0$], in a state of linear symmetric motion (it is one of the "normal" modes of the general problem). Thus, one of the masses, at the center, remains at rest and the system is equivalent to two independent oscillators. It is straightforward to write the exact energy of the ground state of this system:

$$E_0=\hbar\omega \ . \tag{38}$$

The treatment according to (14) and (16) yields here

$$\mathcal{H}^0_{12,23}=\frac{\mathcal{P}^2_{12}}{\mathcal{M}}+\frac{\mathcal{P}^2_{23}}{\mathcal{M}}+\tfrac{1}{2}kx^2_{12}+\tfrac{1}{2}kx^2_{23} \ , \tag{39}$$

$$\mathcal{H}^i_{12,23}=\mathcal{P}_{12}\cdot\mathcal{P}_{23}\mathcal{M}^{-1} \ . \tag{40}$$

The eigenvalue equation for $\mathcal{H}^0_{12,23}$ yields in the zeroth order,

$$E^{(0)}_0=\sqrt{3/2}\hbar\omega \ , \tag{41}$$

$\omega=(k/m)^{1/2}$. Due to the assumed symmetry of the system, the exact Hamiltonian in generalized relative momenta is

$$\mathcal{H}_{12,23}=\frac{1}{2\mathcal{M}}(3\mathcal{P}^2_{12}+3\mathcal{P}^2_{23})+V_{12}+V_{23} \ , \tag{42}$$

which, of course yields the eigenvalue (38) for the ground state. In the zeroth order (39) overestimates the ground-state energy by $0.2247\hbar\omega$. The first-order correction in a conventional perturbation expansion, with $\mathcal{H}'$ given by (40), results in $E^{(1)}_0=-0.125\hbar\omega$ and

$$E_0\cong E^{(0)}_0+E^{(1)}_0=1.0997\hbar\omega \ , \tag{43}$$

within 10% of the exact value.

C. A bound state of three particles with short-range two-body interactions

It is convenient to introduce a "finite-depth" oscillator potential, i.e., a potential such that

$$\begin{aligned} &V=\tfrac{1}{2}kr^2 \quad \text{for } r\le b \ , \\ &V=\tfrac{1}{2}kb^2=V_0 \quad \text{for } r>b \ . \end{aligned} \tag{44}$$

Thus V_0 defines the depth of the finite oscillator well. The depth parameter can be determined such that for the two-particle subsystems a given value of the binding energy is obtained. Introducing some parameters from nuclear systems, such as nucleon masses, deuteron binding energy, and radius, it is possible to determine V_0 for the two-particle system (pseudodeuteron). The Hamiltonian is given by

$$\begin{aligned} &H=\frac{p^2}{2\mu}+\tfrac{1}{2}kr^2, \quad r\le b \\ &H=\frac{p^2}{2\mu}+V_0, \quad r>b \end{aligned} \tag{45}$$

with $\mu=0.504$ mass units (1 m.u.$=931.48$ MeV), $b=1.956$ fm. The condition to yield the binding energy is

$$V_0-E_0=2.225 \text{ MeV} \ , \tag{46}$$

where E_0 is the ground-state energy of (45) for $r\le b$,

$$E_0=\tfrac{3}{2}\hbar\omega \ , \tag{47}$$

with $\omega=\sqrt{k/\mu}$. One obtains $V_0=102.75$ MeV and $E_0=100.53$ MeV. Let us assume now that the dynamics (the forces) in the three-particle system is the same as described above; thus

$$\begin{aligned} &V_{ij}=\frac{V_0}{b^2}r^2_{ij} \text{ for } r_{ij}\le r_m \ , \\ &V_{ij}=\Lambda \text{ for } r_{ij}>r_m \ . \end{aligned} \tag{48}$$

For the calculation of the symmetric ground state of the system we can use results from Sec. V A and recur to the Hamiltonian expressed in redundant variables. For a symmetric system of spherical particles the ratio $r_m/b=1.1$ to a good approximation. The energy of the ground state $E_0=123.06$ MeV, and using $r_m/b=1.1065$, one obtains a binding energy $E_b=2.74$ MeV for each subsystem ($V_{ij}=125.8$ MeV for $r_{ij}=r_m$) and a total binding energy $E^T_b=8.22$ MeV for the system, a "pseudotriton." The treatment of the ground state of the symmetric three-particle system with short-range potentials, formulated in terms of relative coordinates and momenta, is particularly transparent. This author has found an unsuspected wealth of details providing physical insight on the dynamics and relations of parameters governing short-range interactions in quantized systems. The advantages of the Schrödinger equation over the Faddeev approach[7] have been pointed out before.[8] The particular numerical examples chosen here, simple back-of-the-envelope calculations, would have required considerable computations in the conventional approach with Faddeev equations and amplitudes.

VI. THE KAONIC ^{3}He ATOM AND OTHER EXOTIC SYSTEMS

The Hamiltonian given by (14) and particularized for the pair of relative coordinates r_{12} and r_{23} in Eqs. (28)–(31) will be used to treat a kaonic atom consisting of a ^{3}He nucleus and two K^- mesons. The purpose is to show that there is a difference in the results due to the analytic expressions (28)–(31) with respect to the conventional approach. Assuming that the nucleus is designated by the subindex 2, the conventional zeroth-order Hamiltonian is written as

$$\mathcal{H}^0=\frac{p^2_{12}}{2\mu_{12}}+\frac{p^2_{23}}{2\mu_{23}}+V_{12}(r_{12})+V_{23}(r_{23}) \ , \tag{49}$$

where μ is a reduced mass, and the perturbation is

$$\mathcal{H}' = V_{31}(r_{13}) \ . \tag{50}$$

In fact, conventionally, for the ordinary helium atom the subindex designating the nucleus is dropped (the implicit assumption is that of a fixed nucleus). The mathematically correct expressions (30) and (31) differ from the above, and thus a calculation with (49) and (50) will differ generally in all orders and in the limit from the correct values.

For the kaonic system defined above the ratio of the nuclear to the atomic volume is in the range of 10^{-4}; hence it is reasonable to utilize the concept of screening and to carry out a variational calculation to demonstrate the different results. Defining an effective charge factor, $\zeta = Z - s$ (s, screening parameter). The calculations are carried out with the corresponding effective masses of expressions (30) and (31), writing ($M = e = \hbar = 1$)

$$E = -\zeta^2 - 2s\zeta^3\pi^{-1}\int \exp(-2\zeta r_{12}) r_{12}^{-1} dr_{12} + \zeta^6\pi^{-1}\int\int \exp[-2\zeta(r_{12}+r_{23})] r_{13}^{-1} dr_{12} dr_{23} \ , \tag{51}$$

which leads to $E = f(Z,\zeta)$; ζ is the variational parameter to be determined by $\partial E/\partial\zeta = 0$.[13] This choice of ζ yields a zero first-order correction in the perturbation expansion. Such a type of calculation has given the binding and ionization energies to within 1.7% of the experimental values for the ordinary helium atom.[13] Table I summarizes the results for the kaonic atom. It is simple to calculate also the antiprotonic ^{3}He atom (^{3}He nucleus and two antiprotons).

It is well known that the H^- ion is bound and within the frame of the variational calculation; binding can be obtained with a modified wave function for the zeroth-order two-electron system. *Mutatis mutandis,* one *can predict* bound states for exotic systems such as PS^- (already found), $\mu^+\mu^-\mu^-$, $\bar{P}PP$, etc. For calculations of such systems it should be particularly relevant to obtain the effective mass factors and perturbation from Eqs. (30) and (31), avoiding thus the incorrect formulation of Eqs. (49) and (50).

VII. EXTENSION TO n-BODY SYSTEMS

The Hamiltonian can be written almost identically to (10):

$$\mathcal{H} = \sum_{C_n^2} \left[\frac{\mathcal{P}_{ij}^2}{2\mathcal{M}_{ij}} + V_{ij} \right] , \tag{52}$$

where $C_n^2 = n(n-1)/2$, the number of combinations two at a time, without repetition. The Hamiltonian has a number of redundant $\mathbf{r}_{ij}$: $\mathcal{R} = n(n-3)/2+1$ There are a number of C_n^3 closure relations between the $\mathbf{r}_{ij}$'s, and correspondingly between the $\mathcal{P}_{ij}$ which allow the elimination of redundant coordinates. Here again it is possible for harmonic-oscillator potentials to reduce the system to an ensemble of uncoupled oscillators and to obtain a closed-form solution. Of course, a similar technique to that outlined for the three-body problem can be applied to approximate the general solutions of the n-body problem, starting with an "unperturbed" Hamiltonian H_0. Here again one finds the scalar terms proportional to two pairs of two-body momenta, "momentum-dependent potentials," mocking up three-body forces. It seems "*prima facie*" that this formulation provides a systematic approach particularly suitable to the generation of supercomputers arriving now in the market. Thus, aside from the methods based on series expansions, other numerical techniques may be used to solve equations stemming from (52) and the closure equations, which have very simple linear structures.

As a numerical example of an n-body system, let us consider a four-body system with equal masses subject to short-range interactions, in particular, finite-depth harmonic potentials as defined in Sec. V C. A symmetric four-body system with equal masses can be visualized as a tetrahedral structure. In such a case it is also possible to separate the Hamiltonian and the eigenvalue equation into six terms. Taking into account (52) one obtains readily $E_0 = 142.2$ MeV and, with a radius parameter increased by a factor 1.081 with respect to that of the three-body system (in reasonable accord with the $A^{1/3}$ law for nuclei, where the ratio is 1.10 for $A = 3$ and 4), $V_0 = 147$ MeV. Thus the binding energy per term is $E_b = 4.8$ MeV and the four-particle system total binding energy $E_b^t = 28.8$ MeV. This system can be called a "pseudo-alpha particle." There is clearly a good consistency of parameters and binding energies, maintaining a "universal" field of force between pairs of particles.

TABLE I. Kaonic and antiprotonic ^{3}He atoms. The primes indicate the results of a conventional calculation, E are binding energies, and I are ionization energies. There are no experimental data yet on these atoms. Experiment may be enticed by these calculations.

Type of atom	E (keV)	E' (keV)	I (keV)	I' (keV)
Kaonic	−65.12	−63.67	19.38	18.95
Antiprotonic	−113.77	−106.6	33.87	31.74

VIII. SOLUTION IN SYMMETRIC COORDINATES

For simplicity it will be elaborated in the three-body system. Instead of eliminating one redundant coordinate, the three-body equations expressed in relative coordinates and momenta can be solved preserving the symmetry of such formulation. It is convenient to use a vector notation to avoid the lengthy scalar equations. Firstly, for its heuristic value, it is useful to discuss the problem in classical mechanics. The Lagrangian is

$$L = \tfrac{1}{2}\sum \mathcal{M}_{ij}\dot{\mathbf{r}}_{ij}^2 - \sum U_{ij}(|\mathbf{r}_{ij}|) \ , \tag{53}$$

with the condition

$$\sum \mathbf{r}_{ij} = 0 \ . \tag{54}$$

The method of Lagrange multipliers[14] allows one to establish separate equations for each pair $\mathbf{r}_{ij}$, $\mathcal{P}_{ij}=\mathcal{M}_{ij}\dot{\mathbf{r}}_{ij}$. Thus, defining L'

$$L'=\tfrac{1}{2}\sum\mathcal{M}_{ij}\dot{\mathbf{r}}_{ij}^{2}-\sum U_{ij}(|\mathbf{r}_{ij}|)-\boldsymbol{\lambda}\cdot\sum\mathbf{r}_{ij}\ , \tag{55}$$

where $\boldsymbol{\lambda}=\boldsymbol{\lambda}(\lambda_1,\lambda_2,\lambda_3)$ is a multiplier "vector." The Lagrangian L' possesses at each instant the same values as L, but the addition of the term including the condition of constraint allows one to separate the equations for each pair $\mathbf{r}_{ij}$,

$$\frac{d}{dt}\frac{\partial L'}{\partial\dot{\mathbf{r}}_{ij}}+\frac{\partial U_{ij}}{\partial\mathbf{r}_{ij}}-\boldsymbol{\lambda}=0\ . \tag{56}$$

From the solutions $\mathbf{r}_{ij}=\mathbf{r}_{ij}(t,\boldsymbol{\lambda})$, replacing them in (54) one obtains $\boldsymbol{\lambda}=\boldsymbol{\lambda}(t)$ and finally $\mathbf{r}_{ij}=\mathbf{r}_{ij}(t)$. From (54) it is possible to express one of the relative coordinate vectors as a function of the other two.

The solution of the problem in classical mechanics with symmetrical coordinates should have an equivalence in quantum mechanics. Thus one can formulate firstly an equivalent Hamiltonian

$$\mathcal{H}^{e}=\sum\mathcal{H}_{ij}+\boldsymbol{\lambda}\cdot\sum\mathbf{r}_{ij} \tag{57}$$

which has the same values as the original Hamiltonian $\mathcal{H}$ due to (54). As in the Lagrangian treatment it is now permissible to proceed, separating the Hamiltonian into components of each subsystem:

$$\mathcal{H}_{ij}^{e}=\mathcal{H}_{ij}+\boldsymbol{\lambda}\cdot\mathbf{r}_{ij}\ ; \tag{58}$$

the vector $\boldsymbol{\lambda}$ is in general time dependent and allows the separation. The Schrödinger equations are thus also time dependent:

$$\mathcal{H}_{ij}^{e}\psi_{ij}=i\hbar\frac{\partial\psi_{ij}}{\partial t}\ . \tag{59}$$

The solution can be expressed as

$$\psi_{ij}(\mathbf{r}_{ij},t)=A_{ij}e^{-i/\hbar(\mathcal{H}_{ij}t)}e^{-i/\hbar[\mathbf{r}_{ij}\cdot\boldsymbol{\Lambda}(t)]}\ , \tag{60}$$

where

$$\boldsymbol{\Lambda}(t)=\int_0^t\boldsymbol{\lambda}(t)dt\ . \tag{61}$$

The solution for the three-body system is thus

$$\psi=\Pi\psi_{ij}=\Pi A_{ij}e^{-i/\hbar(\mathcal{H}_{ij}t)}e^{-i\hbar^{-1}\left[\sum\mathbf{r}_{ij}\right]\cdot\boldsymbol{\Lambda}(t)} \tag{62}$$

and due to (54) the auxiliary function $\boldsymbol{\Lambda}(t)$ becomes irrelevant for the three-body solution:

$$\psi=\Pi A_{ij}e^{-i/\hbar(\mathcal{H}_{ij}t)}\ . \tag{63}$$

Therefore the problem is now reduced to obtaining the solutions of the uncoupled two-body subsystems with time-independent Hamiltonians applying the condition (54) *a posteriori*. It is also feasible to introduce Jacobi coordinates *a posteriori*. For example,

$$\mathbf{q}=\mathbf{r}_{23}+\frac{M_1}{M_1+M_2}\mathbf{r}_{12}\ . \tag{64}$$

$\mathbf{r}_{12}$ would then be the coordinate $\mathbf{p}$ in the usual notation. It is also apparent that if $\mathbf{r}_{12}$ represents a pair that is bound and $\mathbf{r}_3\to\infty$, $\mathbf{q}$ and $\mathbf{r}_{23}$ are asymptotically equivalent. The simpler form of the solution of the quantum-mechanical problem is perhaps not unexpected. It is certainly due to the form of the time-dependent Schrödinger equation leading to the exponential form of (60). This in turn allows one to dismiss the time dependence of the two-body Hamiltonian operators and find stationary solutions and eigenvalues, which can be regarded as time-averaged values.

The symmetric solution has the advantage that the radial potentials are handled with simplicity, whereas the methods eliminating redundancy of coordinates lead at once to approximate methods, as demonstrated in previous sections.

IX. CONCLUDING REMARKS

The formulation of the n-body problem with the Hamiltonian given in (52) opens interesting alternative avenues for the analysis and solution of three- and n-body systems.[15] The transparency of the resulting equations with respect to the actual dynamics is noteworthy. It allows one to easily obtain intuitive experience and insight into the behavior of multiparticle systems, similar to that normally associated with two-body systems. Long-range potentials r_{ij}^{-1} can be introduced simply in three- and n-body systems.

Note added. The possibility of a solution in terms of symmetric coordinates could be construed as a reason for abandoning the other methods. However, the procedure of eliminating a redundant coordinate has its own merits. The closure condition is included *ab initio* and there is no need to define the coordinate space of the system afterwards. The perturbation terms of type (31) are a bilinear carbon copy of the terms in (30). There is cyclic permutation symmetry of the particle pair subindices; this implies that the choice of the redundant coordinate can be made, in principle, arbitrarily. Common sense will guide the choice in each problem. If all interactions and masses are identical the choice is arbitrary, but the solution should be invariant with respect to the cyclic permutation. If the Hamiltonian containing the pure quadratic terms leads to a solution with eigenfunctions and eigenvalues, then it can be accepted as an axiom that the bilinear terms, themselves not positive definite, will converge when calculated in the eigenfunction basis and will yield corrections smaller than the values of the positive definite pure quadratic terms. For systems with three asymptotically free particles the symmetric Hamiltonian may be preferred. The Jacobi coordinate method is ideally suited to systems consisting of a bound state of a pair and an asymptotically free particle.

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