S-matrix pole trajectories for Yukawa potentials

L. A. L. Roriz and A. Delfino*
Departamento de Física, Universidade Federal de Pernambuco, 50.000 Recife—PE, Brazil
(Received 6 March 1986)

Using the two-body K matrix extended to the whole complex-momentum plane, we construct the S matrix as a function of the strength λ of attractive Yukawa potentials. By varying λ we study pole trajectories for the s, p, and d waves.

In potential scattering theory, bound states are associated with poles of the on-the-energy-shell (on-shell) S matrix on the physical sheet of the complex energy plane. The so-called physical energy sheet corresponds to the upper complex momentum (hereafter, k) half-plane, where poles can occur only on the imaginary axis. However, on the lower complex k half-plane (corresponding to the unphysical sheet of energy), the poles can occur either on the negative imaginary axis (virtual states) or in pairs of complex conjugate numbers (resonances). If such poles are close to the real axis, dramatic effects on the scattering observables like phase shifts and cross sections can be observed. The above-mentioned properties of the S-matrix poles are very well known and are naturally connected with some conditions that the local potential V has to obey.

If we now define a potential $V'=\lambda V$ we can relate all these poles through the variation of the strength parameter λ. By varying λ, we get the trajectories followed by the poles. S-matrix pole trajectories were first of all treated by Nussenzveig.[1] He studied the scattering by a spherically symmetric rectangular potential with depth V_0 and range a. The appropriate strength parameter for the square-well potential is $\lambda=V_0a^2$. We present briefly here his main conclusions for the s-wave case. For $\lambda=V_0a^2\to 0$ the S poles reside at $\pm n\pi-i\infty$, $n=0,1,2\ldots$. With increasing λ the poles move upwards in the complex k plane. The pole for $n=0$ has no partner and moves faster than the pairs of complex poles ($n\neq 0$). This pole ($n=0$) alone becomes the first bound state, crossing the origin for $\lambda=\pi^2/4$. Increasing λ further, all pairs of poles ($n\neq 0$) collide before they reach the origin (at the same point) on the negative imaginary axis for different critical values $\lambda_c(n)$. After each pole collision one pole goes down the imaginary axis while the other moves upwards, becoming first a virtual state (before it crosses the origin) and later an additional bound state (excited state). For p waves ($l=1$) all the complex pole pairs meet at the origin for critical values $\lambda_c(n)$. By increasing λ, successive excited states are created. This is in synthesis the main behavior of the S-pole trajectories for a square-well potential. It was important to describe it because other, not so simple, potentials qualitatively repeat this behavior. Such is the case of the screened Coulomb potential[2] and the square well augmented by a spherically symmetric r^{-2} potential.[3] It is interesting to point out that the cut on the negative imaginary axis originating from r^{-2} completely destroys the existence of virtual states previously obtained from the pure square well.

For all the above-mentioned interactions it is possible to analytically obtain Jost functions from which the S matrix is constructed. This is no longer the case for the Yukawa potential. Perhaps that is why up to now (in spite of its importance in nuclear physics) no S-pole trajectories are known for this potential, as has been pointed out by Newton.[4] In view of this, one has to numerically handle the Lippmann-Schwinger (LS) equation. This equation is very well known on the upper half of the complex k plane, where usually calculations of bound states and phase shifts can be carried out. For our purposes, however, something more has to be done, once we intend to dive into the lower half of the complex k plane. To this aim the LS equation in momentum space is more convenient for beginning the necessary analytical continuation from the physical sheet into the unphysical one. This was first done by Glöckle[5] and afterwards by Fonseca, Adhikari, and Tomio.[6] The former analytically continues the LS equation (before the calculation) into the unphysical sheet of energy, while the latter approximate the solution of the LS equation (calculated in the first sheet of energy) with known analytical properties and continue it analytically into the unphysical sheet of energy.

In this work we develop a K-matrix method in order to be able to extend it to the whole complex k plane. This approach is quite equivalent to that mentioned above. Through the known on-shell relation between the K matrix and the S matrix we obtain the pole trajectories for the pure attractive Yukawa's potential, for s ($l=0$), p ($l=1$), and d ($l=2$) waves.

We start with the partial wave LS equation (in units $\hbar=2m=1$) in momentum space

$$t_l(p,k,k^2)=V_l(p,k)+\frac{2}{\pi}\int_0^\infty \frac{q^2dq\,V_l(p,q)t_l(q,k)}{k^2-q^2+i\epsilon}\,, \tag{1}$$

with

$$V_l(p,k)=\int_0^\infty r^2dr\,j_l(pr)V(r)j_l(kr)\,, \tag{2}$$

where $V(r)$ is a local spherically symmetric potential and j_l is the spherical Bessel function. The t matrix satisfying Eq. (1) has a square-root unitarity cut along the real positive energy axis. Equation (1) can be solved as written just on the physical energy sheet, i.e., on the upper half of the complex k plane. The presence of the unitarity cut causes

Eq. (1) not to be single valued in the whole k^2 plane (energy plane). In order to have the desired analyticity in k^2 one has to map the k values onto two energy sheets, separated by a cut along the real positive axis. The first sheet corresponds to the upper half of the complex k plane and the second to the lower half.

For the sake of illustration we give a simple example. If one uses the nucleon-nucleon Reid $^1S_0$ potential in Eq. (1), no bound state can be found. Definitely no poles of $t(k)$ arise in the upper half of the complex k plane. If, however, Eq. (1) is modified (analytically continued into the lower half of the complex k plane)[5] one pole $k=-i\,|\,k_v\,|$ corresponding to the virtual state $E=+k^2=-0.1218$ MeV is found.

We present here a very simple method in order to work in the complete complex k plane. This method is known as the K-matrix approach, normally used on the physical energy sheet. The method is constructed from Eq. (1) through the use of the identity

$$\lim_{\epsilon\to 0}\frac{1}{k^2-q^2\pm i\epsilon}=\frac{P}{k^2-q^2}\mp i\pi\delta(k^2-q^2) \tag{3}$$

and the definition

$$K_l(p,k,k^2)=V_l(p,k)+\frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{q^2dq\,V_l(p,q)K_l(q,k)}{k^2-q^2}\,, \tag{4}$$

where P is the principal value of the integral. After that the relation between the on-shell elements $t_l(k,k,k^2)\equiv t_l(k)$ and $K_l(k,k,k^2)\equiv K_l(k)$ is given by

$$t_l(k)=K_l(k)-ikK_l(k)t_l(k)\,. \tag{5}$$

To calculate the K matrix from Eq. (4) one subtracts a term identically zero and then the principal value prescription is dropped:

$$K_l(p,k,k^2)=V_l(p,k)+\frac{2}{\pi}\int_0^\infty \frac{dq}{k^2-q^2}[q^2V_l(p,q)K_l(q,k)-k^2V_l(p,k)K_l(k,k)]\,. \tag{6}$$

Equation (6) is nonsingular, and consequently K does not possess the unitarity cut. It makes a great difference whether one starts from Eq. (1) or from Eq. (6) if one intends to dive into the second sheet. While in the former case one has to deform the path of integration in order to move across the cut,[5] Eq. (6) allows one to go into the second sheet just by choosing a complex k belonging to the lower half complex k plane because now there is no cut. Of course our argument supposes that the integrands in Eq. (1) and in Eq. (6) are analytical functions of q except for possible singularities coming from the potential. Summarizing, Eq. (6) is, as written, valid on the complete k plane and relation (5) is preserved.

Now let us briefly discuss the analytical properties of $V_l(p,k)$ for the Yukawa potential. If one has $V(r)=-V_0e^{-\mu r}/\mu r$ there is a cut beginning on the negative imaginary axis for $p=k=-i\mu/2$. On this segment Eq. (6) is not defined. However, Eq. (6) involves $V_l(p,k)$ also for $p\neq k$. In such a case there is a branch point for $(p+k)^2+\mu^2=0$, and once the integration variable q is always real the allowed complex k values cannot have imaginary parts larger than μ. This is the analytical region where Eq. (6) can be defined. It is interesting to point out that Eq. (6) is coupled with a similar equation for $K_l(k,k,k^2)$. We notice also that we tested our method to reproduce some results presented in Refs. 1 and 2.

Before we present our results we indicate how we have obtained the S-matrix poles. By solving Eq. (6) we obtain the K matrix. If k is real or pure imaginary, $K_l(k)$ is real, otherwise $K_l(k)$ is complex. The S-matrix poles that are the same as the t-matrix poles were calculated through Eq. (5). If k is a complex number the search of S-matrix poles is reduced to finding zeros of a function in the complex k plane. This reduces to the search of zero of a function of two variables (Rek and Imk). It has been done by the NAG library programs with a DEC10 computer. The numerical error was less than 1%, which we verified by

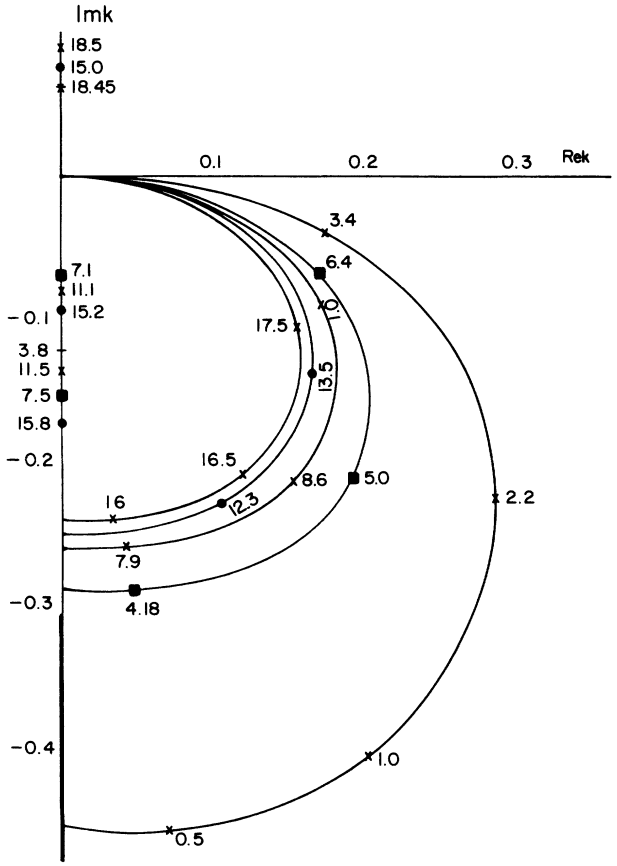

FIG. 1. S-matrix pole trajectories for p-wave scattering by an attractive Yukawa potential of strength λ. The values of λ are shown on the curves. The poles (●) originate the first bound state for $\lambda>3.64$. The other pole trajectories (■, ×, and *) show the formation of excited states. Each trajectory has a symmetric one for negative values of Re(k).

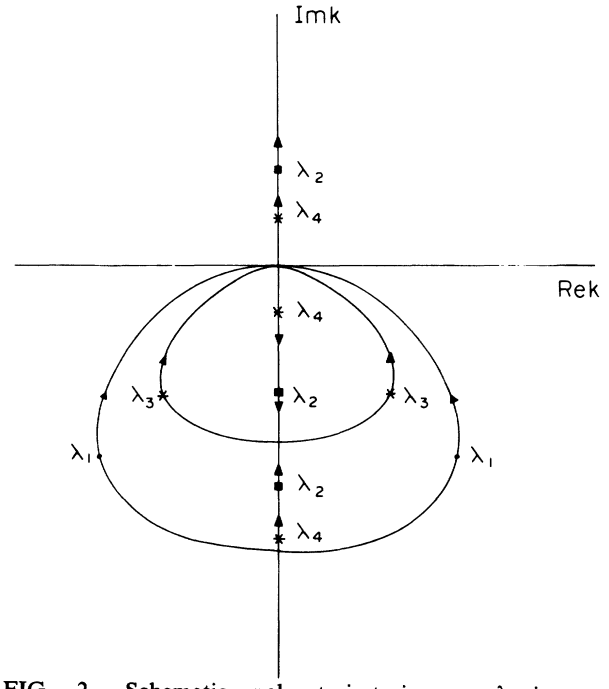

FIG. 2. Schematic pole trajectories as λ increases ($\lambda_1<\lambda_2<\lambda_3<\lambda_4$).

increasing integration mesh points. We made certain that no poles were missing.

The results for $V(r)=-\lambda V_0 e^{-\mu r}/\mu r$, $V_0=41.2942$ MeV, and $\mu=0.6329$ fm^{-1} are as follows.

(1) For $l=0$ no poles were found outside the imaginary axis for values of λ large enough to produce ten excited bound states. In other words, for $\lambda=0.01$ there are no bound states, no virtual states, and no poles were found in the analytical region for complex values of k. By increasing λ the first pole, corresponding to the first virtual state, arises from the cut (located in the second sheet of energy or on the negative imaginary axis of the complex k plane). By further increasing λ this pole crosses the origin, becoming a bound state. It occurs for $\lambda=1.05$. By further increasing λ new poles arise from the cut and move upwards along the imaginary axis to become excited bound states. We see that for $l=0$ the pole trajectories for the attractive Yukawa potential show no similarity to those discussed in Refs. 1–3.

(2) For $l=1$ the trajectories are presented in Fig. 1. For $\lambda<0.3$, no poles were found in the analytical region. For $\lambda>0.3$ a pair of symmetrical complex poles leaves the imaginary axis (still in the cut region) and goes along a curve towards the positive half k plane. On the origin both poles collide and one pole goes up the imaginary k axis while the other goes down. In the meantime by increasing λ another pole arises from the cut (on the imaginary axis) and collides with the one already described (as presented in Fig. 2). For $\lambda=4.14$ they collide, and by increasing λ a new pair of complex poles goes on a new curve towards the origin where these poles collide again. One pole goes up the positive imaginary axis (becoming the first excited state) while the other goes down, and this process repeats itself successively by increasing λ.

(3) For $l=2$ the pole trajectories are presented in Fig. 3 and the same discussion as for the case $l=1$ applies.

(4) From (2) one can better understand the case $l=0$. It is necessary to observe that the curves for increasing λ in

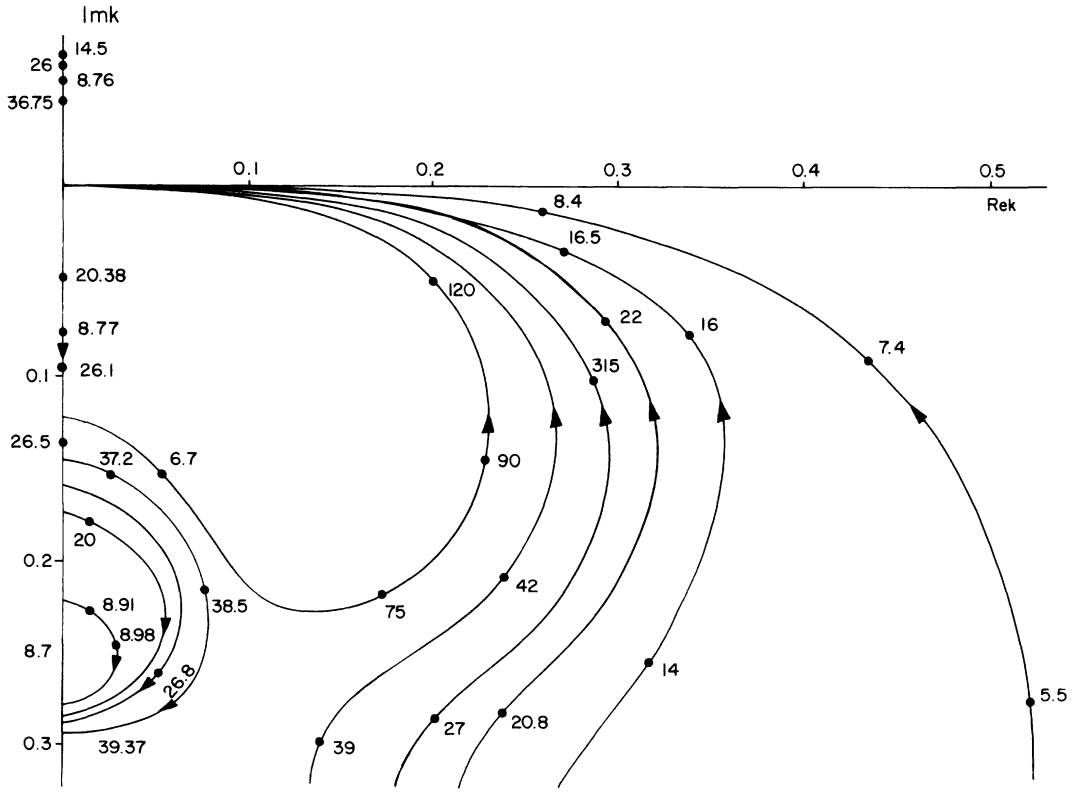

FIG. 3. Same as Fig. 1, for $l=2$.

$l=1$ are closer to the imaginary k axis. The increase of λ does not, of course, change the centrifugal barrier. In this manner $l=0$ can be seen as the limit case where the attractive part is completely dominant. In this case, as one can see from Fig. 1, the curves go to the imaginary axis.

(5) Now let us report that we extended our calculation ($l=0$) for the Malfiet-Tjon potential where a short-range repulsive part is present. For this case our conclusion in (1) is not changed.

Finally let us summarize the main points of our work. First, we have presented a new way to handle the LS equation in the whole complex k plane. Of course, the k-matrix approach is already very well known, but its usefulness in the search of the S-matrix pole was probably overlooked. This technique could be important in many problems in physics. For example, it can be used in the study of virtual states and resonances of some physical systems like $\bar{e}$-HCl (Ref. 3) and proton-deuteron in a two-body model as recently proposed.[7] S-matrix pole trajectories also have importance if one relates the two-body problem to the three-body problem. The works of Sofianos *et al.*[8] and Rupp *et al.*[9] showed that a class of two-body nonlocal interactions overbinds the triton. In other words, by constructing a nonlocal two-nucleon potential (like Tabakin and Beregi) that fits the two-nucleon observables one obtains "triton energies" of about 500 and 7 MeV. The latter could correspond to the experimental triton energy 8.48 MeV, but in the three-body spectrum it corresponds to no more than an excited state. As no argument can be presented to invalidate the calculated 500 MeV, it is easy to conclude that such interactions present a defect. This anomaly is intrinsically associated with two-body S-matrix poles on the real axis of the complex k plane of the nucleon-nucleon system, which can only be detected after a careful study of the S-matrix pole trajectories in the two-nucleon system as has been done in this paper. (Such poles are also called two-body continuum bound states.) If one had studied the S-matrix pole trajectories of Tabakin and Beregi potentials beforehand, one should have found that such potentials do not have the behavior expected of a realistic nucleon-nucleon interaction.

In conclusion, we have presented a new approach to the study of S-matrix pole trajectories which is expected to have interesting applications in the future.

We are grateful to S. K. Adhikari and L. Tomio for very helpful discussions and for reading the manuscript. This work was partially supported by Conselho Nacional de Desenvolvimento Científico e Technológico (Brazil) and Coordenação de Asperfeiçoamento (Brazil).

*Present address: Departamento de Física, Universidade Federal Fluminense, Niterói—RJ, Brazil.

[1]H. M. Nussenzveig, Nucl. Phys. **11**, 499 (1959).
[2]E. Ferreira and A. Teixeira, J. Math. Phys. **7**, 1207 (1966).
[3]H. Estrada and W. Domcke, J. Phys. B **17**, 279 (1984).
[4]R. G. Newton, *Scattering Theory of Waves and Particles*, 2nd ed. (Springer-Verlag, Berlin, 1982).
[5]W. Glöckle, *The Quantum Mechanical Few-Body Problem* (Springer-Verlag, Berlin, 1983).
[6]S. K. Adhikari, A. Fonsec, and L. Tomio, Phys. Rev. C **25**, 77 (1982).
[7]L. Tomio, A. Delfino, and S. K. Adhikari, Phys. Rev. C **35**, 441 (1987).
[8]S. Sofianos, N. J. McGurt, and H. Fiedeldey, Z. Phys. A **286**, 87 (1978).
[9]G. Rupp, L. Streit, and J. A. Tjon, Phys. Rev. C **31**, 2285 (1985).