Functional integral approach to the Lipkin model

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A quantum-mechanical formulation involving both collective and independent-particle motions in many-fermion systems is proposed by using the path-integral technique. A semiclassical method of evaluating the functional integral over both fields is described. As an illustration, the Lipkin model is utilized.

The complex interplay between collective and independent-particle motions has played an important role in the description of nuclear structure. The author has recently proposed a quantum-mechanical treatment[1] by using the path-integral technique for the description of such an interplay from the viewpoint of the mean-field theory. The path integral was then written as a functional integral over collective (Bose) and independent-particle (Fermi) fields, with constraints due to the elimination of the double counting of the degrees of freedom. If the independent-particle degrees of freedom were frozen, it was reduced to the path integral proposed in Ref. 2. Therefore our formulation was a natural extension of their attempts. However, a direct path integral over the Fermi fields is difficult, because the Lagrangian is not quadratic in the Fermi fields and the path integral includes some constraints. In this paper, I present another functional integral treatment to overcome these serious difficulties. With this aim, we employ a well-known Gaussian method[3] which makes the Lagrangian quadratic in the Fermi fields. We have no need to require the constraints in this method. Then the path integral over the Fermi fields is evaluated by using the quantum adiabatic approximation,[4,5] and then the effective Lagrangian of only the collective fields is obtained. As an illustration, we use the Lipkin model. There are several works[6,7] on the semiclassical quantization for this model. Shankar[6] has shown that the energy levels obtained by the Bohr-Sommerfeld quantization condition agree well with the exact ones. However, he has not included the independent-particle degrees of freedom. Therefore it is interesting to have the quantum-mechanical understanding of the independent particle.

The Hamiltonian of the Lipkin model with spin j is given by

$$\hat{H}=2\varepsilon\hat{J}_z-\tfrac{1}{2}V(\hat{J}_+\hat{J}_++\hat{J}_-\hat{J}_-)\ ,$$
$$\hat{J}_+=\sum_{m=1}^{2\Omega}\hat{a}_m^\dagger\hat{b}_m^\dagger=(\hat{J}_-)^\dagger\ , \tag{1}$$

$$\hat{J}_z=\tfrac{1}{2}(\hat{N}-2\Omega)=\tfrac{1}{2}\left[\sum_m(\hat{a}_m^\dagger\hat{a}_m+\hat{b}_m^\dagger\hat{b}_m)-2\Omega\right]\ ,$$
$$\Omega=j+\tfrac{1}{2}\ , \tag{2}$$

where the operators $\hat{a}_m$ and $\hat{b}_m$ create the particle in the upper level and the hole in the lower level, respectively. The quasi-spin operators $(\hat{J}_+,\hat{J}_-,\hat{J}_z)$ satisfy the algebra of SU(2):

$$[\hat{J}_+,\hat{J}_-]=2\hat{J}_z,\quad [\hat{J}_z,\hat{J}_\pm]=\pm\hat{J}_\pm\ . \tag{3}$$

Let us now introduce the fermion coherent state,

$$|c\rangle=\exp\left\{\sum_m[(\hat{a}_m^\dagger a_m-a_m^*\hat{a}_m)+(\hat{b}_m^\dagger b_m-b_m^*\hat{b}_m)]\right\}\ ,$$
$$\hat{a}_m|c\rangle=a_m|c\rangle,\quad \hat{b}_m|c\rangle=b_m|c\rangle\ , \tag{4}$$

where (a_m,a_m^*) and (b_m,b_m^*) are Grassmann numbers which obey the following anticommunication relations:

$$a_ma_n^*=-a_n^*a_m,\quad a_ma_n=-a_na_m\ ,$$
$$b_mb_n^*=-b_n^*b_m,\quad b_mb_n=-b_nb_m\ , \tag{5}$$
$$a_mb_n^*=-b_n^*a_m,\quad a_mb_n=-b_na_m\ .$$

Here the fermion coherent state satisfies the completeness relation,

$$\int d\mu(c)|c\rangle\langle c|=1\ , \tag{6}$$

where the invariant measure $d\mu(c)$ is given by

$$d\mu(c)=\prod_m da_m da_m^* e^{-\sum_m a_m^* a_m}\prod_m db_m db_m^* e^{-\sum_m b_m^* b_m}\ . \tag{7}$$

Following the standard procedure of Feynman's path integral with the use of completeness relation (6), the trace of the time evolution operator can be represented by the Grassmann path integral,

$$\begin{aligned}K(T)&=\mathrm{tr}e^{-i\hat{H}T}\\&=\int d\mu(c)\langle c|e^{-i\hat{H}T}|c\rangle\\&=N\int D[a]D[a^*]D[b]D[b^*]\\&\quad\times\exp\left[i\int_0^T L(t)dt\right]\ ,\end{aligned} \tag{8}$$

where N is a normalization constant and D denotes the functional integral. The Lagrangian is written as

$$L(t)=\frac{i}{2}\sum_m[(a_m^*\dot a_m+a_m\dot a_m^*)+(b_m^*\dot b_m+b_m\dot b_m^*)]-2\varepsilon J_z+\tfrac{1}{2}V(J_+J_++J_-J_-)\,, \tag{9}$$

$$J_+=\sum_m a_m^*b_m^*=(J_-)^\dagger\,,\qquad J_z=\tfrac{1}{2}\left[\sum_m(a_m^*a_m+b_m^*b_m)-2\Omega\right], \tag{10}$$

where the overdot denotes the time derivative. As the Lagrangian $L(t)$ is not quadratic in the Fermi fields, a direct path-integral treatment is difficult. It is, therefore, useful to introduce an equivalent Lagrangian which is quadratic in the Fermi fields. As is well known, a Gaussian method[2] leads to the Lagrangian that is quadratic in the Fermi fields. We introduce the collective fields (ρ_+,ρ_-,ρ_z) and multiply Eq. (8) by

$$\int D[\rho_+]D[\rho_-]D[\rho_z]\exp\left\{-\frac{i}{2}V\int dt[(\rho_+-J_+)^2+(\rho_--J_-)^2]\right\}=\text{constant}\,. \tag{11}$$

Equation (8) then becomes

$$K(T)=N'\int D[\rho_+]D[\rho_-]D[\rho_z]\exp\left[-\frac{i}{2}\int_0^T V(\rho_+^2+\rho_-^2)dt\right]K_i(T)\,, \tag{12}$$

$$K_i(T)=\int D[a]D[a^*]D[b]D[b^*]\exp\left[i\int_0^T L_i dt\right], \tag{13}$$

where $N'=N\times$constant, and the Lagrangian $L_i(t)$ is given by

$$L_i(t)=\frac{i}{2}\sum_m[(a_m^*\dot a_m+a_m\dot a_m^*)+(b_m^*\dot b_m+b_m\dot b_m^*)]-2\varepsilon J_z+V(\rho_+J_++\rho_-J_-)\,, \tag{14}$$

which describes a many-fermion system coupled to the external fields (ρ_+,ρ_-,ρ_z). The Lagrangian L_i is quadratic in the Fermi fields but contains the new fields (collective fields). In connection with the mean-field theory, it is convenient to parametrize auxiliary fields as follows:

$$\rho_+=2r\frac{\zeta^*}{1+\zeta^*\zeta}\,,\qquad \rho_-=2r\frac{\zeta}{1+\zeta^*\zeta}\,,\qquad \rho_z=-r\frac{1-\zeta^*\zeta}{1+\zeta^*\zeta}\,, \tag{15}$$

where ζ is the complex variable and r is the real variable.

The fermion propagator $K_i(T)$ is rewritten as

$$K_i(T)=\int d\mu(c)\langle c\,|\,e^{-i\hat h T}\,|\,c\rangle=\mathrm{tr}\,e^{-i\hat h T}\,,\qquad \hat h=2\varepsilon\hat J_z-\frac{2Vr}{1+\zeta^*\zeta}(\zeta^*\hat J_++\zeta\hat J_-)\,. \tag{16}$$

In the trace of Eq. (16), we use the other fermion coherent state $|\gamma\rangle$ instead of $|c\rangle$:

$$K_i(T)=\int d\mu(\gamma)\langle\gamma\,|\,e^{-i\hat h T}\,|\,\gamma\rangle=\int D[\alpha]D[\alpha^*]D[\beta]D[\beta^*]\exp\left[i\int_0^T\tilde L dt\right]. \tag{17}$$

The fermion coherent state $|\gamma\rangle$ is given by

$$|\gamma\rangle=\exp\left\{\sum_m[(\hat\alpha_m^\dagger\alpha_m-\alpha_m^*\hat\alpha_m)+(\hat\beta_m^\dagger\beta_m-\beta_m^*\hat\beta_m)]\right\}, \tag{18}$$

$$\hat\alpha_m=u\hat a_m-v\hat b_m^\dagger\,,\qquad \hat\beta_m=u\hat b_m+v\hat a_m^\dagger\,, \tag{19}$$

where α_m and β_m are Grassmann numbers and the coefficients are $u=(1+\zeta^*\zeta)^{-1/2}$, $v=(1+\zeta^*\zeta)^{-1/2}\zeta$. Here the Lagrangian $\tilde L(t)$ is given by

$$\tilde L(t)=2\Omega A+2\varepsilon\Omega+\frac{i}{2}\sum_m[(\alpha_m^*\dot\alpha_m+\alpha_m\dot\alpha_m^*)+(\beta_m^*\dot\beta_m+\beta_m\dot\beta_m^*)]-A\sum_m(\alpha_m^*\alpha_m+\beta_m^*\beta_m)+B\sum_m\alpha_m^*\beta_m^*+B^*\sum_m\beta_m\alpha_m\,, \tag{20}$$

where the coefficients A and B are

$$A=\frac{i}{2}\frac{\zeta^*\dot\zeta-\dot\zeta^*\zeta}{1+\zeta^*\zeta}+\varepsilon\frac{1-\zeta^*\zeta}{1+\zeta^*\zeta}+2Vr\frac{\zeta^{*2}+\zeta^2}{(1+\zeta^*\zeta)^2}\,,\qquad B=\frac{i\dot\zeta}{1+\zeta^*\zeta}-2\varepsilon\frac{\zeta}{1+\zeta^*\zeta}+2Vr\frac{\zeta^*-\zeta^3}{(1+\zeta^*\zeta)^2}\,. \tag{21}$$

Path integral (12) can then be written as

$$K(T)=N'\int D[r]D[\zeta]D[\zeta^*]\exp\left[i\int_0^T L_0(t)dt\right]K_f(T)\ , \tag{22}$$

$$K_f(T)=\int D[\alpha]D[\alpha^*]D[\beta]D[\beta^*]\exp\left[i\int_0^T L_f(t)dt\right]\ . \tag{23}$$

Here the Lagrangians are defined by

$$L_0(t)=2\Omega A+2\varepsilon\Omega-2Vr^2\frac{\zeta^{*2}+\zeta^2}{(1+\zeta^*\zeta)^2}\ , \tag{24}$$

$$L_f(t)=\frac{i}{2}\sum_m[(\alpha_m^*\dot\alpha_m+\alpha_m\dot\alpha_m^*)+(\beta_m^*\dot\beta_m+\beta_m\dot\beta_m^*)]-A\sum_m(\alpha_m^*\alpha_m+\beta_m^*\beta_m)+B\sum_m\alpha_m^*\beta_m^*+B^*\sum_m\beta_m\alpha_m\ . \tag{25}$$

If the degrees of freedom α_m, α_m^*, β_m, and β_m^* are frozen in Eq. (22), the propagator $K(T)$ can be reduced as

$$K_0(T)=N'\int D[r]D[\zeta]D[\zeta^*]\exp\left[i\int_0^T L_0(t)dt\right]\ . \tag{26}$$

Let us now examine the classical limit. The classical propagator $K_0^{\rm cl}$ is obtained through the stationary phase approximation (SPA):

$$K_0^{\rm cl}\propto\exp(iS_0^{\rm cl})\ , \tag{27}$$

where the classical action $S_0^{\rm cl}$ satisfies the variational equation $\delta S_0=0$, and then the classical equation of motion becomes

$$\begin{aligned} &r=\Omega\ ,\\ &i\dot\zeta=2\varepsilon\zeta-2Vr\frac{\zeta^*-\zeta^3}{1+\zeta^*\zeta}\ ,\\ &i\dot\zeta^*=-2\varepsilon\zeta^*+2Vr\frac{\zeta-\zeta^{*3}}{1+\zeta^*\zeta}\ . \end{aligned} \tag{28}$$

Under the parametrization

$$\zeta=\tan(\theta/2)\exp(-i\psi)\ ,$$

the preceding equation is rewritten as

$$\begin{aligned} &\dot\theta=-2V\Omega\sin\theta\sin(2\psi)\ ,\\ &\dot\psi=2\varepsilon-2V\Omega\cos\theta\cos(2\psi)\ . \end{aligned} \tag{29}$$

This is nothing but the conventional time-dependent Hartree-Fock (TDHF) equation.[8] Thus the TDHF equation is involved as the classical solution of the collective field ζ in the functional integral $K_0(T)$.[9] Therefore we can see that the Lagrangian L_f of Eq. (25) describes a system of independent particles moving in the collective field.

Next let us proceed to the semiclassical quantization[10] for such a system. We first set up the following conditions for Eq. (23):

$$B=0,\quad B^*=0\ . \tag{30}$$

We will find that these conditions coincide with classical equation (36) later on. Under the quantum adiabatic approximation, the Grassmann functional integral is evaluated as[5]

$$K_f(T)=e^{i\alpha/2}(1+e^{-i\alpha})\ , \tag{31}$$

$$\alpha=\int A(t)dt\ . \tag{32}$$

Substituting Eq. (31) into Eq. (22), we obtain the propagator,

$$K(T)=\sum_n\frac{(2\Omega)!}{(2\Omega-n)!n!}\int D[r]D[\zeta]D[\zeta^*]\exp(iS_{\rm eff})\ , \tag{33}$$

$$S_{\rm eff}=i\int_0^T\left[2(\Omega-n)A-2Vr^2\frac{\zeta^{*2}+\zeta^2}{(1+\zeta^*\zeta)^2}+2\varepsilon\Omega\right]dt\ . \tag{34}$$

Thus the contribution of the Fermi fields is entirely absorbed in the effective action $S_{\rm eff}$. We can now apply the SPA to Eq. (33), and get the semiclassical propagator

$$K^{\rm cl}(T)\propto\sum_n\frac{(2\Omega)!}{(2\Omega-n)!n!}\exp(iS_{\rm eff}^{\rm cl})\ , \tag{35}$$

where the classical action $S_{\rm eff}^{\rm cl}$ satisfies the variational equation $\delta S_{\rm eff}=0$:

$$\begin{aligned} &r=\Omega-n\ ,\\ &i\dot\zeta=2\varepsilon\zeta-2Vr\frac{2\Omega-2n-r}{\Omega-n}\frac{\zeta^*-\zeta^3}{1+\zeta^*\zeta}\ ,\\ &i\dot\zeta^*=-2\varepsilon\zeta^*+2Vr\frac{2\Omega-2n-r}{\Omega-n}\frac{\zeta-\zeta^{*3}}{1+\zeta^*\zeta}\ . \end{aligned} \tag{36}$$

The above result agrees with condition (30) for $r=\Omega-n$. Therefore we find that condition (30) is self-consistent with the semiclassical approximation. Equation (36) is just the classical equation of path integral (22). By comparing with TDHF equation (29),[8] we see that the preceding equation involves the contribution from the independent particles. For this reason, Eq. (36) is just an extension of the conventional TDHF equation (28). Substituting Eq. (35) into the propagator

$$K(E)=i\int_0^\infty e^{iET}K(T)dT\ , \tag{37}$$

and using the SPA about T, we find the classical propagator

$$K^{\rm cl}(E) \propto \sum_n \frac{(2\Omega)!}{(2\Omega-n)!n!} e^{iW} , \quad (38)$$

$$W = i(\Omega-n)\int_0^T \frac{\dot{\zeta}\zeta^* - \dot{\zeta}^*\zeta}{1+\zeta^*\zeta} dt . \quad (39)$$

Following the standard procedure of the semiclassical quantization, the propagator $K^{\rm cl}(E)$ has poles about

$$i(\Omega-n)\int_0^T \frac{1}{1+\zeta^*\zeta}(\zeta^*\dot{\zeta} - \zeta\dot{\zeta}^*)dt = 2m\pi . \quad (40)$$

This condition is essentially equivalent to the semiclassical quantization condition derived by Shanker,[6] except that it includes the independent-particle degrees of freedom. Furthermore, we notice that the energy levels given by Eq. (40) are characterized by two integers m and n. The quantum number m represents, in semiclassical language, the number of full waves fitted along the classical orbit satisfying Eq. (36), while n labels the excitations of the independent particles and denotes the seniority numbers.

This model is quite familiar and soluble. Although we are dealing with the Lipkin model, we can easily make an extension to more complicated systems. The formulation of the general case will be given in a subsequent paper.

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